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Design of Optimal Controller with Integral and Preview Actions for Discrete-Time System

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Abstract. This paper is concerned with a method of designing a type one servomechanism for a discrete-time system subject to a time-varying demand and an unmeasurable constant disturbance. It is assumed that the time-varying demand is previewable in the sense that some finite future as well as present and past values of the demands are available at each time. An optimal controller with state feedback plus integral and preview actions is derived by applying a linear quadratic integral technique (Tomizuka and Rosenthal 1979). It is shown under the stabilizability and detectability conditions that the closed loop system achieves a complete regulation in the presence of small perturbations in system parameters, eliminating the effect of disturbance. An observer based controller is also considered.

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1. Introduction

In many practical control systems design, it is required that the outputs, or the controlled variables, track without steady state error the demand signals in the presence of unmeasurable disturbances. For more than a decade, there has been much interest in tracking or servomechanism problems for linear time-invariant multivariable systems (Davison 1972, Smith and Davison 1972, Young and Willems 1972, Bradshaw and Porter 1976, Furuta and Kamiya 1982). Furthermore, design problems of robust servomechanism have extensively been studied by the state-space and frequency domain approaches (Davison and Goldenberg 1975, Davison 1976, Francis and Wonham 1976, Ferreira 1976). An overview of the state of knowledge on the robust servomechanism problem is presented by Desoer and Wang (1980).

In most papers mentioned above, however, it is assumed that the desired signals as well as disturbances are constants, or ramp functions, or more generally, the outputs of some free time-invariant linear systems. More recently, assuming that the disturbances are previewable, Tomizuka and Rosenthal (1979) have developed a digital controller with state feedback plus integral and preview actions for a discrete-time system with a constant demand input; they have shown that the preview of future disturbances is very effective for improving the transient responses of the closed loop system. A related finite preview control problem for a continuous-time system is also considered by Tomizuka (1975).

This paper deals with a tracking problem for a discrete-time system in the presence of unmeasurable disturbances. It is assumed that the desired signal is rather arbitrary but eventually converges to a constant vector, and that finite future values of the demand signal are available at each instant of time. These assumptions may not be unrealistic in many practical control problems. For example, in power plant control, we must keep the outputs at constant levels over a period of time, where the constant levels, or the set points, may change from time to time according to the load demand, for which a local future information is available. We wish to present a method of designing an optimal type one servomechnism for a discrete-time system by extending the linear quadratic integral (LQI) technique due to Tomizuka and Rosenthal (1979).

This paper is organized as follows. In section 2, we formulate the tracking problem as an LQI problem by defining an appropriate performance index and an augmented state-space model that includes the available future demands as a part of the state vector. The optimal controller with state feedback plus integral and preview actions is derived in section 3. Section 4 presents some preliminary lemmas. In section 5, we show that the closed loop system is asymptotically stable and hence a complete regulation occurs under the conditions of stabilizability (or reachability) and detectability (or observability). We also show that a complete regulation occurs in the presence of small perturbations in system parameters. Section 6 is devoted to the stability analysis of the overall system when an observer is incorporated into the state feedback loop.

A numerical example from a power plant control is presented elsewhere (Katayama, Ohki, Inoue and Nakayama 1984).

2. Problem statement

We consider a time-invariant linear discrete system described by

$$x(k+1) = Ax(k) + Bu(k) + Ew(k) \tag{1}$$

$$y(k) = Cx(k) \tag{2}$$

where $x(k)$ is the $n \times 1$ state vector, $u(k)$ is the $r \times 1$ control vector, $y(k)$ is the $p \times 1$ output vector to be controlled, and $w(k)$ is the $q \times 1$ inaccessible constant disturbance. The A, B, C, E are constant matrices of dimensions $n \times n$, $n \times r$, $p \times n$, $n \times q$, respectively. It is assumed that rank $B = r$, rank $C = p$, and rank $E = q$.

Let $y_d(k)$ be the $p \times 1$ desired output, or the demand vector, for which we assume that there exists a constant vector $\bar{y}_d$ such that $\lim_{k \to \infty} y_d(k) = \bar{y}_d$. This implies that the demand vector is an arbitrary time-varying function, except that it reaches a steady state. We further assume that the demand is previewable in the sense that at each time k, N_L future values $y_d(k+1), \ldots, y_d(k+N_L)$ as well as the present and past values of the demand are available. The future values of the desired output beyond time $k+N_L$ are approximated by $y_d(k+N_L)$, namely

$$y_d(k+i) = y_d(k+N_L), \quad i = N_L+1, \ldots \tag{3}$$

The basic design problem considered in this paper is to find a controller such that:

1. In the steady state, the output $y(k)$ tracks the desired output $y_d(k)$ in the presence of disturbance $w(k)$.

2. The closed loop system is asymptotically stable and exibits acceptable transient responses.

In order to meet with the above requirements, it is desired to introduce integrators to eliminate the tracking error $e(k) = y(k) - y_d(k)$. In other words, we must design a type one servomechanism for the system of (1) and (2) such that the asymptotic regulation occurs: $e(k) \to \infty$ as $k \to \infty$, while keeping the transient responses satisfactory in some sense. To this end, we employ the LQI technique (Athans 1971, Smith and Davison 1972, Tomizuka and Rosenthal 1979).

Let the incremental state vector be $\Delta x(k) = x(k) - x(k-1)$, and let the incremental control vector be $\Delta u(k) = u(k) - u(k-1)$. It is well known (Athans 1971) that the integral action of the controller is introduced by including the incremental control in the performance index. Therefore we wish to obtain the optimal controller $u(k)$ such that the performance index

$$J = \sum_{i=k}^{\infty} [e^T(i)Q_e e(i) + \Delta x^T(i)Q_x \Delta x(i) + \Delta u^T(i)R\Delta u(i)] \tag{4}$$

is minimized at each time k, where Q_e and R are $p \times p$ and $r \times r$ symmetric positive definite matrices, respectively, and Q_x is an $n \times n$ symmetric nonnegative definite matrix, and where i denotes the dummy time index, and the superscript $(\cdot)^T$ denotes the transpose.

It should be noted that the term $e^T(i)Q_e e(i)$ represents the loss due to tracking error, and that $\Delta x^T(i)Q_x\Delta x(i)$ and $\Delta u^T(i)R\Delta u(i)$ represent the losses due to the incremental state and control vectors, respectively. Thus the physical interpretation of J is to achieve the asymptotic regulation without excessive rate of change in the state and control vectors. The quadratic term for the rate of change in state vector, which is not used in Tomizuka and Rosenthal (1979), will make the design technique more flexible, allowing us to directly regulate the transient responses of the state variables.

3. Design of optimal controller

We derive an augmented state-space description that includes the future information on the demand signal as well as the error $e(i)$, the incremental state vector $\Delta x(i)$, and the incremental control vector $\Delta u(i)$. From (1), the incremental state is described by

$$\Delta x(i+1) = A\Delta x(i) + B\Delta u(i), \quad i = k,k+1,... \tag{5}$$

where we note that the incremental disturbance $\Delta w(k)$ does not appear, because the disturbance is a step function. Also, we see from (2) and (5) that the tracking error satisfies

$$e(i+1) = e(i) + CA\Delta x(i) + CB\Delta u(i) - \Delta y_d(i+1), \quad i = k,k+1,... \tag{6}$$

where the incremental demand is defined by

$$\Delta y_d(i) = y_d(i) - y_d(i-1) \tag{7}$$

Combining (5) and (6) yields

$$\begin{bmatrix} e(i+1) \\ \Delta x(i+1) \end{bmatrix} = \begin{bmatrix} I_p & CA \\ 0 & A \end{bmatrix} \begin{bmatrix} e(i) \\ \Delta x(i) \end{bmatrix} + \begin{bmatrix} CB \\ B \end{bmatrix} \Delta u(i) + \begin{bmatrix} -I_p \\ 0 \end{bmatrix} \Delta y_d(i+1), \tag{8}$$

where $i = k, k+1,...,$ and I_p denotes the $p \times p$ unit matrix.

Since N_L future demands $y_d(i)$, $i = k,k+1,...,k+N_L$, are available at time k, the relevant informantion on the incremental demand can be summarized as the $pN_L \times 1$ vector:

$$x_d(k) = [\Delta y_d^T(k),...,\Delta y_d^T(k+N_L)]^T \tag{9}$$

It follows from the assumption of (3) that $x_d(i)$ satisfies

$$x_d(i+1) = A_d x_d(i), \quad i = k, k+1,... \tag{10}$$

where

$$A_d = \begin{bmatrix} 0 & I_p & & \mathbf{0} \\ & 0 & \ddots & \\ & & \ddots & I_p \\ \mathbf{0} & & & 0 \end{bmatrix} \quad (pN_L \times pN_L) \tag{11}$$

Now define the $(p+n+pN_L) \times 1$ augmented state vector

$$\bar{x}(i) = [e^T(i) \;\; \Delta x^T(i) \;\; x_d{}^T(i)]^T \tag{12}$$

Putting together (8) and (10) yields

$$\bar{x}(i+1) = \left[\begin{array}{cc|cccc} I_p & CA & -I_p & 0 & \dots & 0 \\ 0 & A & 0 & 0 & \dots & 0 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{4}{c}{A_d} \end{array}\right] \bar{x}(i) + \left[\begin{array}{c} CB \\ B \\ \hline 0 \end{array}\right] \Delta u(i), \quad i = k, k+1, \dots \tag{14}$$

On the other hand, in terms of the augmented state vector $\bar{x}(i)$, the performance index J of (4) is expressed as

$$J = \sum_{i=k}^{\infty} \left\{ \bar{x}(i) \begin{bmatrix} Q_e & 0 & 0 \\ 0 & Q_x & 0 \\ 0 & 0 & 0 \end{bmatrix} \bar{x}(i) + \Delta u^T(i) R \Delta u(i) \right\} \tag{15}$$

Therefore, the optimal controller can be derived by solving the optimal control problem that minimizes the performance index J of (15) subject to the dynamic constraint of (14).

For the sake of simplicity, we define

$$\tilde{B} = \begin{bmatrix} CB \\ B \end{bmatrix}, \quad \tilde{I} = \begin{bmatrix} I_p \\ 0 \end{bmatrix}, \quad \tilde{F} = \begin{bmatrix} CA \\ A \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} Q_e & 0 \\ 0 & Q_x \end{bmatrix}, \quad \tilde{A} = [\tilde{I} \;\; \tilde{F}] \tag{16}$$

<u>Theorem 1</u>

The optimal incremental control $\Delta u^o(k)$ is given by

$$\Delta u^o(k) = - G_I e(k) - G_x \Delta x(k) - \sum_{\ell=1}^{N_L} G_d(\ell) \Delta y_d(k+\ell) \tag{17}$$

where

$$G_I = [R + \tilde{B}^T \tilde{K} \tilde{B}]^{-1} \tilde{B}^T \tilde{K} \tilde{I} \tag{18a}$$

$$G_x = [R + \tilde{B}^T \tilde{K} \tilde{B}]^{-1} \tilde{B}^T \tilde{K} \tilde{F} \tag{18b}$$

$$G_d(1) = - G_I \tag{18c}$$

$$G_d(\ell) = [R + \tilde{B}^T \tilde{K} \tilde{B}]^{-1} \tilde{B}^T \tilde{X}(\ell-1), \quad \ell = 2, \dots, N_L \tag{18d}$$

and where the $(p+n) \times (p+n)$ matrix $\tilde{K}$ is the nonnegative definite solution of the algebraic Riccati equation

$$\tilde{K} = \tilde{A}^T\tilde{K}\tilde{A} - \tilde{A}^T\tilde{K}\tilde{B}[R + \tilde{B}^T\tilde{K}\tilde{B}]^{-1}\tilde{B}^T\tilde{K}\tilde{A} + \tilde{Q} \qquad (19)$$

Furthermore, the $(p+n) \times p$ matrices $\tilde{X}(\ell)$ are given by

$$\tilde{X}(\ell) = \tilde{A}_c^T \tilde{X}(\ell-1), \quad \ell = 2,\ldots, N_L; \quad \tilde{X}(1) = - \tilde{A}_c^T \tilde{K}\tilde{I} \qquad (20)$$

where $\tilde{A}_c$ is the closed loop matrix defined by

$$\tilde{A}_c = \tilde{A} - \tilde{B}[R + \tilde{B}^T\tilde{K}\tilde{B}]^{-1}\tilde{B}^T\tilde{K}\tilde{A} \qquad (21)$$

Proof

A proof is elementary, and is omitted. □

Theorem 2

The optimal controller $u^o(k)$ is given by

$$u^o(k) = - G_I \sum_{i=0}^{k} e(i) - G_x x(k) - \sum_{\ell=1}^{N_L} G_d(\ell) y_d(k+\ell) \qquad (22)$$

where it is assumed that $y(k) = y_d(k) = 0$, $x(k) = 0$ for $k = 0, -1, \ldots .$

Proof

A proof is immediate from Theorem 1. □

It should be noted that the optimal controller $u^o(k)$ of (22) consists of three terms; the first term represents the integral action on the tracking error, the second term represents the state feedback, and the third term is the feedforward or preview action based on the local future information on the demand vector.

We observe that if $N_L = 0$, then the preview action disappears from (22), so that $u^o(k)$ becomes

$$u^o(k) = - G_I \sum_{i=0}^{k} e(i) - G_x x(k) \qquad (23)$$

Moreover, since $G_d(1) = -G_I$, if $N_L = 1$, then we have

$$u^o(k) = -G_I \sum_{i=0}^{k} e(i) - G_x x(k) - G_d(1) y_d(k+1) \tag{24a}$$

$$= -G_I \sum_{i=0}^{k} [y(i) - y_d(i+1)] - G_x x(k) \tag{24b}$$

This is a state feedback controller with integral and feedforward actions.

Let $v(k)$ be the discrete integral of tracking error $e(k)$, namely

$$v(k) = v(k-1) + e(k) \tag{25a}$$

or

$$v(k) = \frac{z}{z-1} e(k) \tag{25b}$$

Thus it follows from (22) and (25) that the optimal controller is expressed as

$$u^o(k) = -G_I v(k) - G_x x(k) - \sum_{\ell=1}^{N_L} G_d(\ell) y_d(k+\ell) \tag{26}$$

Noting that $e(k) = y(k) - y_d(k)$, it follows from (1), (2) and (25a) that

$$v(k+1) = v(k) + CAx(k) + CBu(k) + CEw(k) - y_d(k+1) \tag{27}$$

Combining (1) and (27) gives

$$\begin{bmatrix} v(k+1) \\ x(k+1) \end{bmatrix} = \tilde{A} \begin{bmatrix} v(k) \\ x(k) \end{bmatrix} + \tilde{B}u(k) + \tilde{E}w(k) - \tilde{I}y_d(k+1) \tag{28}$$

where

$$\tilde{E} = \begin{bmatrix} CE \\ E \end{bmatrix} \tag{29}$$

Substituting (26) into (28) yields

$$\begin{bmatrix} v(k+1) \\ x(k+1) \end{bmatrix} = \tilde{A}_c \begin{bmatrix} v(k) \\ x(k) \end{bmatrix} - \tilde{B} \sum_{\ell=1}^{N_L} G_d(\ell) y_d(k+\ell) - \tilde{I} y_d(k+1) + \tilde{E} w(k) \qquad (30)$$

where $\tilde{A}_c$ is given by (20).

Therefore we observe that the closed loop characteristic is determined by $\tilde{A}_c$, or the state feedback and the integral action, so that the stability of the overall system is independent of the preview action. It should be noted that the controller $u^o(k)$ is independent of the matrix E; thus the exact knowledge of the disturbance matrix is not necessary for designing the optimal controller. Note that this is not the case if the state vector is not directly accessible (see section 6.).

4. Preliminary lemmas

In order to prove the asymptotic stability of the closed loop system, we need some preliminary lemmas for stabilizability (or reachability) and detectability (or observability).

Lemma 1a

The pair $(\tilde{A}, \tilde{B})$ is stabilizable if and only if (A, B) is stabilizable and the following rank condition holds:

$$\text{rank} \begin{bmatrix} 0 & C \\ B & A - I_n \end{bmatrix} = p + n \qquad (31)$$

Proof

For the proofs of this and following lemmas, the PBH rank test (Kailath 1980) is employed. Assume that (A, B) is stabilizable. For the stabilizability of $(\tilde{A}, \tilde{B})$, it suffices to show that for any complex $|\lambda| \geqq 1$

$$\operatorname{rank}[\tilde{A} - \lambda I_{p+n} \;\vdots\; \tilde{B}] = \operatorname{rank}\begin{bmatrix} (1-\lambda)I_p & CA & CB \\ 0 & A - \lambda I_n & B \end{bmatrix} = p + n \qquad (32)$$

Since $\operatorname{rank}[A - \lambda I_n \;\vdots\; B] = n$ for any complex $|\lambda| \geqq 1$, we see that (32) holds for $\lambda \neq 1$. For the case of $\lambda = 1$, it follows from (31) that

$$\operatorname{rank}\begin{bmatrix} CA & CB \\ A - I_n & B \end{bmatrix} = \operatorname{rank}\left\{\begin{bmatrix} I_p & C \\ 0 & I_n \end{bmatrix}\begin{bmatrix} 0 & C \\ B & A - I_n \end{bmatrix}\right\} = p + n \qquad (33)$$

Thus we have shown that (32) holds for any complex $|\lambda| \geqq 1$.

Now assume that $(\tilde{A}, \tilde{B})$ is stabilizable, so that (32) holds for any complex $|\lambda| \geqq 1$. Since the matrix $[\tilde{A} - \lambda I_{p+n} \;\vdots\; \tilde{B}]$ has a maximal row rank for any complex $|\lambda| \geqq 1$, we see that $\operatorname{rank}[A - \lambda I_n \;\vdots\; B] = n$ for any complex $|\lambda| \geqq 1$. Letting $\lambda = 1$ in (32), and using (33), we have (31). □

A continuous-time version of Lemma 1a has been proved by Smith and Davison (1972) by manipulating the controllability matrix. It is also well known that the rank condition of (31) implies that the system (C, A, B) has no transmission zeros at $z = 1$ (Davison 1976).

<u>Lemma 1b</u>

The pair $(\tilde{A}, \tilde{B})$ is reachable if and only if (A, B) is reachable and the rank condition of (31) holds.

<u>Proof</u>

For the reachability of $(\tilde{A}, \tilde{B})$, it suffices to show that (32) holds for any complex λ. Assume that (A, B) is reachable. Then it follows that $\operatorname{rank}[A - \lambda I_n \;\vdots\; B] = n$ for any complex λ. Thus, for $\lambda \neq 1$, we can easily see that (32) holds. Moreover, for $\lambda = 1$, (33) holds as shown above. This implies that $(\tilde{A}, \tilde{B})$ is reachable. On the other hand, if $(\tilde{A}, \tilde{B})$ is reachable, then (32) holds for any complex λ. Hence, as in the proof of Lemma 1a, we have $\text{rank}[A - \lambda I_n \mid B] = n$ for any complex λ, and (31). This completes the proof of Lemma 1b. □

By manipulating the controllability matrix, Seraji (1983) has proved Lemma 1b, and Young and Willems (1972) and Smith and Davison (1972) have proved the continuous-time version of Lemma 1b.

Now let H_e and H_x be $p \times p$ and $n \times n$ matrices such that $Q_e = H_e^T H_e$ and $Q_x = H_x^T H_x$, respectively. Then we have

$$Q = \tilde{H}^T \tilde{H} \tag{34}$$

where

$$\tilde{H} = \begin{bmatrix} H_e & 0 \\ 0 & H_x \end{bmatrix} \tag{35}$$

Lemma 2a

Let Q_e be positive definite. If (C, A) is detectable, then $(\tilde{H}, \tilde{A})$ is detectable.

Proof

We can easily see that (CA, A) is detectable if and only if (C, A) is detectable. For the detectability of $(\tilde{H}, \tilde{A})$, it suffices to show that for any complex $|\lambda| \geqq 1$,

$$\text{rank}\begin{bmatrix} \tilde{H} \\ \tilde{A} - \lambda I_{p+n} \end{bmatrix} = \text{rank}\begin{bmatrix} H_e & 0 \\ 0 & H_x \\ (1-\lambda) I_p & CA \\ 0 & A - \lambda I_n \end{bmatrix} = p + n \tag{36}$$

Suppose that (C, A) is detectable, and hence (CA, A) is detectable. Then for any complex $|\lambda| \geqq 1$,

$$\text{rank} \begin{bmatrix} CA \\ A - \lambda I_n \end{bmatrix} = n \tag{37}$$

But since rank H_e = rank Q_e = p, it follows from (37) that (36) holds for any complex $|\lambda| \geqq 1$. □

Lemma 2b

Let Q_e be positive definite, and assume that A is nonsingular. Then if (C, A) is observable, $(\tilde{H}, \tilde{A})$ is observable.

Proof

Suppose that (C, A) is observable. Since A is nonsingular, (CA, A) is observable if and only if (C, A) is observable. Therefore (37) holds for any complex λ, so that we see from rank H_e = p that (36) holds for any complex λ. This implies that $(\tilde{H}, \tilde{A})$ is observable. □

It should be noted that if $Q_x = 0$, then Lemmas 2a and 2b give necessary and sufficient conditions for the detectability and observability of $(\tilde{H}, \tilde{A})$, respectively.

5. Property of feedback system

In this section, we consider the stability of the closed loop system described by (30).

Theorem 3a

Suppose that the following conditions are satisfied:

a) Q_e and R are positive definite;

b) The rank condition of (31) holds:

$$\text{rank} \begin{bmatrix} 0 & C \\ B & A - I_n \end{bmatrix} = p + n; \tag{31}$$

c) (A, B) is stabilizable;

d) (C, A) is detectable.

Then the algebraic Riccati equation of (19) has the unique nonnegative definite solution $\tilde{K}$, and the eigenvalues of $\tilde{A}_c$ of (21) are all inside the unit circle in the complex plane, namely, $\tilde{A}_c$ is asymptotically and exponentially stable. □

Proof

From Lemmas 1a and 2a, it follows that $(\tilde{A}, \tilde{B})$ is stabilizable and $(\tilde{H}, \tilde{A})$ is detectable. Furthermore, since R is positive definite the algebraic Riccati equation

$$\tilde{K} = \tilde{A}^T\tilde{K}\tilde{A} - \tilde{A}^T\tilde{K}\tilde{B}[R + \tilde{B}^T\tilde{K}\tilde{B}]^{-1}\tilde{B}^T\tilde{K}\tilde{A} + \tilde{H}^T\tilde{H} \qquad (19')$$

is well defined. Thus the theorem is proved by applying the well known theorem for the linear quadratic regulator (Kucera 1972, Kwakernaak and Sivan 1972). □

Theorem 3b

Suppose that the conditions of a), b) of Theorem 3a are satisfied. Moreover, assume that

c') (A, B) is reachable;

d') (C, A) is observable and A is nonsingular.

Then the statement of Theorem 3a holds, except that the algebraic Riccati equation has the unique positive definite solution.

Proof

It follows from Lemmas 1b and 2b that $(\tilde{A}, \tilde{B})$ is reachable and $(\tilde{H}, \tilde{A})$ is observable. The rest of the proof is standard (Kucera 1972, Kwakernaak and Sivan 1972). □

Remark 1

It should be noted that the condition of (31) implies that $r \geqq p$. Thus for $\tilde{A}_c$ to be asymptotically stable, the number of control variables must be greater than or equal to that of the output variables to be controlled. This is quite common in practical control problems.

Remark 2

It follows from (18c), (18d) and (20) that the preview gains are given by

$$G_d(\ell) = -[R + \tilde{B}^T\tilde{K}\tilde{B}]^{-1}\tilde{B}^T(\tilde{A}_c{}^T)^{\ell-1}\tilde{K}\tilde{I}, \quad \ell = 1,\ldots, N_L \tag{38}$$

Thus, under the assumption of Theorem 3a or 3b, the information on the future values of the demand vector becomes less important as ℓ increases, since $\tilde{A}_c$ is exponentially stable.

Now we show that under the assumption of Theorem 3a or 3b, a complete regulation occurs for the optimal closed loop system.

Theorem 4

Assume that the conditions of either Theorem 3a or 3b are satisfied. If the demand vector is a step function, then a complete regulation occurs:

$$\lim_{k\to\infty} e(k) = 0 \quad \text{(exponentially)} \tag{39}$$

and also

$$\lim_{k\to\infty} x(k) = \bar{x} \quad \text{and} \quad \lim_{k\to\infty} u^o(k) = \bar{u} \tag{40}$$

where $\bar{x}$ and $\bar{u}$ are constant vectors related by

$$\begin{aligned} \bar{x} &= A\bar{x} + B\bar{u} + E\bar{w} \\ \bar{y}_d &= C\bar{x} \end{aligned} \tag{41}$$

and where $w(k) = \bar{w}$ for $k > 0$.

Proof

By taking the increment of (30), or by substituting $\Delta u^{o}(k)$ from (17) into (8), it follows that

$$\xi(k+1) = \tilde{A}_c \xi(k) + f(k) \qquad (42)$$

where $\xi(k) = [e^T(k) \quad \Delta x^T(k)]^T$, and

$$f(k) = - \tilde{I}\Delta y_d(k+1) - \tilde{B} \sum_{\ell=1}^{N_L} G_d(\ell)\Delta y_d(k+\ell) \qquad (43)$$

Since the demand vector is a step function, we have $\Delta y_d(k+\ell) = 0$ for any ℓ. Thus $f(k) \equiv 0$, so that (42) reduces to $\xi(k+1) = \tilde{A}_c\xi(k)$. But since $\tilde{A}_c$ is exponentially stable from Theorem 3a or 3b, it follows that $\lim_{k\to\infty} \xi(k) = 0$, so that $\lim_{k\to\infty} e(k) = 0$ and $\lim_{k\to\infty} \Delta x(k) = 0$. By using (17), we have $\lim_{k\to\infty} \Delta u^{o}(k) = 0$ Thus we have shown (40); moreover from (1) and (2), we have (41). □

Theorem 5

Assume that the conditions of either Theorem 3a or 3b are satisfied. If the demand vector satisfies

$$\lim_{k\to\infty} y_d(k) = \bar{y}_d \qquad (44)$$

then a complete regulation also occurs: $e(k) \to \infty$ as $k \to \infty$, and we have (40) and (41). The convergence of $e(k)$ is, however, not necessarily exponential, since it depends on the rate of convergnece of demand $y_d(k)$.

Proof

A proof is immediate by noting that $\tilde{A}_c$ is exponentially stable and that $f(k) \to 0$ as $k \to \infty$ in (42). □

Remark 3

It may be noted that since (41) can be written as

$$\begin{bmatrix} 0 & C \\ B & A - I_n \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{x} \end{bmatrix} = \begin{bmatrix} \bar{y}_d \\ -E\bar{w} \end{bmatrix} \quad (45)$$

the rank condition of (31) implies that there exist $\bar{u}$ and $\bar{x}$ for given $\bar{y}_d$ and $\bar{w}$. Note that if $p = r$, then the steady states $\bar{u}$ and $\bar{x}$ are independent of the quadratic weights $\tilde{Q}$ and R. The transient responses, however, heavily depend on the quadratic weights. It should be also noted that if $r > p$, namely, the number of control variables are greater than that of the output variables to be controlled, then the steady states $\bar{u}$ and $\bar{x}$ will be affected by the quadratic weights.

Remark 4

We note here that the asymptotic stability of a dynamic system is generally preserved for small perturbations in the system parameters. Thus it follows from Theorem 4 or 5 that a complete regulation occurs for the closed loop system of (30) in the presence of small perturbations in A, B, C, E matrices, namely, the controller is insensitive to small change in system parameters. Furthermore, the arbitrary perturbations are allowed as long as the closed loop system is asymptotically stable.

6. Observer based controller

When the state vector $x(k)$ is not directly measurable, we are led to the introduction of an observer or a Kalman filter to obtain the estimate of the state vector (O'Reilly 1983). In this section, we assume that the measurable output vector is given by

$$y_m(k) = C_m x(k) \quad (46)$$

where $y_m(k)$ is the $m \times 1$ measurable output vector, and C_m is the $m \times m$ constant matrix. Usually the $p \times 1$ output vector to be regulated is a part of the measurable output vector, so that there exists the $p \times m$ matrix M such that $C = MC_m$. This is called the readability condition (Francis and Wonham 1976).

Since $w(k)$ is constant, we have

$$\begin{bmatrix} x(k+1) \\ w(k+1) \end{bmatrix} = \begin{bmatrix} A & E \\ 0 & I_q \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u^o(k) \tag{47}$$

$$y_m(k) = [C_m \quad 0] \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \tag{48}$$

Let Y_m^{k-1} be the measurements up to $k-1$, namely, $Y_m^{k-1} = \{y_m(0), y_m(1), \ldots, y_m(k-1)\}$. Let $\hat{x}(k)$ and $\hat{w}(k)$ be the estimates of $x(k)$ and $w(k)$ based on the measurements Y_m^{k-1}, respectively. Then the full order observer for the system of (47) and (48) is given by

$$\begin{bmatrix} \hat{x}(k+1) \\ \hat{w}(k+1) \end{bmatrix} = \begin{bmatrix} A & E \\ 0 & I_q \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{w}(k) \end{bmatrix} + \begin{bmatrix} L_x \\ L_w \end{bmatrix} [y_m(k) - C_m\hat{x}(k)] + \begin{bmatrix} B \\ 0 \end{bmatrix} \hat{u}^o(k) \tag{49}$$

where L_x and L_w are $n \times m$ and $q \times m$ constant gain matrices, respectively, which are determined so that the $(n+q) \times (n+q)$ matrix

$$\tilde{A}_L = \begin{bmatrix} A - L_xC_m & E \\ -L_wC_m & I_q \end{bmatrix} \tag{50}$$

is asymptotically stable (O'Reilly 1983). It should be noted that $\hat{u}^o(k)$ in (49) is obtained by replacing $x(k)$ by $\hat{x}(k)$ in (22) or (26).

Lemma 3

Suppose that the following rank condition holds:

$$\text{rank}\begin{bmatrix} C_m & 0 \\ I_n - A & E \end{bmatrix} = n + q \tag{51}$$

If (C_m, A) is detectable, then we can find suitable gains L_x and L_w such that $\tilde{A}_L$ of (48) is asymptotically stable.

Proof

We show that the pair

$$\left\{[C_m \quad 0], \begin{bmatrix} A & E \\ 0 & I_q \end{bmatrix}\right\} \tag{52}$$

is detectable. It suffices to show that for any complex $|\lambda| \geq 1$,

$$\text{rank}\begin{bmatrix} C_m & 0 \\ \lambda I_n - A & -E \\ 0 & (\lambda-1)I_q \end{bmatrix} = n + q \tag{53}$$

Since (C_m, A) is detectable, for any complex $|\lambda| \geq 1$,

$$\text{rank}\begin{bmatrix} C_m \\ \lambda I_n - A \end{bmatrix} = n \tag{54}$$

Thus (53) holds for any complex $\lambda \neq 1$. If $\lambda = 1$, (53) also holds from the condition of (51). The rest of the proof is obvious from the definition of detectability. □

Note that the detectability (observability) of (C_m, A) follows from the detectability (observability) of (C, A). It should be also noted that the rank condition of (51) implies that the system (C_m, A, E) has no transmission zeros at $z = 1$ (Davison 1976), and that $m \geq q$, namely, the number of output variables is not less than that of the unmeasurable disturbances.

Now define the estimation errors by $\tilde{x}(k) = x(k) - \hat{x}(k)$ and $\tilde{w}(k) = w(k) - \hat{w}(k)$. Then, from (47)-(50), we have

$$\begin{bmatrix} \tilde{x}(k+1) \\ \tilde{w}(k+1) \end{bmatrix} = \tilde{A}_L \begin{bmatrix} \tilde{x}(k) \\ \tilde{w}(k) \end{bmatrix} \tag{55}$$

If we employ the estimate $\hat{x}(k)$ in place of the state vector $x(k)$ in the controller of (26), then we have

$$\hat{u}^o(k) = u^o(k) + G_x\tilde{x}(k) \tag{56}$$

since $\hat{x}(k) = x(k) - \tilde{x}(k)$. However, if $\tilde{A}_L$ is asymptotically stable, $\tilde{x}(k)$ converges to zero, so that the controller $\hat{u}^o(k)$ is asymptotically equivalent to $u^o(k)$.

Substituting (56) into (28) and combining the resultant system with (55) yield

$$\begin{bmatrix} v(k+1) \\ x(k+1) \\ \hdashline \tilde{x}(k+1) \\ \tilde{w}(k+1) \end{bmatrix} = \left[\begin{array}{c:c:c} \tilde{A}_c & \tilde{B}G_x & 0 \\ \hdashline 0 & \multicolumn{2}{c}{\tilde{A}_L} \end{array}\right] \begin{bmatrix} v(k) \\ x(k) \\ \hdashline \tilde{x}(k) \\ \tilde{w}(k) \end{bmatrix} + \begin{bmatrix} \tilde{E} \\ \hdashline 0 \end{bmatrix} w(k)$$

$$+ \begin{bmatrix} -\tilde{B} \\ \hdashline 0 \end{bmatrix} \sum_{\ell=1}^{N_L} G_d(\ell) y_d(k+\ell) - \begin{bmatrix} \tilde{I} \\ \hdashline 0 \end{bmatrix} y_d(k+\ell) \tag{57}$$

Therefore we have the following theorem.

<u>Theorem 6</u>

Suppose that the conditions of Theorem 3a are satisfied. If the rank condition of (51) holds, and if the demand vector $y_d(k)$ converges to $\bar{y}_d$, then there exist $\bar{v}$ and $\bar{x}$ such that

$$\lim_{k\to\infty} v(k) = \bar{v} \quad \text{and} \quad \lim_{k\to\infty} x(k) = \bar{x} \tag{58}$$

Hence a complete regulation is achieved under the observer based controller of $\hat{u}^o(k)$.

Proof

A proof is omitted. □

Remark 5

Conditions b), c), d) of Theorem 3a together with the readability conditions ($C = MC_m$) are equivalent to the necessary and sufficient conditions for the existence of a robust controller for the system of (1), (2) and (46) (Davison and Goldenberg 1975, Davison 1976).

Remark 6

As in Remark 4, the observer based controller of (56) achieves a complete regulation under small perturbations of system parameters. However, it is to be noted that the robustness of the LQ regulator is not preserved for the case when a state observer or a Kalman filter is introduced into the state feedback loop (Doyle and Stein 1979, O'Reilly 1983).

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