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Ground states of the massless Dereziński–Gérard model

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We consider the massless Dereziński–Gérard model introduced by Dereziński and Gérard in 1999. We give a sufficient condition for the existence of a ground state of the massless Dereziński–Gérard model without the assumption that the Hamiltonian of particles has compact resolvent. © *2009 American Institute of Physics.* [doi:10.1063/1.3253976]

I. INTRODUCTION

The Dereziński–Gérard model was introduced by Dereziński and Gérard[5] as an abstract model of particle-field interaction (they call it the Pauli–Fierz model, but we call it the Dereziński–Gérard model for the sake of clarity.). The Hamiltonian of the Dereziński–Gérard model is given by

$$H := A \otimes I + I \otimes H_{\mathrm{b}} + \widetilde{\phi}(v).$$

The first and second terms mean Hamiltonians of particles and a quantum field, respectively. The third term means the interaction between particles and a quantum field.

The existence of a ground state of the massive Dereziński–Gérard model was investigated by Dereziński and Gérard[5] and Miyao and Sasaki.[10] Dereziński and Gérard[5] showed the existence of a ground state of the massive Dereziński–Gérard model by specifying the essential spectrum of the Hamiltonian, which is an analog of the HVZ theorem in many body problems. Miyao and Sasaki[10] showed the existence of a ground state of the massive Dereziński–Gérard model by applying the method developed by Arai and Hirokawa.[2]

In Ref. 7, Georgescu *et al.* discussed spectral analysis by means of Mourre estimate, limiting absorption principle, and local finiteness of eigenvalues in the massless Dereziński–Gérard model. In Ref. 8, Gérard showed the existence of a ground state of the massless Dereziński–Gérard model under the assumption that the Hamiltonian of particles *A* has compact resolvent. In Ref. 4, Bruneau and Dereziński extended the results of Refs. 5 and 8 by using quadratic form techniques under the same assumption as that of Ref. 8.

In the present paper, we consider the same problem as that of Ref. 8 without the above assumption. This is a big difference between Refs. 8 and 4 and the present paper.

The present paper is organized as follows. In Sec. II we describe the model considered in this paper and state the main result. In Sec. III, we first prove an energy bound for the Dereziński–Gérard model. Second, we state commutation relations for the generalized annihilation/creation operators and relative boundedness of them. Finally, we prove the self-adjointness of the total Hamiltonian. Section IV is a preliminary to Sec. V. In Sec. IV, we prove some estimates which are used to prove the existence of a ground state of the massless Dereziński–Gérard model. Section V is devoted to the proof of the existence of a ground state of the massless Dereziński–Gérard model.

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II. DEFINITION OF THE MODEL

In the present paper, we denote the inner product and the norm of a Hilbert space $\mathcal{X}$ by $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{X}}$, respectively. The inner product is antilinear in the first variable. If there is no danger of confusion, then we omit the subscript $\mathcal{X}$ in $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{X}}$.

For a linear operator T on a Hilbert space, we denote its domain by $D(T)$. If T is densely defined, the adjoint of T is denoted by T^*. For linear operators S and T on a Hilbert space, $D(S+T) := D(S) \cap D(T)$ unless otherwise stated.

For a self-adjoint operator S on a Hilbert space, we denote its spectrum and its essential spectrum by $\sigma(S)$ and $\sigma_{\mathrm{ess}}(S)$, respectively. The spectral measure associated with S is denoted by $E_S(\cdot)$. If S is bounded below, then we set

$$E_0(S) := \inf \sigma(S)$$

and call it the ground state energy of S. We say that S has a ground state if $E_0(S)$ is an eigenvalue of S; in this case, each nonzero vector in $\ker(S - E_0(S))$ is called a ground state of S. The number $m(S) := \dim \ker(S - E_0(S))$ is called the multiplicity of the ground state of S.

To describe the Bose field, one uses the boson Fock space over a separable complex Hilbert space $\mathcal{X}$,

$$\mathcal{F}_{\mathrm{b}}(\mathcal{X}) := \bigoplus_{n=0}^{\infty} \bigotimes_s^n \mathcal{X} = \left\{ \psi \in \{\psi^{(n)}\}_{n=0}^{\infty} \middle| n \geq 0, \psi^{(n)} \in \bigotimes_s^n \mathcal{X}, \sum_{n=0}^{\infty} \|\psi^{(n)}\|^2 < \infty \right\},$$

where $\otimes_s^n \mathcal{X}$ denotes the n-fold symmetric tensor product of $\mathcal{X}$ with $\otimes_s^0 := \mathbb{C}$ (the space of complex numbers). The vector $\Omega := \{1, 0, \ldots\}$ is called the Fock vacuum in $\mathcal{F}_{\mathrm{b}}(\mathcal{X})$.

One of the main objects on $\mathcal{F}_{\mathrm{b}}(\mathcal{X})$ is the annihilation operator $a(f)$ which is a densely defined closed linear operator on $\mathcal{F}_{\mathrm{b}}(\mathcal{X})$, such that for all $\eta = \{\eta_n\}_{n=0}^{\infty} \in D(a(f)^*)$, $(a(f)^* \eta)^{(0)} = 0$, and $(a(f)^* \eta)^{(n)} = \sqrt{n} S_n(f \otimes \eta^{n-1)})$, $n \geq 1$, where S_n is the symmetrization operator on $\otimes^n \mathcal{X}$. The adjoint $a(f)^*$, which is called the creation operator, and the annihilation operator $a(g)$ obey the canonical commutation relations

$$[a(f), a(g)^*] = \langle f, g \rangle, \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0$$

for all $f, g \in \mathcal{X}$ on the dense subspace,

$$\mathcal{F}_0(\mathcal{X}) := \{\eta \in \mathcal{F}_{\mathrm{b}}(\mathcal{X}) | \text{there exists a number } n_0 \text{ such that } \eta^{(n)} = 0 \text{ for all } n \geq n_0\},$$

where $[\cdot, \cdot]$ means the commutator.

For every self-adjoint operator S on $\mathcal{X}$, one can define a self-adjoint operator $d\Gamma(S)$, called the second quantization of S, by

$$d\Gamma(S) := \bigoplus_{n=0}^{\infty} S^{(n)},$$

with $S^{(0)} := 0$ and $S^{(n)}$ is the closure of

$$\left. \left(\sum_{j=1}^n \underbrace{I \otimes \cdots \otimes \overset{j}{\overbrace{S}} \otimes \cdots I}_{n} \right) \right|_{\hat{\otimes}^n D(S)},$$

where I denotes the identity and $\hat{\otimes}^n D(S)$ algebraic tensor product of $D(S)$. If S is non-negative, then so is $d\Gamma(S)$. The second quantization $N_{\mathrm{b}} := d\Gamma(I)$ of the identity is called the number operator.

As the state space of the Dereziński–Gérard model, we take the tensor product Hilbert space,

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\mathrm{b}}(\mathcal{K}),$$

where $\mathcal{H}$ is a separable Hilbert space and $\mathcal{K} = L^2(\mathbb{R}^d)$. The Hilbert space $\mathcal{F}$ is identified with the space

$$\bigoplus_{n=0}^{\infty} [\mathcal{H} \otimes \otimes_s^n \mathcal{K}],$$

and we use this identification freely in what follows.

The subspace $\mathcal{D}_0$ of $\mathcal{F}$ is defined as follows:

$$\mathcal{D}_0 := \{\psi \in \mathcal{F} | \text{there exists an } n_0 \text{ such that, for all } n \geq n_0, \ \psi^{(n)} = 0\}.$$

Let A be a self-adjoint operator on $\mathcal{H}$. Let ω be a non-negative, injective, and self-adjoint operator on $\mathcal{K}$. A typical example of ω is the multiplication operator by the function $\omega(k) = \sqrt{m^2 + |k|^2}$. Here $m \geq 0$ is the mass of the boson. The function ω represents a dispersion relation of one free boson associated with the Bose field under consideration.

We define an operator H_{b} by

$$H_{\mathrm{b}} := d\Gamma(\omega)$$

acting on $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$. The free Hamiltonian of the Dereziński–Gérard model is given by

$$H_0 := A \otimes I + I \otimes H_{\mathrm{b}}.$$

To define the interaction part of the Dereziński–Gérard model, we introduce an analog $\tilde{\phi}(v)$ of the usual Segal's field operator for a bounded operator v from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{K}$. To do this, we first define the operator $\tilde{a}^*(v)$, which is an analog of the usual creation operator. The domain and the operation of $\tilde{a}^*(v)$ is as follows (Refs. [5] and [10]):

$$D(\tilde{a}^*(v)) := \left\{ \psi = (\psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} \Big| \sum_{n=0}^{\infty} n \|(I_{\mathcal{H}} \otimes S_n)(v \otimes I_{\otimes_s^{n-1} \mathcal{K}}) \psi^{(n-1)}\|^2 < \infty \right\},$$

$$(\tilde{a}^*(v)\psi)^{(0)} := 0, \quad (\tilde{a}^*(v)\psi)^{(n)} := \sqrt{n}(I_{\mathcal{H}} \otimes S_n)(v \otimes I_{\otimes_s^{n-1} \mathcal{K}}) \psi^{(n-1)}, \quad n \geq 1.$$

It is easy to see that $D(\tilde{a}^*(v)) \supset \mathcal{D}_0$. Thus the operator $\tilde{a}^*(v)$ is a densely defined. We set

$$\tilde{a}(v) := (\tilde{a}^*(v))^*.$$

The domain and the operation of $\tilde{a}(v)$ is as follows:

$$D(\tilde{a}(v)) := \left\{ \psi = (\psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} \Big| \sum_{n=0}^{\infty} (n+1) \|(I_{\mathcal{H}} \otimes S_n)(v^* \otimes I_{\otimes_s^n \mathcal{K}}) \psi^{(n+1)}\|^2 < \infty \right\},$$

$$(\tilde{a}(v)\psi)^{(n)} := \sqrt{n+1}(I_{\mathcal{H}} \otimes S_n)(v^* \otimes I_{\otimes_s^n \mathcal{K}}) \psi^{(n+1)}.$$

It is easily verified that $\tilde{a}^\sharp(v)$ ($\tilde{a}^\sharp = \tilde{a}$ or $\tilde{a}^*$) is closed and $\mathcal{D}_0$ is a core for $\tilde{a}^\sharp(v)$.

We define an analog of Segal's field operator as follows:

$$\tilde{\phi}(v) := \frac{1}{\sqrt{2}}(\tilde{a}(v) + \tilde{a}^*(v)).$$

The total Hamiltonian of the Dereziński–Gérard model is defined as follows:

$$H := H_0 + \tilde{\phi}(v).$$

Throughout this paper, we assume the following conditions.

(A.1) A is bounded below.

(A.2) (Gap condition) $\Sigma - E_0(A) > 0$, where $\Sigma := \inf \sigma_{\text{ess}}(A)$.

For $m > 0$, we set

$$\omega_m := \omega + m,$$

$$H_m := A \otimes I + I \otimes d\Gamma(\omega_m) + \tilde{\phi}(v)$$
$$= H + mI \otimes N_{\text{b}}. \tag{2.1}$$

(A.3) There exists an $m_0 > 0$, such that if $m \in (0, m_0]$, H_m has a ground state.

Remark 1: We make some comments on the assumption (A.3). As stated in Sec I, there are sufficient conditions that the massive Dereziński–Gérard Hamiltonians have a ground state. In spite of these results, the reason why we adopt the assumption (A.3) is to make the present paper self-contained.

To state the following assumption, for $s \in \mathbb{R}$, we introduce a subspace V_ω^s of $B(\mathcal{H}, \mathcal{H} \otimes \mathcal{K})$ by

$$V_\omega^s := \{v \in B(\mathcal{H}, \mathcal{H} \otimes \mathcal{K}) | (I \otimes \omega)^s v \in B(\mathcal{H}, \mathcal{H} \otimes \mathcal{K})\}.$$

In what follows, we abbreviate $(I \otimes \omega)^s v$ as $\omega^s v$.

Remark 2: If ω is strictly positive, then $V_\omega^s = B(\mathcal{H}, \mathcal{H} \otimes \mathcal{K})$ for $s \le 0$.

(A.4) $v \in V_\omega^{-1}$.

Remark 3: One can easily see that $v \in V_\omega^{-1}$ implies $v \in V_\omega^{-1/2}$. The referee remarked this fact to the author. The author is thankful to the referee for it.

The following theorem is the main theorem of the present paper.

Theorem 2.1: (Existence of a ground state for the massless case) *Assume (A.1)–(A.4). Suppose that*

$$\frac{1}{4(\Sigma - E_0(H))^2} \|v\|^2 \|\omega^{-1} v\|^2 + \frac{\|\omega^{-1} v\|^2}{2} < 1.$$

Then the massless Dereziński–Gérard model has a ground state.

III. ELEMENTARY RESULTS FOR THE DEREZIŃSKI–GÉRARD MODEL

In this section, we establish some elementary results for the Dereziński–Gérard model. These results hold independent of whether H is massive or massless. We use these results freely throughout this paper.

Proposition 3.1: (Energy bound for the Dereziński–Gérard Hamiltonian)

$$E_0(H) \le E_0(A).$$

Proof: Let $u \in D(A)$ with $\|u\| = 1$. Then, by the variational principle,

$$E_0(H) \le \langle u \otimes \Omega, Hu \otimes \Omega \rangle = \langle u, Au \rangle.$$

Using the variational principle again, we obtain the desired result. ■

Proposition 3.1 means that the interaction does not raise the ground state energy in the Dereziński–Gérard model.

Let $v \in B(\mathcal{H}, \mathcal{H} \otimes \mathcal{K})$. The operator v corresponds to the $B(\mathcal{H})$-valued function $v(\cdot)$ on $\mathbb{R}^d$ defined by

$$v(k)\xi := (v\xi)(k), \quad \xi \in \mathcal{H}, \quad k \in \mathbb{R}^d.$$

As commutation relations for $\tilde{a}^{\sharp}(v)$, the following commutation relations hold.

Theorem 3.2: (Reference 1) *Let* $v_1, v_2 \in B(\mathcal{H}, \mathcal{H} \otimes \mathcal{K})$. *If* $[v_1(k)^*, v_2(k')] = 0$ *for a.e.* $k, k' \in \mathbb{R}^d$, *then the following commutation relations hold on* $\mathcal{D}_0$:

$$[\tilde{a}(v_1), \tilde{a}^*(v_2)] = v_1^* v_2 \otimes I, \quad [\tilde{a}(v_1), \tilde{a}(v_2)] = 0, \quad [\tilde{a}^*(v_1), \tilde{a}^*(v_2)] = 0.$$

The following relative boundedness of $\tilde{a}(v)$ is fundamental.

Theorem 3.3: (Reference 6) *For* $v \in V_{\omega}^{-1/2}$, $D(\tilde{a}(v)) \supset D(I \otimes H_{\mathrm{b}}^{1/2})$, *and, for* $\psi \in D(I \otimes H_{\mathrm{b}}^{1/2})$,

$$\|\tilde{a}(v)\psi\| \le \|\omega^{-1/2}v\| \|I \otimes H_{\mathrm{b}}^{1/2}\psi\|.$$

Substituting I into ω in the above theorem, we obtain the following relative boundedness of $\tilde{a}(v)$ with respect to $I \otimes N_{\mathrm{b}}^{1/2}$.

Corollary 3.4: For $v \in B(\mathcal{H}, \mathcal{H} \otimes \mathcal{K})$, $D(\tilde{a}(v)) \supset D(I \otimes N_{\mathrm{b}}^{1/2})$, *and, for* $\psi \in D(I \otimes N_{\mathrm{b}}^{1/2})$,

$$\|\tilde{a}(v)\psi\| \le \|v\| \|I \otimes N_{\mathrm{b}}^{1/2}\psi\|.$$

Theorem 3.3 and the Kato–Rellich theorem imply the self-adjointness of the total Hamiltonian.

Theorem 3.5: (Self-adjointness of the total Hamiltonian) *Let* $v \in V_{\omega}^{-1/2}$. *Then the total Hamiltonian H is self-adjoint and bounded below. Moreover, any core of* H_0 *is also a core of H.*

IV. SOME LEMMAS FOR THE MASSIVE CASE

We devote this section to the preparation of some lemmas needed in the proof of Theorem 2.1. In this section, we assume that the operator ω is strictly positive.

Combining Lemma 4.2 in Ref. 2 with Theorem VIII.4 in Ref. 11.

Lemma 4.1: Let $\{f_l\} \subset D(\omega^{1/2})$ *be a complete orthonormal system of* $\mathcal{K}$. *Then for all* $\psi \in D(I \otimes H_{\mathrm{b}})$,

$$\sum_{l=1}^{\infty} \langle I \otimes a(\omega^{-1/2}f_l)\psi, I \otimes a(\omega^{1/2}f_l)\psi \rangle = \langle \psi, I \otimes N_{\mathrm{b}}\psi \rangle.$$

We introduce a subspace $\mathcal{F}_{\mathrm{fin}}(\omega)$ of $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$ and a subspace $\mathcal{D}_{\omega}$ of $\mathcal{F}$ as follows:

$$\mathcal{F}_{\mathrm{fin}}(\omega) := \mathcal{L}\{\Omega, a(g_1)^* \cdots a(g_n)^* \Omega | n \ge 1, g_1 \cdots g_n \in D(\omega)\},$$

$$\mathcal{D}_{\omega} := D(A) \hat{\otimes} \mathcal{F}_{\mathrm{fin}}(\omega).$$

For $f \in \mathcal{K}$, we define a bounded operator v_f from $\mathcal{H}$ into $\mathcal{H} \otimes \mathcal{K}$ by

$$v_f \xi := \xi \otimes f, \quad \xi \in \mathcal{H}.$$

Note that

$$\tilde{a}(v_f) = I \otimes a(f).$$

Lemma 4.2: For $f \in D(\omega)$, *we define*

$$T(f) := I \otimes a(\omega f) + \frac{1}{\sqrt{2}} v_f^* v \otimes I.$$

Let ψ_E *be an eigenvector of H with eigenvalue E. Then,* $\psi_E \in D(T(f)) \cap D(I \otimes a(f))$, $I \otimes a(f)\psi_E \in D(H)$, *and* $(H - E)I \otimes a(f)\psi_E = -T(f)\psi_E$.

Proof: For $\psi \in \mathcal{D}_{\omega}$,

$$I \otimes a(f)^*H\psi = [I \otimes a(f)^*, H]\psi + HI \otimes a(f)^*\psi$$

$$= \{[I \otimes a(f)^*, I \otimes H_b] + [I \otimes a(f)^*, \tilde{\phi}(v)]\}\psi + HI \otimes a(f)^*\psi$$

$$= -I \otimes a(\omega f)^*\psi - \frac{1}{\sqrt{2}}v^*v_f \otimes I\psi + HI \otimes a(f)^*\psi$$

$$= -T(f)^*\psi + HI \otimes a(f)^*\psi.$$

Therefore, we have

$$\langle I \otimes a(f)\psi_E, H\psi \rangle = \langle \psi_E, I \otimes a(f)^*H\psi \rangle$$

$$= \langle \psi_E, -T(f)^*\psi \rangle + \langle \psi_E, HI \otimes a(f)^*\psi \rangle$$

$$= -\langle T(f)\psi_E, \psi \rangle + \langle E\psi_E, I \otimes a(f)^*\psi \rangle$$

$$= -\langle T(f)\psi_E, \psi \rangle + E\langle I \otimes a(f)\psi_E, \psi \rangle.$$

Since $\mathcal{D}_\omega$ is a core of H, the above equality holds for $\psi \in D(H)$. Hence, we obtain

$$I \otimes a(f)\psi_E \in D(H)$$

and

$$(H - E)I \otimes a(f)\psi_E = -T(f)\psi_E.$$

∎

The next two lemmas are key lemmas to prove the main theorem.

Lemma 4.3: Let $\tilde{\psi}_0$ be a normalized ground state of H. Then

$$\langle \tilde{\psi}_0, I \otimes N_b\tilde{\psi}_0 \rangle \leq \frac{\|\omega^{-1}v\|^2}{2}.$$

Proof: For $f \in D(\omega)$, by Lemma 4.2, we have

$$0 \leq \langle I \otimes a(f)\tilde{\psi}_0, (H - E_0(H))I \otimes a(f)\tilde{\psi}_0 \rangle$$

$$= -\langle I \otimes a(f)\tilde{\psi}_0, T(f)\tilde{\psi}_0 \rangle$$

$$= -\langle I \otimes a(f)\tilde{\psi}_0, I \otimes a(\omega f)\tilde{\psi}_0 \rangle$$

$$- \frac{1}{\sqrt{2}}\langle I \otimes a(f)\tilde{\psi}_0, (v_f^*v \otimes I)\tilde{\psi}_0 \rangle.$$

Substituting $\omega^{-1/2}f_l$ into f, we have

$$\langle I \otimes a(\omega^{-1/2}f_l)\tilde{\psi}_0, I \otimes a(\omega^{1/2}f_l)\tilde{\psi}_0 \rangle \leq -\frac{1}{\sqrt{2}}\langle I \otimes a(\omega^{-1/2}f_l)\tilde{\psi}_0, (v_{\omega^{-1/2}f_l}^*v \otimes I)\tilde{\psi}_0 \rangle. \quad (4.1)$$

Summing up with respect to l, we have

$$\sum_{l=1}^{\infty} \langle I \otimes a(\omega^{-1/2}f_l)\tilde{\psi}_0, I \otimes a(\omega^{1/2}f_l)\tilde{\psi}_0 \rangle = \langle \tilde{\psi}_0, I \otimes N_b\tilde{\psi}_0 \rangle \quad (4.2)$$

by Lemma 4.1.

On the other hand,

$$\sum_{l=1}^{\infty} \langle I \otimes a(\omega^{-1/2}f_l)\widetilde{\psi}_0, (v^*_{\omega^{-1/2}f_l}v \otimes I)\widetilde{\psi}_0 \rangle = \sum_{l=1}^{\infty} \langle \widetilde{a}(v_{\omega^{-1/2}f_l})\widetilde{\psi}_0, (v^*_{\omega^{-1/2}f_l}v \otimes I)\widetilde{\psi}_0 \rangle$$

$$= \sum_{l=1}^{\infty} \langle (v^* v_{\omega^{-1/2}f_l} \otimes I)\widetilde{a}(v_{\omega^{-1/2}f_l})\widetilde{\psi}_0, \widetilde{\psi}_0 \rangle$$

$$= \sum_{l=1}^{\infty} \langle \widetilde{a}(v_{\omega^{-1/2}f_l}v^*_{\omega^{-1/2}f_l}v)\widetilde{\psi}_0, \widetilde{\psi}_0 \rangle = \langle \widetilde{a}(\omega^{-1}v)\widetilde{\psi}_0, \widetilde{\psi}_0 \rangle.$$

$$(4.3)$$

In the last equality, we used the fact that

$$\sum_{l=1}^{\infty} v_{\omega^{-1/2}f_l}v^*_{\omega^{-1/2}f_l} = (I \otimes \omega)^{-1}.$$

By Corollary 3.4, we have

$$|\langle \widetilde{a}(\omega^{-1}v)\widetilde{\psi}_0, \widetilde{\psi}_0 \rangle| \le \|\omega^{-1}v\| \|I \otimes N_b^{1/2}\widetilde{\psi}_0\| = \|\omega^{-1}v\| \langle \widetilde{\psi}_0, I \otimes N_b \widetilde{\psi}_0 \rangle^{1/2}. \qquad (4.4)$$

Thus, (4.1)–(4.4) imply that

$$\langle \widetilde{\psi}_0, I \otimes N_b \widetilde{\psi}_0 \rangle \le \frac{\|\omega^{-1}v\|^2}{2}.$$

This proves the lemma. ∎

Let E_A be the spectral measure associated with A. We set $\{\mu_r\}_{r=1}^q := \sigma(A) \cap [E_0(A), \Sigma)(q \le \infty)$. We define an operator Q_r on $\mathcal{H}$ by $Q_r := E_A([E_0(A), \mu_r))$ and set $Q_r^\perp := I - Q_r$. We denote the orthogonal projection onto the subspace spanned by the Fock vacuum Ω by P_Ω.

Lemma 4.4: Let $\widetilde{\psi}_0$ be as above. Then

$$\langle \widetilde{\psi}_0, Q_r^\perp \otimes P_\Omega \widetilde{\psi}_0 \rangle \le \frac{1}{4(\mu_{r+1} - E_0(H))^2} \|v\|^2 \|\omega^{-1}v\|^2.$$

Proof: Simple calculations show that

$$Q_r^\perp \otimes P_\Omega H = Q_r^\perp A \otimes P_\Omega + \frac{1}{\sqrt{2}} Q_r^\perp \otimes P_\Omega \widetilde{a}(v)$$

and

$$Q_r^\perp \otimes P_\Omega(H - E_0(H))\widetilde{\psi}_0 = 0.$$

Thus, we have

$$\langle \widetilde{\psi}_0, Q_r^\perp(A - E_0(H)) \otimes P_\Omega \widetilde{\psi}_0 \rangle = -\frac{1}{\sqrt{2}} \langle Q_r^\perp \otimes P_\Omega \widetilde{\psi}_0, \widetilde{a}(v)\widetilde{\psi}_0 \rangle.$$

By using the spectral theorem, we have

$$\langle \widetilde{\psi}_0, Q_r^\perp(A - E_0(H)) \otimes P_\Omega \widetilde{\psi}_0 \rangle \ge (\mu_{r+1} - E_0(H))\langle \widetilde{\psi}_0, Q_r^\perp \otimes P_\Omega \widetilde{\psi}_0 \rangle.$$

Hence, we have

$$\langle \widetilde{\psi}_0, Q_r^{\perp} \otimes P_{\Omega} \widetilde{\psi}_0 \rangle \leq \frac{1}{\sqrt{2(\mu_{r+1} - E_0(H))}} \|Q_r^{\perp} \otimes P_{\Omega} \widetilde{\psi}_0\| \|\widetilde{a}(v) \widetilde{\psi}_0\|.$$

Therefore, we obtain

$$\langle \widetilde{\psi}, Q_r^{\perp} \otimes P_{\Omega} \widetilde{\psi}_0 \rangle \leq \frac{1}{2(\mu_{r+1} - E_0(H))^2} \|\widetilde{a}(v) \widetilde{\psi}_0\|^2.$$

By applying Lemma 4.3 and the inequality

$$\|\widetilde{a}(v) \widetilde{\psi}_0\|^2 \leq \|v\|^2 \|I \otimes N_{\mathrm{b}}^{1/2} \widetilde{\psi}_0\|^2,$$

we obtain

$$\langle \widetilde{\psi}_0, Q_r^{\perp} \otimes P_{\Omega} \widetilde{\psi}_0 \rangle \leq \frac{1}{4(\mu_{r+1} - E_0(H))^2} \|v\|^2 \|\omega^{-1} v\|^2.$$

∎

V. EXISTENCE OF A GROUND STATE OF THE MASSLESS DEREZIŃSKI–GÉRARD MODEL

This section is the main part of the present paper. We shall prove the existence of a ground state of the massless Dereziński–Gérard model by applying Lemma 5.1 below. Lemma 5.1 was established by Arai and Hirokawa in Ref. 2. Since then, this lemma has been widely used to prove the existence of a ground state of massless models by many researchers.[2,3,8,9]

Lemma 5.1: (Reference 2*) Let* A_n *and* A *be self-adjoint operators on a Hilbert space* $\mathcal{H}$ *which are bounded below. Suppose that* A_n *has a ground state* ψ_n, $E_0(A_n)$ *converges to* $E_0(A)$, *and* A_n *and* A *have a common core* $\mathcal{D}$, *such that* $A_n \psi$ *converges to* $A\psi$ *for all* $\psi \in \mathcal{D}$. *If there exists a nonzero vector* $\psi_0 \in \mathcal{H}$ *such that* ψ_n *converges to* ψ_0 *weakly, then* ψ_0 *is a ground state of* A.

Let H be the Dereziński–Gérard Hamiltonian and H_m be the operator defined by (2.1). By the assumption (A.3), there exists a normalized ground state $\psi(m)$ of H_m. By the weak compactness of the closed unit ball of the Hilbert space $\mathcal{F}$, there exists $\psi_0 \in \mathcal{F}$ and a subsequence $\{\psi(m_j)\}$ which converges to ψ_0 weakly with $m_j \searrow 0$. To apply Lemma 5.1 above, we must check that (1) $\psi_0 \neq 0$, (2) $E_0(H_m)$ converges to $E_0(H)$ as $m \to 0$, and (3) the existence of a common core of H and H_m on which H_m converges to H strongly.

Recall the subspace $\mathcal{D}_{\omega}$ introduced in Sec. IV. By the general theory of tensor product of operators, $\mathcal{D}_{\omega}$ is a core of H_0. This fact, Theorem 3.5 and Eq. (2.1) imply the following fact.

Lemma 5.2: $\mathcal{D}_{\omega}$ *is a common core of* H *and* H_m *and* $H_m \psi \to H\psi$ *for all* $\psi \in \mathcal{D}_{\omega}$.

Lemma 5.3:

$$E_0(H_m) \to E_0(H).$$

Proof: By Lemma 5.2 above, H_m converges to H in the strong resolvent sense. Hence

$$\limsup_{m \to 0} E_0(H_m) \leq E_0(H).$$

On the other hand,

$$H_m \geq H \quad \text{for all} \ m > 0.$$

Therefore,

$$E_0(H_m) \geq E_0(H) \quad \text{for all} \ m > 0.$$

So we obtain

$$\liminf_{m\to 0} E_0(H_m) \geq E_0(H).$$

Thus, we have the desired result. ∎

In the proof of Theorem 2.1, we use the following operator inequality.

Lemma 5.4: (Reference 2)

$$Q_r \otimes P_\Omega \geq I - I \otimes N_b - Q_r^\perp \otimes P_\Omega.$$

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1: The conditions (2) and (3) above correspond to Lemma 5.3 and Lemma 5.2, respectively. So, we have only to check the condition (1).

By Lemmas 4.3, 4.4, and 5.4, we have

$$\langle \psi(m_j), Q_r \otimes P_\Omega \psi(m_j)\rangle \geq 1 - \langle \psi(m_j), I \otimes N_b \psi(m_j)\rangle - \langle \psi(m_j), Q_r^\perp \otimes P_\Omega \psi(m_j)\rangle$$

$$\geq 1 - \frac{\|\omega_{m_j}^{-1} v\|^2}{2} - \frac{1}{4(\mu_{r+1} - E_0(H))^2}\|v\|^2 \|\omega_{m_j}^{-1} v\|^2$$

$$\geq 1 - \frac{\|\omega^{-1} v\|^2}{2} - \frac{1}{4(\mu_{r+1} - E_0(H))^2}\|v\|^2 \|\omega^{-1} v\|^2.$$

Tending $m_j \to \infty$, we have

$$\langle \psi_0, Q_r \otimes P_\Omega \psi_0\rangle \geq 1 - \frac{\|\omega^{-1} v\|^2}{2} - \frac{1}{4(\mu_{r+1} - E_0(H))^2}\|v\|^2 \|\omega^{-1} v\|^2.$$

The right hand side of the above inequality converges to

$$1 - \frac{\|\omega^{-1} v\|^2}{2} - \frac{1}{4(\Sigma - E_0(H))^2}\|v\|^2 \|\omega^{-1} v\|^2$$

as $r \to \infty$. Therefore, for sufficiently large r,

$$\langle \psi_0, Q_r \otimes P_\Omega \psi_0\rangle > 0.$$

Hence $\psi_0 \neq 0$. By applying Lemma 5.1, we have the desired result. ∎

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