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# Self-duality and Integrable Systems

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## §0. Introduction

In his lectures (1984-85) at Kyoto University, Professor M.Sato presented a program for generalizing the soliton theory ([9]; c.f. [10]). The Kadomtsev-Petviashvili (KP) equation is a typical example of the soliton theory. The KP equation is written in the form of deformation equations of a linear ordinary differential equation. The time evolutions of a solution are interpreted as dynamical motions on an infinite dimensional Grassmann manifold ([7],[9]). The Lie algebra of microdifferential operators of one variable acts on this manifold transitively. He conjectured that any integrable systems can be written in the form of deformation equations of a linear systems, and proposed to investigate a deformation of differential equations in higher dimensions. He showed a simple example of a deformation of holonomic systems in higher dimensions ([9]), and its generalization is treated in [4]. In this paper we study a deformation of  $\mathcal{D}$ -modules in higher dimensions.

First we review the KP equation. We denote by  $\mathcal{E}$  the ring of microdifferential operators of one variable  $x$ . We fix a microdifferential operator  $P$ , and denote by  $t_P$  a time variable with respect to  $P$ . We study the following evolution equation associated to  $P$ :

$$\frac{\partial W}{\partial t_P} + WP = (WPW^{-1})_+ W, \quad (0.1)$$

where  $W = W(x, D_x) = 1 + \sum_{j < 0} w_j(x) D_x^j \in \mathcal{E}$ . We denote by  $\mathcal{W}$  the set of such operators  $W$ . This space  $\mathcal{W}$  is a group by the composition of  $\mathcal{E}$ . We get the KP-hierarchy taking  $P = D_x^n$  ( $n = 1, 2, 3, \dots$ ) in (0.1). The equation (0.1) defines a dynamical motion on  $\mathcal{W}$ . This infinitesimal action of the Lie algebra  $\mathcal{E}$  on  $\mathcal{W}$  is transitive.

The purpose of this article is to give a foundation for higher dimensional generalization of the KP hierarchy. Let now  $\mathcal{E}$  be the ring of microdifferential operators in several variables. Similarly to the one dimensional case, fixing an operator  $P \in \mathcal{E}$ , we shall study the following equation

$$\frac{\partial W}{\partial t_P} + WP = (WPW^{-1})_+ W, \quad (0.2)$$

where the operator  $W$  is a 0-th order microdifferential operator. Here we choose a decomposition  $\mathcal{E} = \mathcal{D} \oplus \mathcal{E}_\phi$  and  $(WPW^{-1})_+ \in \mathcal{D}$  is the component

of  $WPW^{-1}$  according to this decomposition. In general the equation (0.2) imposes some constraints on the initial value  $W(t_P = 0)$ , since the vector field defined by (0.2) is not tangent to the space  $\mathcal{E}(0)$ . There is no operator  $W_0$  such that the equation (0.2) has a solution  $W(t) \in \mathcal{E}$  with the initial value  $W_0$  for any  $P \in \mathcal{E}$ . We take generators  $P$  in (0.2) only in the Lie subalgebra  $V$  of  $\mathcal{E}$

$$V = \{F_0(x', D_0)x_0 + \sum_{0 \leq j < r} F_j(x', D_0)D_j + E(x', D_0); \\ F_k(x', D_0), E(x', D_0) \in \mathcal{E} \text{ for } 0 \leq k < r \},$$

where  $x' = (x_1, x_2, \dots, x_{r-1})$ . This Lie algebra contains the transformation groups both of the self-dual Yang-Mills equations and of the self-dual Einstein equations (see [7],[8]). In §2 we will determine the subspace  $\mathcal{W}$  of  $\mathcal{E}(0)$  so that the vector field defined by (0.2) for any  $P \in V$  is tangent to  $\mathcal{W}$ . The space  $\mathcal{W}$  is a subgroup in  $\mathcal{E}$ . The Lie algebra  $V$  acts on  $\mathcal{W}$  transitively.

In the case of  $r = 3$ , our integrable system is nothing but a composed system of the self-dual Yang-Mills equations and the equations of self-dual metrics on Riemannian manifolds of dimension four. The Lie algebra  $V$  acts transitively on the space of self-dual connections on self-dual spaces. Thus we obtain a group-theoretical description of the twistor theory ([1],[5]).

*Notations.* We use the following notations:  $\mathbb{Z}$  denotes the set of integers.  $\mathbb{N}$  denotes the set of non-negative integers. We denote by  $\mathbb{C}$  the complex number field. We denote by  $1_n$  the unit matrix of size  $n \times n$ .

## §1. Deformations of $\mathcal{D}$ -modules

Throughout this paper we shall work in the category of formal power series,  $\mathcal{O} = \mathbb{C}[[x]] = \mathbb{C}[[x_0, x_1, \dots, x_{r-1}]]$  ( $r \geq 2$ ). Let  $\mathcal{D}$  be the ring of differential operators with coefficients in  $\mathcal{O}$ . Then every differential operator  $P$  of order  $m$  can be written as:

$$P = \sum_{\alpha \in \mathbb{N}^r, |\alpha| \leq m} a_\alpha(x) D_x^\alpha,$$

where  $a_\alpha(x)$  are elements of  $\mathcal{O}$ ,  $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{r-1}) \in \mathbb{N}^r$ ,  $|\alpha| = \alpha_0 + \alpha_1 + \dots + \alpha_{r-1}$ ,  $D_x^\alpha = D_0^{\alpha_0} D_1^{\alpha_1} \dots D_{r-1}^{\alpha_{r-1}}$  and  $D_j = \frac{\partial}{\partial x_j}$  ( $j = 0, 1, \dots, r-1$ ).

The ring  $\mathcal{E}$  of formal microdifferential operators is a set of formal Laurent series in  $D_0, D_1, \dots, D_{r-1}$  with only non-negative powers of  $D_1, \dots, D_{r-1}$ . The precise definition is as follows. We denote by  $\mathcal{E}(m)$  the space of formal series :

$$P = \sum_{\alpha \in \mathbb{Z} \times \mathbb{N}^{r-1}, |\alpha| \leq m} a_\alpha(x) D_x^\alpha$$

where  $a_\alpha$ 's are elements of  $\mathcal{O}$ , and the summation is taken through  $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{r-1})$ , where  $\alpha_0 \in \mathbb{Z}, \alpha_1 \in \mathbb{N}, \dots, \alpha_{r-1} \in \mathbb{N}$ . We set

$$\mathcal{E} = \bigcup_{m \in \mathbb{Z}} \mathcal{E}(m).$$

We endow the  $\mathcal{O}$ -module  $\mathcal{E}$  with a structure of ring by extending the Leibniz formula. For two elements  $P = \sum_\alpha a_\alpha D_x^\alpha$  and  $Q = \sum_\beta b_\beta D_x^\beta$  of  $\mathcal{E}$ , we define the composition  $P \circ Q$  by

$$P \circ Q = \sum_{\alpha, \beta \in \mathbb{Z} \times \mathbb{N}^{r-1}, \gamma \in \mathbb{N}^r} \binom{\alpha}{\gamma} a_\alpha b_\beta^{(\gamma)} D_x^{\alpha + \beta - \gamma},$$

where  $b_\beta^{(\gamma)} = D_x^\gamma(b_\beta)$ . The ring  $\mathcal{E}$  has an increasing filtration by subspaces  $\{\mathcal{E}(m)\}_{m \in \mathbb{Z}}$ . We have

$$\mathcal{E}(m)\mathcal{E}(n) = \mathcal{E}(n + m).$$

For any  $\mathcal{O}$ -submodule  $\mathcal{L}$  of  $\mathcal{E}$  we define the induced filtration  $\{\mathcal{L}(m)\}_{m \in \mathbb{Z}}$  of  $\mathcal{L}$  by  $\mathcal{L}(m) = \mathcal{L} \cap \mathcal{E}(m)$ .

Let  $\mathcal{E}_\phi$  be the  $\mathcal{O}$ -module consisting of the formal microdifferential operators of the form

$$\sum_{\alpha_0 < 0} a_\alpha(x) D_x^\alpha,$$

The ring  $\mathcal{E}$  is the direct sum of  $\mathcal{D}$  and  $\mathcal{E}_\phi$ . For any  $P \in \mathcal{E}$ , we define  $P_+ \in \mathcal{D}$  and  $P_- \in \mathcal{E}_\phi$  by the decomposition of  $\mathcal{E}$ :

$$\begin{aligned} \mathcal{E} &= \mathcal{D} \oplus \mathcal{E}_\phi \\ P &= P_+ + P_- \end{aligned} .$$

For any  $\mathcal{O}$ -submodule  $\mathcal{L}$  of  $\mathcal{E}$  we define the  $\mathcal{O}$ -module  $\mathcal{L}_-$  by  $\mathcal{L}_- = \mathcal{L} \cap \mathcal{E}_\phi$ . Remark that  $\mathcal{E}(0) = \mathcal{O} \oplus \mathcal{E}_\phi(0)$ .

In the following we shall study a left  $\mathcal{D}$ -submodule  $I$  of  $\mathcal{E}$  which satisfies the following condition:

$$\mathcal{E}(m) = I(m) \oplus \mathcal{E}_\phi(m) \quad \text{for any } m \in \mathbb{Z}. \quad (1.1)$$

For example  $I = \mathcal{D}$  satisfies (1.1). We make clear the structure of such a  $\mathcal{D}$ -submodule  $I$ .

**Lemma 1.1** Suppose that a  $\mathcal{D}$ -submodule  $I$  of  $\mathcal{E}$  satisfies the condition (1.1). Then  $I$  is generated as  $\mathcal{D}$ -module by a unique operator  $W$  such that

$$W \in \mathcal{E}(0) \quad \text{and} \quad W_+ = 1. \quad (1.2)$$

*Proof.* The operator  $W$  is obtained by decomposing the identity operator  $1 \in \mathcal{E}(0)$  into the sum of an operator in  $I(0)$  and an operator in  $\mathcal{E}_\phi(0)$  according to the condition (1.1):

$$\begin{aligned} \mathcal{E}(0) &= I(0) \oplus \mathcal{E}_\phi(0) \\ 1 &= W + U \end{aligned}$$

It is evident that  $W$  is contained in  $\mathcal{E}(0)$  and that  $W_+ = 1$ . Since we have

$$\mathcal{E}(m)W = \mathcal{E}(m), \quad \mathcal{E}_\phi(m)W = \mathcal{E}_\phi(m)$$

for any  $m \in \mathbb{Z}$ , we obtain that

$$\mathcal{E}(m) = \mathcal{D}(m)W \oplus \mathcal{E}_\phi(m) \quad \text{for any } m \in \mathbb{Z}.$$

Thus the  $\mathcal{D}$ -module  $I' = \mathcal{D}W$  also satisfies (1.1). Because  $I$  contains  $I'$  and both satisfy (1.1),  $I$  coincides with  $I'$ . The uniqueness is clear.  $\square$

Remark that  $W$  in Lemma 1.1 is invertible by (1.2).

We investigate nonlinear evolution equations according to the program of M. Sato ([9],[10]). For any  $P \in \mathcal{E}$  and any  $\mathcal{D}$ -submodule  $I_0$  of  $\mathcal{E}$  we define the time evolution  $I_t$  of  $I_0$  by the following differential equations:

$$\frac{\partial V(t)}{\partial t} + V(t)P \in I_t \quad \text{for any } V(t) \in I_t. \quad (1.3)$$

We call  $P \in \mathcal{E}$  the generator of the evolution equation (1.3).

In general we cannot find any  $\mathcal{D}$ -submodule  $I_t$  which solves (1.3). In this paper we shall study the case that we can find a solution  $I_t$  of (1.3) which is a  $\mathcal{D}$ -module satisfying (1.1) for any  $t$ . Then  $I_t$  is generated by an operator  $W(t) \in \mathcal{E}(0)$  by Lemma 1.1 and we can rewrite the equation (1.3) in terms of the generator  $W(t)$ .

**Lemma 1.2** We fix an operator  $P \in \mathcal{E}$ . We assume that the solution  $I_t$  of the evolution equation (1.3) is a  $\mathcal{D}$ -submodule which satisfies (1.1) for any  $t$ . Then the equation (1.3) reduces to the following equation

$$\frac{\partial W(t)}{\partial t} + W(t)P = (W(t)PW(t)^{-1})_+ W(t), \quad (1.4)$$

where the operator  $W(t)$  is the generator of  $I_t$  in Lemma 1.1.

*Proof.* From the equation (1.3) there exists an operator  $B(t) \in \mathcal{D}$  such that

$$\frac{\partial W(t)}{\partial t} + W(t)P = B(t)W(t).$$



Thus we have

$$B(t) = W(t)PW(t)^{-1} + \frac{\partial W(t)}{\partial t}W(t)^{-1}.$$

Since the operator  $W(t)$  is contained in  $1 + \mathcal{E}_\phi$ , the operator  $\frac{\partial W(t)}{\partial t}W(t)^{-1}$  is contained in  $\mathcal{E}_\phi$ . Thus we obtain that  $B(t) = (W(t)PW(t)^{-1})_+$ .  $\square$

*Remark.* The equation (1.4) is rewritten as

$$\begin{aligned} \frac{\partial W}{\partial t} &= (WPW^{-1})_+W - WP \\ &= -(WPW^{-1})_-W. \end{aligned} \tag{1.5}$$

The evolution equation (1.4) is associated with an infinitesimal action  $\rho$  of  $\mathcal{E}$  on the space  $\mathcal{E}$ . For  $P \in \mathcal{E}$  the vector field  $\rho(P)$  is given as follows.

$$W \longrightarrow -(WPW^{-1})_-W \in T_W\mathcal{E},$$

where the tangent space  $T_W\mathcal{E}$  is identified with  $\mathcal{E}$  by the structure of vector space of  $\mathcal{E}$ .

**Theorem 1.3** For any  $P, Q \in \mathcal{E}$  we have

$$\rho([P, Q]) = -[\rho(P), \rho(Q)].$$

*Proof.* We denote by  $\epsilon_1$  and  $\epsilon_2$  the time parameters with respect to  $P$  and  $Q$ , respectively. We set  $\tilde{P} = WPW^{-1}$  and  $\tilde{Q} = WQW^{-1}$ . We have

$$\begin{aligned} \exp(\epsilon_1\rho(P))W &\equiv (1 - \epsilon_1(\tilde{P})_-)W \pmod{\epsilon_1^2}, \\ \exp(\epsilon_2\rho(Q))W &\equiv (1 - \epsilon_2(\tilde{Q})_-)W \pmod{\epsilon_2^2}. \end{aligned}$$

Hence we have, modulo  $\epsilon_1^2, \epsilon_2^2$

$$\begin{aligned} \exp(\epsilon_1 \rho(P))(1 - \epsilon_2(WQW^{-1})_-) \\ \equiv 1 - \epsilon_2((1 - \epsilon_1 \tilde{P}_-) \tilde{Q}(1 + \epsilon_1 \tilde{P}_-))_- \\ \equiv 1 - \epsilon_2 \tilde{Q}_- + \epsilon_1 \epsilon_2 [\tilde{P}_-, \tilde{Q}]_-. \end{aligned}$$

Thus we obtain, modulo  $\epsilon_1^2, \epsilon_2^2$

$$\begin{aligned} \exp(\epsilon_1 \rho(P)) \exp(\epsilon_2 \rho(Q)) W \\ \equiv \exp(\epsilon_1 \rho(P))(1 - \epsilon_2(WQW^{-1})_-) W \\ \equiv (1 - \epsilon_2 \tilde{Q}_- + \epsilon_1 \epsilon_2 [\tilde{P}_-, \tilde{Q}]_-)(1 - \epsilon_1 \tilde{P}_-) W \\ \equiv (1 - \epsilon_1 \tilde{P}_- - \epsilon_2 \tilde{Q}_- + \epsilon_1 \epsilon_2 ([\tilde{P}_-, \tilde{Q}]_- + \tilde{Q}_- \tilde{P}_-)) W. \end{aligned}$$

Similarly we have, modulo  $\epsilon_1^2, \epsilon_2^2$

$$\begin{aligned} \exp(\epsilon_2 \rho(Q)) \exp(\epsilon_1 \rho(P)) W \\ \equiv (1 - \epsilon_2 \tilde{Q}_- - \epsilon_1 \tilde{P}_- + \epsilon_1 \epsilon_2 ([\tilde{Q}_-, \tilde{P}]_- + \tilde{P}_- \tilde{Q}_-)) W. \end{aligned}$$

By the formula

$$\begin{aligned} \exp(\epsilon_1 \rho(P)) \exp(\epsilon_2 \rho(Q)) W - \exp(\epsilon_2 \rho(Q)) \exp(\epsilon_1 \rho(P)) W \\ \equiv \epsilon_1 \epsilon_2 [\rho(P), \rho(Q)] W \quad \text{mod } \epsilon_1^2, \epsilon_2^2, \end{aligned}$$

we obtain

$$\begin{aligned} [\rho(P), \rho(Q)] W &= \{[\tilde{P}_-, \tilde{Q}]_- + \tilde{Q}_- \tilde{P}_- - [\tilde{Q}_-, \tilde{P}]_- - \tilde{P}_- \tilde{Q}_-\} W \\ &= \{[\tilde{P}_-, \tilde{Q}_+ + \tilde{Q}_-]_- - [\tilde{Q}_-, \tilde{P}_+ + \tilde{P}_-]_- + [\tilde{Q}_-, \tilde{P}_-]\} W. \end{aligned} \tag{1.6}$$

Since  $[\tilde{P}_+, \tilde{Q}_+]_- = 0$ , the right hand side of (1.6) is equal to

$$[\tilde{P}, \tilde{Q}]_- W = -\rho([P, Q]) W. \quad \square$$

When  $P, Q \in \mathcal{E}$  commute with each other, the following equations are compatible by Theorem 1.3:

$$\begin{aligned}\frac{\partial W}{\partial s} &= (WPW^{-1})_+W - WP, \\ \frac{\partial W}{\partial t} &= (WQW^{-1})_+W - WQ,\end{aligned}$$

where  $W = W(s, t)$ .

In the case  $r = 1$ , for any  $P \in \mathcal{E}$  and for any  $I_0 = \mathcal{D}W(0)$  the solution  $I_t$  of the equation (1.3) satisfies the condition (1.1) for any  $t$ . With the choice  $P = D_0^n$  ( $n = 1, 2, \dots$ ), we obtain the KP-hierarchy ([7],[8]):

$$\frac{\partial W(t)}{\partial t} + W(t)D_0^n = (W(t)D_0^n W(t)^{-1})_+W(t).$$

In higher dimensional case, we must choose a nice pair of the generator  $P$  and  $I_0 = \mathcal{D}W(0)$  in order that  $I_t$  satisfies the condition (1.1) for any  $t$ . We shall see the evolution equation (1.4) constrains the initial value  $W(0)$  in the following example.

*Example 1.1* We consider (1.4) in the case  $r = 2$ . We take  $D_0^2$  as the generator of the equation (1.4). We write

$$W(t) = \sum_{i+j \leq 0, j \geq 0} w_{i,j} D_0^i D_1^j, \quad w_{0,0} \equiv 1.$$

The operator  $W(t)D_0^2$  is decomposed into the sum

$$\begin{aligned}\mathcal{E} &= \mathcal{D}W \quad \oplus \quad \mathcal{E}_\phi \\ W(t)D_0^2 &= (W(t)D_0^2 W(t)^{-1})_+W \quad + \quad U \quad .\end{aligned}$$

Then we have

$$\begin{aligned}(W(t)D_0^2 W(t)^{-1})_+ &= (D_0^2 - 2\frac{\partial w_{-1,1}}{\partial x_0} D_1 - 2\frac{\partial w_{-1,0}}{\partial x_0}), \\ U &= \sum_{i+j \leq 1, i < 0} \left( -2\frac{\partial w_{i-1,j}}{\partial x_0} + 2\frac{\partial w_{-1,1}}{\partial x_0} w_{i,j-1} - \frac{\partial^2 w_{i,j}}{\partial x_0^2} \right. \\ &\quad \left. + 2\frac{\partial w_{-1,0}}{\partial x_0} w_{i,j} + 2\frac{\partial w_{-1,1}}{\partial x_0} \frac{\partial w_{i,j}}{\partial x_1} \right) D_0^i D_1^j.\end{aligned}$$

The equation (1.4) is equivalent to the following:

$$\begin{aligned} \frac{\partial w_{i,j}}{\partial t} + u_{i,j} &= 0 && \text{for } i + j \leq 0, \\ \frac{\partial w_{i-1,j}}{\partial x_0} - \frac{\partial w_{-1,1}}{\partial x_0} w_{i,j-1} &= 0, && \text{for } i + j = 1. \end{aligned}$$

The second equation constrains the initial data  $W(0)$ .

## §2. Integrable systems in higher dimensions

In the example 1.1 we have considered the equation (1.3) for one generator  $P = D_0^2$ . In this section we will introduce a space  $V$  of generators, and determine the space of  $\mathcal{D}$ -submodules  $I$  of  $\mathcal{E}$  such that the condition (1.1) is preserved under the time evolution (1.3) for any  $P \in V$ .

First we review two known examples, the self-dual Yang-Mills equations and the self-dual Einstein equations. We can interpret both the equations as integrable systems of three variables (see [14]).

*Example 2.1* Self-dual Yang-Mills equations (see [15],[11]).

The self-dual Yang-Mills equations are written in the following form

$$\begin{aligned} \frac{\partial A_1}{\partial x_2}(x_1, x_2, s, t) &= \frac{\partial A_2}{\partial x_1}(x_1, x_2, s, t), \\ \left[ \frac{\partial}{\partial s} + A_1(x_1, x_2, s, t), \frac{\partial}{\partial t} + A_2(x_1, x_2, s, t) \right] &= 0 \end{aligned} \quad (2.1)$$

for gauge fields  $A_1, A_2 \in \text{Mat}(n \times n)$  on four-dimensional manifolds.

The evolution equation (1.4) is generalized to the case that  $W$  and  $P$  have matrix coefficients. We introduce the space  $\mathcal{W}_{YM}(n)$  and the Lie algebra  $V_{YM}(n)$ :

$$\begin{aligned} \mathcal{W}_{YM}(n) &= \{W(x_1, x_2, D_0) = \sum_{i \in \mathbb{N}} w_i(x_1, x_2) D_0^{-i}; \\ &w_i \in \text{Mat}(n \times n, \mathbb{C}[[x_1, x_2]]), w_0 \equiv 1_n\}. \end{aligned}$$

$$V_{YM}(n) = \{F_1(x_1, x_2, D_0)D_1 + F_2(x_1, x_2, D_0)D_2 + E(x_1, x_2, D_0); \\ F_1, F_2 \in \text{Mat}(n \times n, \mathcal{D}), E \in \text{Mat}(n \times n, \mathcal{E})\}.$$

The evolution equation (1.4) for  $P \in V_{YM}(n)$  with any initial value  $W \in \mathcal{W}_{YM}(n)$  has a solution in  $\mathcal{W}_{YM}(n)$ . We consider the equation (1.4) for  $P = D_0D_1, D_0D_2 \in V_{YM}(n)$ :

$$\begin{aligned} \frac{\partial W}{\partial s} + WD_0D_1 &= (WD_0D_1W^{-1})_+W, \\ \frac{\partial W}{\partial t} + WD_0D_2 &= (WD_0D_2W^{-1})_+W. \end{aligned} \quad (2.2)$$

In terms of the coefficients  $w_j$  of  $W$  the equation (2.2) is written in the form

$$\begin{aligned} \frac{\partial w_i}{\partial s} &= \frac{\partial w_{i+1}}{\partial x_1} - \frac{\partial w_1}{\partial x_1} w_i, \\ \frac{\partial w_i}{\partial t} &= \frac{\partial w_{i+1}}{\partial x_2} - \frac{\partial w_1}{\partial x_2} w_i, \end{aligned} \quad (2.3)$$

for  $i > 0$ . We set  $A_j = \frac{\partial w_1}{\partial x_j}$  ( $j = 1, 2$ ). By eliminating  $w_2$  from the equation (2.3) for  $i = 1$ , we obtain the equation (2.1).

*Example 2.2* Self-dual Einstein equations (see [2],[12]).

The self-dual Einstein equations are written in the form (see [6])

$$\begin{aligned} \frac{\partial A_1}{\partial x_2}(x_1, x_2, s, t) &= \frac{\partial A_2}{\partial x_1}(x_1, x_2, s, t), \\ \frac{\partial B_1}{\partial x_2}(x_1, x_2, s, t) &= \frac{\partial B_2}{\partial x_1}(x_1, x_2, s, t), \\ \left[ \frac{\partial}{\partial s} + A_1(x_1, x_2, s, t) \frac{\partial}{\partial x_1} + B_1(x_1, x_2, s, t) \frac{\partial}{\partial x_2}, \right. \\ &\quad \left. \frac{\partial}{\partial t} + A_2(x_1, x_2, s, t) \frac{\partial}{\partial x_1} + B_2(x_1, x_2, s, t) \frac{\partial}{\partial x_2} \right] = 0, \\ \frac{\partial B_j}{\partial x_2}(x_1, x_2, s, t) + \frac{\partial A_j}{\partial x_1}(x_1, x_2, s, t) &= 0 \quad (j = 1, 2). \end{aligned} \quad (2.4)$$



In the following we forget the last equation in (2.4) for simplicity.

We introduce the space  $\mathcal{W}_E$  and the Lie algebra  $V_E$ :

$$\mathcal{W}_E = \left\{ W = \sum_{j,k \in \mathbb{N}} \frac{1}{j!k!} G_1^j G_2^k D_1^j D_2^k; \quad G_i = \sum_{j < 0} g_{i,j}(x_1, x_2) D_0^j (i = 1, 2) \right\}.$$

$$V_E = \{ F_1(x_1, x_2, D_0) D_1 + F_2(x_1, x_2, D_0) D_2; \quad F_1, F_2 \in \mathcal{E} \}.$$

The evolution equation (1.4) for  $P \in V_E$  with any initial value  $W \in \mathcal{W}_E$  has a solution in  $\mathcal{W}_E$ . We consider (1.4) for  $P = D_0 D_1, D_0 D_2 \in V_E$ :

$$\begin{aligned} \frac{\partial W}{\partial s} + W D_0 D_1 &= (W D_0 D_1 W^{-1})_+ W, \\ \frac{\partial W}{\partial t} + W D_0 D_2 &= (W D_0 D_2 W^{-1})_+ W. \end{aligned} \quad (2.5)$$

In terms of the coefficients  $g_{i,j}$  the equation (2.5) is written in the form:

$$\begin{aligned} \frac{\partial g_{i,j}}{\partial s} &= \frac{\partial g_{i,j-1}}{\partial x_1} - \frac{\partial g_{1,-1}}{\partial x_1} \frac{\partial g_{i,j}}{\partial x_1} - \frac{\partial g_{2,-1}}{\partial x_1} \frac{\partial g_{i,j}}{\partial x_2}, \\ \frac{\partial g_{i,j}}{\partial t} &= \frac{\partial g_{i,j-1}}{\partial x_2} - \frac{\partial g_{1,-1}}{\partial x_2} \frac{\partial g_{i,j}}{\partial x_1} - \frac{\partial g_{2,-1}}{\partial x_2} \frac{\partial g_{i,j}}{\partial x_2} \quad \text{for } i = 1, 2, j < 0. \end{aligned} \quad (2.6)$$

We set  $A_j = -\frac{\partial g_{1,-1}}{\partial x_j}, B_j = -\frac{\partial g_{2,-1}}{\partial x_j}$  ( $j = 1, 2$ ). By eliminating  $g_{i,-2}$  ( $i = 1, 2$ ) from the equation (2.6) for  $i = 1, 2, j = -1$ , we obtain the equation (2.4).

We shall unify these two examples and obtain more general systems. We introduce the Lie subalgebra  $V$  of  $\mathcal{E}$  which contains both the Lie algebras  $V_{YM}$  and  $V_E$ :

$$\begin{aligned} V = \{ F_0(x', D_0) x_0 + \sum_{0 < j < r} F_j(x', D_0) D_j + E(x', D_0); \\ F_k(x', D_0), E(x', D_0) \in \mathcal{E}, (0 \leq k < r) \}, \end{aligned}$$

where  $x' = (x_1, x_2, \dots, x_{r-1})$ .

**Lemma 2.1**  $V$  is isomorphic to  $\mathcal{D}(1) \otimes_{\mathcal{O}} \mathbb{C}[[x]][x_0^{-1}]$  as a Lie algebra.

*Proof.* Set  $X = \mathbb{C}^r = \{(z_0, z_1, \dots, z_{r-1}) \in \mathbb{C}^r\}$ , and  $Y = \{z \in X; z_0 = 0\}$ . We take the transformation

$$x_0 = z_0^2 D_{z_0}, \quad D_{x_0} = z_0^{-1}, \quad x' = z', \quad D_{x'} = D_{z'}.$$

Then the transform of  $V$  is  $\mathcal{D}(1) \otimes \mathbb{C}[[z]][z_0^{-1}]$ . □

The Lie algebra  $\mathcal{D}(1) \otimes \mathbb{C}[[x]][x_0^{-1}]$  is the direct sum of the Lie algebra  $\Theta \otimes_{\mathcal{O}} \mathbb{C}[[x]][x_0^{-1}]$  of vector fields and the commutative Lie algebra  $\mathbb{C}[[x]][x_0^{-1}]$ , where the Lie algebra  $\Theta$  is defined by

$$\Theta = \sum_{0 \leq j < r} \mathcal{O} D_j.$$

The Lie algebra  $\Theta \otimes_{\mathcal{O}} \mathbb{C}[[x]][x_0^{-1}]$  corresponds to the infinitesimal coordinate transformations, and the Lie algebra  $\mathbb{C}[[x]][x_0^{-1}]$  corresponds to the infinitesimal gauge transformations of a line bundle.

Now we shall determine the set of  $\mathcal{D}$ -submodules in  $\mathcal{E}$  such that the condition (1.1) is preserved under (1.3) with respect to any  $P \in V$ . We introduce the subspaces  $\mathcal{W}_m$  and  $\mathcal{W}_{YM}$  of  $\mathcal{E}$ :

$$\mathcal{W}_m = \{W(x, D_x) = \sum_{\alpha \in \mathbb{N}^r} \frac{1}{\alpha!} G^\alpha x_0^{\alpha_0} D_x^\alpha \in \mathcal{E};$$

$$G^\alpha = G_0^{\alpha_0} G_1^{\alpha_1} \dots G_{r-1}^{\alpha_{r-1}} \quad \text{where } G_j = G_j(x', D_0) \in \mathcal{E}(-1)\},$$

$$\mathcal{W}_{YM} = \{W(x', D_0) = \sum_{i \in \mathbb{N}} w_i(x') D_0^{-i}; w_i \in \mathbb{C}[[x_1, \dots, x_{r-1}]], w_0 \equiv 1.\}.$$

**Proposition 2.2** Let  $I_0 = \mathcal{D}W_0$  be a  $\mathcal{D}$ -submodule of  $\mathcal{E}$ . Assume that the time evolution of  $I_0$  for any  $P \in V$  also satisfies the condition (1.1). Then  $W_0$  factorizes into the product of  $W_m \in \mathcal{W}_m$  and  $W_{YM} \in \mathcal{W}_{YM}$ ;

$$W_0 = W_m W_{YM}.$$

*Remark.* The operator  $W_m$  corresponds to the equation of self-dual metrics (see §3).

*Proof of Proposition 2.2.*

*Step 1.*

**Lemma 2.3** Let  $W(t)$  be the solution of (1.4) for  $P \in V$  with  $W(0) = W_0$ . Then  $(W(t)PW(t)^{-1})_-$  is contained in  $\mathcal{E}_\phi(0)$ .

*Proof.*

It follows from the equation (1.5) that

$$\frac{\partial W(t)}{\partial t} W(t)^{-1} = -(W(t)PW(t)^{-1})_-.$$

Since the operator  $\frac{\partial W(t)}{\partial t} W(t)^{-1}$  is contained in  $\mathcal{E}(0)$ , we obtain Lemma 2.3.  $\square$

*Step 2.* For any  $P \in \mathcal{E}$ , let  $\tilde{P}$  be  $W_0 P W_0^{-1}$ .

**Lemma 2.4** Suppose that  $P, Q \in V$  commute with each other. The operator  $[\tilde{P}_+, \tilde{Q}_-]_-$  is contained in  $\mathcal{E}_\phi(0)$ .

*Proof.* We consider the solution  $W = W(s, t)$  of the equation (1.5) for the operators  $P, Q$ :

$$\begin{aligned} \frac{\partial W}{\partial s} &= -(WPW^{-1})_- W, \\ \frac{\partial W}{\partial t} &= -(WQW^{-1})_- W, \end{aligned} \tag{2.7}$$

where  $W = W(s, t)$ . Since  $[P, Q] = 0$ , the system (2.7) is compatible by Theorem 1.3. It follows from Lemma 2.3 that  $WPW^{-1}$  is contained in  $\mathcal{D} + \mathcal{E}_\phi(0)$ . We have

$$\begin{aligned} \frac{\partial}{\partial t}(WPW^{-1}) &= \frac{\partial W}{\partial t} PW^{-1} - WPW^{-1} \frac{\partial W}{\partial t} W^{-1} \\ &= -(WQW^{-1})_- WPW^{-1} + WPW^{-1} (WQW^{-1})_- \\ &= [WPW^{-1}, (WQW^{-1})_-]. \end{aligned}$$

Hence  $[\tilde{P}, \tilde{Q}_-]$  is contained in  $\mathcal{D} + \mathcal{E}_\phi(0)$ . Since  $[\tilde{P}_-, \tilde{Q}_-]$  is contained in  $\mathcal{E}_\phi(0)$  by Lemma 2.3,  $[\tilde{P}_+, \tilde{Q}_-]$  is contained in  $\mathcal{D} + \mathcal{E}_\phi(0)$ .  $\square$

*Step 3* Recall that  $\tilde{P}$  denotes  $W_0 P W_0^{-1}$  for any  $P \in \mathcal{E}$ . We define the operators  $G_i, H_i \in \mathcal{E}$  ( $0 \leq i < r$ ) as follows:

$$\begin{aligned}\tilde{D}_0 &= D_0 - G_0(x, D_x), \\ \tilde{x}_i &= x_i + G_i(x, D_x) \quad \text{for } i = 1, 2, \dots, r-1, \\ \tilde{x}_0 &= x_0 + H_0(x, D_x), \\ \tilde{D}_i &= D_i + H_i(x, D_x) \quad \text{for } i = 1, 2, \dots, r-1.\end{aligned}\tag{2.8}$$

**Lemma 2.5** Assume that  $W_0$  satisfies the condition in Proposition 2.2. The operators  $G_i, H_i$  are written in the following form:

$$\begin{aligned}G_i &= G_i(x', D_0) \in \mathcal{E}(-1), \\ H_i &= \sum_{0 < j < r} K_{ij}(x', D_0) D_j + L_i(x, D_0) \quad \text{for } 0 \leq i < r, \\ &\text{where } K_{ij}(x', D_0) \in \mathcal{E}, \quad L_i(x, D_0) \in \mathcal{E}(-1), \\ &\text{and } x' = (x_1, \dots, x_{r-1}).\end{aligned}$$

*Proof.* We shall show the following statement  $(2.9)_n$  by the induction on  $n$ :

$$\begin{aligned}G_i &\stackrel{\dagger}{=} G'_i + G''_i, \quad H_i = H'_i + H''_i \\ \text{for some } G'_i, H'_i, G''_i &\text{ and } H''_i \text{ (} 0 \leq i < r \text{) such that} \\ G''_0, H''_i &\in \mathcal{E}(-n), \quad G''_i, H''_0 \in \mathcal{E}(-n-1) \quad \text{for } 0 < i < r, \\ G'_i &= G'_i(x', D_0) \in \mathcal{E}(-1), \\ H'_i &= \sum_{0 < j < r} K_{ij}(x', D_0) D_j + L_i(x, D_0) \quad \text{for } 0 \leq i < r \\ \text{with } K_{ij}(x', D_0), L_i(x, D_0) &\in \mathcal{E}(-1).\end{aligned}\tag{2.9}_n$$

It is evident that the statement  $(2.9)_0$  is true. By assuming  $(2.9)_n$  we shall prove  $(2.9)_{n+1}$ .

We expand the operators:

$$\begin{aligned} G_j'' &= \sum_{k \leq 0, \alpha' \in \mathbb{N}^{r-1}} g_{k, \alpha'}^{(j)} D_0^k D_{x'}^{\alpha'}, \\ H_j'' &= \sum_{k \leq 0, \alpha' \in \mathbb{N}^{r-1}} h_{k, \alpha'}^{(j)} D_0^k D_{x'}^{\alpha'}. \end{aligned} \quad (2.10)$$

Since  $G_0''$  belongs to  $\mathcal{E}(-n)$ , we have

$$\begin{aligned} \tilde{D}_0^{n+2} &= (D_0 - G_0' - G_0'')^{n+2} \\ &\equiv (D_0 - G_0')^{n+2} - (n+2)G_0''(D_0 - G_0')^{n+1} \\ &\equiv (D_0 - G_0')^{n+2} - (n+2)G_0''D_0^{n+1} \quad \text{modulo } \mathcal{E}(0). \end{aligned}$$

The operator  $(D_0 - G_0')^{n+2}$  belongs to  $\mathcal{D} + \mathcal{E}(0)$ , because  $G_0'$  does not contain  $D_j$  ( $0 < j < r$ ). Hence we obtain

$$\tilde{D}_0^{n+2} \equiv -(n+2)G_0''D_0^{n+1} \quad \text{modulo } \mathcal{D} + \mathcal{E}(0).$$

Since  $\tilde{D}_0^{n+2}$  is contained in  $\mathcal{D} + \mathcal{E}(0)$  for any  $n \in \mathbb{Z}$  by Lemma 2.3,  $G_0''D_0^{n+1}$  is also contained in  $\mathcal{D} + \mathcal{E}(0)$ . Therefore we obtain

$$g_{-n-|\alpha'|, \alpha'}^{(0)} = 0 \quad \text{for } |\alpha'| \geq 2. \quad (2.11)$$

Similarly we have, modulo  $\mathcal{D} + \mathcal{E}(0)$

$$\begin{aligned} \tilde{D}_0^{n+1} \tilde{D}_j &\equiv -(n+1)G_0''D_0^n D_j + H_j''D_0^{n+1}, \\ \tilde{x}_0 \tilde{D}_0^{n+2} &\equiv -(n+2)x_0 G_0''D_0^{n+1} + H_0''D_0^{n+2}, \\ \tilde{x}_j \tilde{D}_0^{n+1} \tilde{D}_k &\equiv G_j''D_0^{n+1} D_k + x_j (H_k''D_0^{n+1} - (n+1)G_0''D_0^n D_k). \end{aligned}$$



By Lemma 2.3,  $\tilde{D}_0^{n+1}\tilde{D}_j$ ,  $\tilde{x}_0\tilde{D}_0^{n+2}$  and  $\tilde{x}_j\tilde{D}_0^{n+1}\tilde{D}_k$  are contained in  $\mathcal{D}+\mathcal{E}(0)$ . Thus we obtain

$$\begin{aligned}
h_{-|\alpha'|-n,\alpha'}^{(j)} &= 0 \quad \text{for } |\alpha'| \geq 3, \\
h_{-2-n,\alpha'+e_j}^{(j)} &= (n+1)g_{-1-n,\alpha'}^{(0)} \quad \text{for } |\alpha'| = 1, \\
h_{-|\alpha'|-n-1,\alpha'}^{(0)} &= 0 \quad \text{for } |\alpha'| \geq 2, \\
g_{-|\alpha'|-n-1,\alpha'}^{(j)} &= 0 \quad \text{for } |\alpha'| \geq 2,
\end{aligned} \tag{2.12}$$

where  $j = 1, 2, \dots, r-1$  and  $e_j = (0, \dots, \underset{j}{1}, \dots, 0) \in \mathbb{N}^{r-1}$ .

We denote by  $\mathcal{V}$  and  $\mathcal{V}'$  the subspaces of  $\mathcal{D}+\mathcal{E}(0)$ :

$$\begin{aligned}
\mathcal{V} &= \left\{ \sum_{0 < k < r} F_k(x, D_0)D_k + E(x, D_0); \quad F_k(x, D_0), E(x, D_0) \in \mathcal{E} \right\}, \\
\mathcal{V}' &= \left\{ \sum_{0 < k < r} F_k(x', D_0)D_k + E(x, D_0); \quad F_k(x', D_0), E(x, D_0) \in \mathcal{E} \right\}.
\end{aligned}$$

For  $P, Q \in \mathcal{V}'$  the commutator  $[P, Q]$  is contained in  $\mathcal{V}$ . Since  $H'_j$  ( $0 \leq j < r$ ) are contained in  $\mathcal{V}'$ ,  $[H'_i, H'_j]$   $[x_0, H'_j]$  ( $0 \leq i, j < r$ ) are contained in  $\mathcal{V}$ .

For  $j > 0$  we have

$$\begin{aligned}
[\tilde{x}_0, \tilde{D}_j] &= [x_0, H_j] - \frac{\partial H_0}{\partial x_j} + [H_0, H_j] \\
&= [x_0, H_j''] - \frac{\partial H_0''}{\partial x_j} + ([H_0'', H_j''] + [H_0', H_j''] + [H_0'', H_j']) \\
&\quad + ([x_0, H_j'] - \frac{\partial H_0'}{\partial x_j} + [H_0', H_j']) \\
&\equiv [x_0, H_j''] - \frac{\partial H_0''}{\partial x_j} \quad \text{modulo } \mathcal{E}(-n-2) + \mathcal{V}.
\end{aligned}$$

Since  $[\tilde{x}_0, \tilde{D}_j] = 0$ ,  $[x_0, H_j''] - \frac{\partial H_0''}{\partial x_j}$  belongs to  $\mathcal{E}(-n-2) + \mathcal{V}$ . By (2.10) we

have, modulo  $\mathcal{E}(-n-2) + \mathcal{V}$

$$\begin{aligned} [x_0, H_j''] - \frac{\partial H_0''}{\partial x_j} \\ \equiv \sum_{|\alpha'| \geq 2} ((n + |\alpha'|) h_{-n-|\alpha'|, \alpha'}^{(j)} - \frac{\partial h_{-n-1-|\alpha'|, \alpha'}^{(0)}}{\partial x_j}) D_0^{-1-n-|\alpha'|} D_{x'}^{\alpha'} \end{aligned} \quad (2.13)$$

We obtain from (2.12) and (2.13) that

$$\begin{aligned} h_{-2-n, \alpha'}^{(j)} &= 0 \quad \text{for } |\alpha'| = 2, \quad 0 < j < r, \\ g_{-1-n, \alpha'}^{(0)} &= 0 \quad \text{for } |\alpha'| = 1. \end{aligned} \quad (2.14)$$

Similarly by the equations

$$\begin{aligned} [\tilde{D}_0, \tilde{D}_j] &= \frac{\partial H_j}{\partial x_0} + \frac{\partial G_0}{\partial x_j} - [G_0, H_j] = 0 \quad \text{for } 0 < j < r, \\ [\tilde{D}_0, \tilde{x}_j] &= \frac{\partial G_j}{\partial x_0} + [G_j, G_0] - [G_0, x_j] = 0 \quad \text{for } 0 < j < r, \\ [\tilde{D}_0, \tilde{x}_0] &= 1 + [D_0, H_0] - [G_0, x_0] + [G_0, H_0] = 1, \end{aligned}$$

we obtain

$$\frac{\partial h_{-1-n, e_k}^{(j)}}{\partial x_0} = 0, \quad \frac{\partial h_{-2-n, e_k}^{(0)}}{\partial x_0} = 0, \quad \frac{\partial g_{-1-n, 0}^{(j)}}{\partial x_0} = 0. \quad (2.15)$$

For  $0 < i < r$  we have modulo  $\mathcal{E}(-1)$

$$\begin{aligned} \tilde{D}_0^n \tilde{D}_i &= (D_0 - G'_0 - G''_0)^n (D_i + H'_i + H''_i) \\ &\equiv ((D_0 - G'_0)^n - nG''_0(D_0 - G'_0)^{n-1})(D_i + H'_i + H''_i) \\ &\equiv -nG''_0 D_0^{n-1} D_i + (D_0 - G'_0)^n (D_i + H'_i) + H''_i D_0^n. \end{aligned} \quad (2.16)$$

By (2.12) and (2.14) we have

$$H_i'' D_0^n \equiv \sum_{0 < j < r} h_{-1-n, e_j}^{(i)} D_0^{-1} D_j \text{ modulo } \mathcal{E}(-1),$$

$$-n G_0'' D_0^{n-1} D_i \equiv -n g_{-n, 0}^{(0)} D_0^{-1} D_i \text{ modulo } \mathcal{E}(-1).$$

We set  $K_{ij}$  and  $L_i$  in  $(2.9)_n$  as follows:

$$K_{ij}(x', D_0) = \sum_{m < 0} k_{i,j,m}(x') D_0^m,$$

$$L_i(x, D_0) = \sum_{m < 0} l_{i,m}(x') D_0^m$$

By  $(2.9)_n$  we have

$$\begin{aligned} (\tilde{D}_0 \tilde{D}_i)_+ &= (D_0 - G_0)(D_i + H_i' + H_i'') \\ &= D_0 D_i + \{D_0(H_i' + \sum_{1 \leq j < r} K_{ij} + L_i)\}_+ \\ &= D_0 D_i + \sum_{1 \leq j < r} (h_{-1, e_j}^{(i)} + k_{i,j,-1}) D_j + l_{i,-1} \end{aligned} \quad (2.17)$$

**Sublemma.** We have

$$\frac{\partial g_{-n, 0}^{(0)}}{\partial x_0} = 0. \quad (2.18)$$

*Proof.* The commutator  $[(\tilde{D}_0 \tilde{D}_i)_+, \tilde{D}_0^n \tilde{D}_i]$  is contained in  $\mathcal{D} + \mathcal{E}(0)$  by Lemma 2.4. By (2.16) and (2.17) we have, modulo  $\mathcal{E}(0)$

$$\begin{aligned} & [(\tilde{D}_0 \tilde{D}_i)_+, \tilde{D}_0^n \tilde{D}_i] \\ & \equiv [(\tilde{D}_0 \tilde{D}_i)_+, -n g_{-n, 0}^{(0)} D_0^{-1} D_i \\ & \quad + \sum_{0 < j < r} h_{-1-n, e_j}^{(i)} D_0^{-1} D_j + (D_0 - G_0')^n (D_i + H_i')] \\ & \equiv [D_0 D_i, -n g_{-n, 0}^{(0)} D_0^{-1} D_i] \\ & \quad + [(\tilde{D}_0 \tilde{D}_i)_+, \sum_{0 < j < r} h_{-1-n, e_j}^{(i)} D_0^{-1} D_j + (D_0 - G_0')^n (D_i + H_i')]. \end{aligned}$$

Since  $(\tilde{D}_0 \tilde{D}_i)_+$  and  $\sum_{0 < j < r} h_{-1-n, e_j}^{(i)} D_0^{-1} D_j + (D_0 - G'_0)^n (D_i + H'_i)$  are contained in  $\mathcal{V}$ , the operator

$$\begin{aligned} & [D_0 D_i, -n g_{-n, 0}^{(0)} D_0^{-1} D_i] \\ &= -n \frac{\partial g_{-n, 0}^{(0)}}{\partial x_0} D_0^{-1} D_i^2 - n \frac{\partial g_{-n, 0}^{(0)}}{\partial x_i} D_i - n \frac{\partial^2 g_{-n, 0}^{(0)}}{\partial x_0 \partial x_i} \end{aligned}$$

is contained in  $\mathcal{D} + \mathcal{E}(0)$ . This implies (2.18).  $\square$

It follows from (2.11), (2.12) and (2.14) that we have

$$\begin{aligned} G'_0 - g_{-n, 0}^{(0)} D_0^{-n} &\in \mathcal{E}(-n-1), \\ G'_i - g_{-n-1, 0}^{(i)} D_0^{-n-1} &\in \mathcal{E}(-n-2), \\ H'_0 - \left( \sum_{0 < k < r} h_{-n-2, e_k}^{(0)} D_0^{-n-2} D_k + h_{-n-1, 0}^{(0)} D_0^{-n-1} \right) &\in \mathcal{E}(-n-2), \\ H'_i - \left( \sum_{0 < k < r} h_{-n-1, e_k}^{(i)} D_0^{-n-1} D_k + h_{-n, 0}^{(i)} D_0^{-n} \right) &\in \mathcal{E}(-n-1). \end{aligned}$$

for  $0 < i < r$ . Thus we obtain  $(2.9)_{n+1}$  from (2.15) and (2.18).  $\square$

*Step 4* Now we shall prove Proposition 2.2. We introduce a micro-differential operator

$$W_m = \sum_{\alpha \in \mathbb{N}^r} \frac{1}{\alpha!} G^\alpha x_0^{\alpha_0} D_{x'}^{\alpha'},$$

where  $G^\alpha = G_0^{\alpha_0} G_1^{\alpha_1} \cdots G_{r-1}^{\alpha_{r-1}}$  is given in Lemma 2.5. Then we have

$$\begin{aligned} [D_0, W_m] &= \sum_{\alpha \in \mathbb{N}^r} \frac{\alpha_0}{\alpha!} G^\alpha x_0^{\alpha_0-1} D_{x'}^{\alpha'} \\ &= G_0 \sum_{\alpha \in \mathbb{N}^r} \frac{1}{\alpha!} G^\alpha x_0^{\alpha_0} D_{x'}^{\alpha'} = G_0 W_m. \end{aligned}$$

Hence we get

$$W_m D_0 W_m^{-1} = D_0 - G_0. \quad (2.20)$$

Similarly we obtain that

$$W_m x_j W_m^{-1} = x_j + G_j \quad (1 \leq j < r). \quad (2.21)$$

Set  $W_{YM} = W_m^{-1} W$ . Then it follows from (2.8), (2.20) and (2.21) that  $W_{YM}$  commutes with  $D_0, x_1, \dots, x_{r-1}$ . Therefore  $W_{YM}$  is contained in  $\mathcal{W}_{YM}$ .

Thus we have completed the proof of Proposition 2.2.  $\square$

We set

$$\mathcal{W} = \{W_m W_{YM}; \quad W_m \in \mathcal{W}_m, W_{YM} \in \mathcal{W}_{YM}\}.$$

We shall investigate the structure of  $\mathcal{W}, \mathcal{W}_m$  and  $\mathcal{W}_{YM}$ .

**Proposition 2.6** The spaces  $\mathcal{W}, \mathcal{W}_m$  and  $\mathcal{W}_{YM}$  are groups by the composition of microdifferential operators, and  $\mathcal{W}_{YM}$  is a normal subgroup in  $\mathcal{W}$ .

*Proof.* It is evident that  $\mathcal{W}_{YM}$  is an abelian group.

Let  $W_m = \sum_{\alpha} \frac{1}{\alpha!} G^{\alpha} x_0^{\alpha_0} D_{x'}^{\alpha'}$  and  $W'_m = \sum_{\beta} \frac{1}{\beta!} \tilde{F}^{\beta} x_0^{\beta_0} D_{x'}^{\beta'}$  be operators in  $\mathcal{W}_m$ . For any microdifferential operator  $P(x', D_0)$ , we set

$$\tilde{P} := W_m P W_m^{-1} = P(x_1 + G_1, \dots, x_{r-1} + G_{r-1}, D_0 - G_0).$$

The last equality follows from (2.20) and (2.21). Noting that  $\tilde{P}$  commutes with  $G_j$ , the composition

$$\begin{aligned} W_m W'_m &= \sum_{\gamma} \frac{1}{\beta!} \tilde{F}^{\beta} W_m x_0^{\beta_0} D_{x'}^{\beta'} \\ &= \sum_{\alpha, \beta} \frac{1}{\beta!} \tilde{F}^{\beta} \frac{1}{\alpha!} G^{\alpha} x_0^{\alpha_0 + \beta_0} D_{x'}^{\alpha' + \beta'} \\ &= \sum_{\gamma} \frac{1}{\gamma!} (\tilde{F} + G)^{\gamma} x_0^{\gamma_0} D_{x'}^{\gamma'} \end{aligned} \quad (2.22)$$



is contained in  $\mathcal{W}_m$ . For  $W_{YM} = \sum_i w_i(x')D_0^{-i} \in \mathcal{W}_{YM}$  the operator

$$\begin{aligned} & W_m W_{YM} W_m^{-1} \\ &= \sum_{i \geq 0} w_i(x_1 + G_1, x_2 + G_2, \dots, x_{r-1} + G_{r-1})(D_0 - G_0)^{-i} \end{aligned} \quad (2.23)$$

is contained in  $\mathcal{W}_{YM}$ . Since  $\mathcal{W} = \mathcal{W}_m \mathcal{W}_{YM}$ , it follows from (2.22) and (2.23) that  $\mathcal{W}$  is a group and that  $\mathcal{W}_{YM}$  is a normal subgroup of  $\mathcal{W}$ .  $\square$

We define the Lie subalgebras  $V_m$  and  $V_{YM}$  of  $V$

$$\begin{aligned} V_m &= \{F_0(x', D_0)x_0 + \sum_{0 < j < r} F_j(x', D_0)D_j; \\ & \quad F_k(x', D_0) \in \mathcal{E}, (0 \leq k < r)\}, \\ V_{YM} &= \{E(x', D_0); E(x', D_0) \in \mathcal{E}\}. \end{aligned}$$

We have

$$V = V_m \oplus V_{YM}.$$

**Proposition 2.7** For any  $P \in V$  (resp.  $V_m, V_{YM}$ ) and  $W \in \mathcal{W}$  (resp.  $\mathcal{W}_m, \mathcal{W}_{YM}$ ), we have  $WPW^{-1} \in V$  (resp.  $V_m, V_{YM}$ ).

*Proof.* For  $W = \sum_\alpha \frac{1}{\alpha!} G^\alpha x_0^{\alpha_0} D_{x'}^{\alpha'} \in \mathcal{W}_m$  we have

$$\begin{aligned} [D_i, W] &= \sum_{0 \leq j < r} \sum_\alpha \frac{\alpha_j}{\alpha!} G^\alpha \frac{\partial G_j}{\partial x_i} x_0^{\alpha_0} D_{x'}^{\alpha'} \\ &= \sum_{0 < j < r} \frac{\partial G_j}{\partial x_i} W D_j + \frac{\partial G_0}{\partial x_i} W x_0 \end{aligned}$$

for any  $i = 1, 2, \dots, r-1$ . Similarly we have

$$[x_0, W] = \sum_{0 < j < r} [x_0, G_j] W D_j + [x_0, G_0] W x_0.$$

Hence we obtain

$$\begin{aligned} WD_iW^{-1} &= D_i - \sum_{0 < j < r} \frac{\partial G_j}{\partial x_i} WD_jW^{-1} - \frac{\partial G_0}{\partial x_i} Wx_0W^{-1} \quad \text{for } 0 < i < r, \\ Wx_0W^{-1} &= x_0 - \sum_{0 < j < r} [x_0, G_j] WD_jW^{-1} - [x_0, G_0] Wx_0W^{-1}. \end{aligned} \quad (2.24)$$

We set

$$G_{ij} = \begin{cases} \frac{\partial G_j}{\partial x_i} & \text{for } 0 < i < r, \\ [x_0, G_j] & i = 0, \end{cases}$$

for  $0 \leq j < r$ . Then we have

$$\begin{aligned} x_0 &= (1 + G_{00})Wx_0W^{-1} + \sum_{0 < j < r} G_{0j}WD_jW^{-1}, \\ D_i &= G_{i0}Wx_0W^{-1} + \sum_{0 < j < r} (\delta_{ij} + G_{ij})WD_jW^{-1} \quad \text{for } 0 < i < r. \end{aligned} \quad (2.25)$$

Since  $G_{ij} \in \mathcal{E}(-1)$ , The matrix  $(\delta_{ij} + G_{ij})_{0 \leq i, j < r}$  is invertible. Hence  $WD_iW^{-1}$  ( $0 < i < r$ ) and  $Wx_0W^{-1}$  are contained in  $V_m$ , because  $G_{ij}$  is independent of  $x_0, D_1, \dots, D_{r-1}$ . For any operator  $P = P(x', D_0) \in \mathcal{E}$ ,  $WPW^{-1}$  is independent of  $x_0, D_1, \dots, D_{r-1}$  by (2.20) and (2.21). Hence we get Proposition 2.7 for  $V_m$  and  $\mathcal{W}_m$ .

The proposition is evident for  $V_{YM}$  and  $\mathcal{W}_{YM}$ . For  $W_m \in \mathcal{W}_m$  and  $E \in V_{YM}$  the operator  $W_mE W_m^{-1}$  is contained in  $V_{YM}$  by (2.20) and (2.21). For  $W_{YM} \in \mathcal{W}_{YM}$  we have

$$\begin{aligned} W_{YM}D_iW_{YM}^{-1} &= D_i - \frac{\partial W_{YM}}{\partial x_i} W_{YM}^{-1} \quad \text{for } 0 < i < r, \\ W_{YM}x_0W_{YM}^{-1} &= x_0 - [x_0, W_{YM}]W_{YM}^{-1}. \end{aligned}$$

Since  $W_{YM}$  commutes with  $D_0, x_1, \dots, x_{r-1}$ , the operator  $W_{YM}PW_{YM}^{-1}$  is contained in  $V$  for any  $P \in V$ . Since  $V = V_m \oplus V_{YM}$  and  $\mathcal{W} = \mathcal{W}_m \mathcal{W}_{YM}$ , we get Proposition 2.7.  $\square$

*Example 2.3* We shall write down the evolution equations (1.4) for  $W \in \mathcal{W}$  in the case  $r = 3$ . We take  $D_0D_1, D_0D_2$  as generators of (1.4).

We set

$$W = \left( \sum_{\alpha \in \mathbb{N}^3} \frac{1}{\alpha!} G^\alpha x_0^{\alpha_0} D_{x'}^{\alpha'} \right) \left( \sum_i w_i(x') D_0^{-i} \right),$$

where  $G_i(x_1, x_2, D_0) = \sum_{j < 0} g_{i,j}(x_1, x_2) D_0^j \quad (i = 1, 2).$

Then we have

$$\begin{aligned} (\tilde{D}_0 \tilde{D}_1)_+ &= D_0 D_1 - \frac{\partial g_{0,-1}}{\partial x_1} x_0 - \sum_{i=1,2} \frac{\partial g_{i,-1}}{\partial x_1} D_i - \frac{\partial w_1}{\partial x_1}, \\ (\tilde{D}_0 \tilde{D}_2)_+ &= D_0 D_2 - \frac{\partial g_{0,-1}}{\partial x_2} x_0 - \sum_{i=1,2} \frac{\partial g_{i,-1}}{\partial x_2} D_i - \frac{\partial w_1}{\partial x_2}. \end{aligned}$$

Taking time parameters  $s$  and  $t$  with respect to  $D_0 D_1$  and  $D_0 D_2$ , respectively, we obtain the evolution equations

$$\begin{aligned} \frac{\partial W}{\partial s} + W D_0 D_1 &= \left( D_0 D_1 - \frac{\partial g_{0,-1}}{\partial x_1} x_0 - \sum_{i=1,2} \frac{\partial g_{i,-1}}{\partial x_1} D_i - \frac{\partial w_1}{\partial x_1} \right) W, \\ \frac{\partial W}{\partial t} + W D_0 D_2 &= \left( D_0 D_2 - \frac{\partial g_{0,-1}}{\partial x_2} x_0 - \sum_{i=1,2} \frac{\partial g_{i,-1}}{\partial x_2} D_i - \frac{\partial w_1}{\partial x_2} \right) W. \end{aligned} \quad (2.26)$$

It follows from (2.26) that we obtain the Zakharov-Shabat type equation

$$\begin{aligned} \left[ \frac{\partial}{\partial s} - D_0 D_1 + \frac{\partial g_{0,-1}}{\partial x_1} x_0 + \sum_{i=1,2} \frac{\partial g_{i,-1}}{\partial x_1} D_i + \frac{\partial w_1}{\partial x_1}, \right. \\ \left. \frac{\partial}{\partial t} - D_0 D_2 + \frac{\partial g_{0,-1}}{\partial x_2} x_0 + \sum_{i=1,2} \frac{\partial g_{i,-1}}{\partial x_2} D_i + \frac{\partial w_1}{\partial x_2} \right] = 0. \end{aligned} \quad (2.27)$$

We shall investigate the infinitesimal action  $\rho$  of the Lie subalgebra  $V$  of  $\mathcal{E}$ . Remark that the Lie algebra of the group  $\mathcal{W}$  (resp.  $\mathcal{W}_m, \mathcal{W}_{YM}$ ) is

canonically isomorphic to the Lie subalgebra  $V_- = V \cap \mathcal{E}_\phi$  (resp.  $(V_m)_-$ ,  $(V_{YM})_-$ ).

**Theorem 2.8** The action  $\rho$  of  $V$  (resp.  $V_m, V_{YM}$ ) preserves the space  $\mathcal{W}$  (resp.  $\mathcal{W}_m, \mathcal{W}_{YM}$ ). The action of  $V_-$  (resp.  $(V_m)_-, (V_{YM})_-$ ) coincides with the infinitesimal right action of on the group  $\mathcal{W}$  (resp.  $\mathcal{W}_m, \mathcal{W}_{YM}$ ).

*Proof.* For any element  $P \in V$  (resp.  $V_m, V_{YM}$ ) and any operator  $W \in \mathcal{W}$  (resp.  $\mathcal{W}_m, \mathcal{W}_{YM}$ ), we have

$$\rho(P) = -(WPW^{-1})_-W \in T_W\mathcal{E}.$$

By Proposition 2.7,  $-(WPW^{-1})_-W$  is contained in the tangent space of  $\mathcal{W}$  (resp.  $\mathcal{W}_m, \mathcal{W}_{YM}$ ) at  $W$ . Taking  $P \in V_-$  (resp.  $(V_m)_-, (V_{YM})_-$ ), we have  $\rho(P) = -WP$ . Hence the action  $\rho$  is the right action of vector fields.  $\square$

By Theorem 2.8 the Lie algebra  $V$  (resp.  $V_m, V_{YM}$ ) acts on  $\mathcal{W}$  (resp.  $\mathcal{W}_m, \mathcal{W}_{YM}$ ) transitively.

### §3. Twistor theory and integrable systems

On oriented Riemannian manifolds of dimension four, the Weyl curvature tensor  $C$  decomposes into two components, the self-dual part  $C_+$  and the anti-self-dual part  $C_-$ . A manifold is called self-dual (resp. anti-self-dual) when  $C_-$  (resp.  $C_+$ ) vanishes. Penrose [5] showed that the vanishing of the anti-self-dual part of the Weyl tensor is precisely the integrability condition of the existence of a curved twistor space.

In this section we prove that the equation  $C_+=0$  is the compatibility condition of the deformation equations of filtered  $\mathcal{D}$ -submodules in  $\mathcal{E}$  (See [13] in which the Frobenius integrability condition of the equations of self dual metrics is discussed). We get the equations of self dual metrics from the equation (2.26) for  $W \in \mathcal{W}_m$ .

Let  $M$  be a complex four-manifold and  $g$  a holomorphic metric, i.e. a non-degenerate symmetric holomorphic covariant two-tensor on  $M$ . We shall choose a holomorphic orientation on  $M$  which is necessary to define the complex Hodge \*-operator. Our discussion being only local, we can assume the existence of two complex vector bundles  $S_+$  and  $S_-$ : the bundles of self-dual and anti-self-dual spinors.

Let  $\{e_j\}_{j=1,2,3,4}$  denote a local coframe on  $M$  such that  $g = e_1e_2 + e_3e_4$ . We can write them in spinor language as

$$\begin{bmatrix} e_4 & e_2 \\ -e_1 & e_3 \end{bmatrix} = \begin{bmatrix} \psi_1\phi_1 & \psi_1\phi_2 \\ \psi_2\phi_1 & \psi_2\phi_2 \end{bmatrix} \quad (3.1)$$

where  $\psi_1, \psi_2$  (resp.  $\phi_1, \phi_2$ ) are the bases of self-dual (resp. anti-self-dual) spinor coframes.

We take  $P = P(S_-)$ , the projective bundle of the rank two vector bundle  $S_-$ . We parametrize  $S_-$  locally by

$$(x, \mu_1, \mu_2) \longrightarrow \mu_1\phi_1(x) + \mu_2\phi_2(x),$$

and  $\mu = \mu_1/\mu_2$  is an affine coordinate for  $\mu_2 \neq 0$ .

**Theorem 3.1** ([5]) The Riemannian manifold  $(M, g)$  is self-dual iff the

following Pfaffian system  $\Omega$  on  $P$  is integrable:

$$\begin{aligned} \theta &:= d\mu + \omega_{21}\mu^2 - (\omega_{22} - \omega_{11})\mu - \omega_{12} = 0, \\ \Omega : \quad \sigma_1 &:= \mu e_4 + e_2 = 0, \\ \sigma_2 &:= -\mu e_1 + e_4 = 0, \end{aligned} \tag{3.2}$$

where  $\omega_{ij}$  is the connection form of  $S_-$  with respect to the frame  $\phi_1$  and  $\phi_2$ .

Let  $A(\Omega)$  be the sheaf of vector fields orthogonal to the Pfaffian system  $\Omega$ . The sheaf  $A(\Omega)$  is a Lie algebra iff  $\Omega$  is integrable. In this case there exists a local basis  $(v_1, v_2)$  of  $A(\Omega)$  such that  $[v_1, v_2] = 0$ .

**Proposition 3.2** Assume  $(M, g)$  self-dual. With appropriate coordinates  $(\lambda, x_1, x_2, s, t)$  of  $P$ , there exists a commuting basis  $(v_1, v_2)$  of  $A(\Omega)$ , in the following form

$$\begin{aligned} v_1 &= \frac{\partial}{\partial s} - \lambda \frac{\partial}{\partial x_1} - \left( \frac{\partial R}{\partial x_1} \frac{\partial}{\partial x_1} + \frac{\partial S}{\partial x_1} \frac{\partial}{\partial x_2} + \frac{\partial T}{\partial x_1} \frac{\partial}{\partial \lambda} \right), \\ v_2 &= \frac{\partial}{\partial t} - \lambda \frac{\partial}{\partial x_2} - \left( \frac{\partial R}{\partial x_2} \frac{\partial}{\partial x_1} + \frac{\partial S}{\partial x_2} \frac{\partial}{\partial x_2} + \frac{\partial T}{\partial x_2} \frac{\partial}{\partial \lambda} \right), \end{aligned} \tag{3.3}$$

where the functions  $R, S$  and  $T$  do not depend on  $\lambda$ .

*Proof.* First we notice the following lemma.

**Lemma 3.3** ([3]) Let  $(M, g)$  be a self-dual Riemannian four-manifold. Then there exist local coordinates  $(p_1, p_2, q_1, q_2)$  of  $M$  such that

$$g = \sum_{i,j=1,2} P_{ij}(p, q) dp_i dq_j.$$

It follows from Lemma 3.3 that we can take local frames  $\{e_j\}_{j=1,2,3,4}$  as follows:

$$\begin{aligned} e_1 &= -dp_1, \quad e_2 = -(P_{11}dq_1 + P_{12}dq_2), \\ e_4 &= dp_2, \quad e_3 = P_{21}dq_1 + P_{22}dq_2. \end{aligned}$$

By Theorem 3.1 the ideal  $\mathcal{I}$  generated by  $(\theta, \sigma_1, \sigma_2)$  is closed under the exterior derivative  $d$ . Thus we have

$$\begin{aligned} d\sigma_1 \wedge \theta \wedge \sigma_1 \wedge \sigma_2 &= 0, \\ d\sigma_2 \wedge \theta \wedge \sigma_1 \wedge \sigma_2 &= 0. \end{aligned} \tag{3.4}$$

By direct calculations we have

$$\begin{aligned} d\sigma_1 \wedge \theta \wedge \sigma_1 \wedge \sigma_2 &= \left( \left( \frac{\partial P_{12}}{\partial q_1} - \frac{\partial P_{11}}{\partial q_2} \right) \mu^2 + K\mu + L \right) d\mu \wedge dp_1 \wedge dp_2 \wedge dq_1 \wedge dq_2, \\ d\sigma_2 \wedge \theta \wedge \sigma_1 \wedge \sigma_2 &= \left( \left( \frac{\partial P_{22}}{\partial q_1} - \frac{\partial P_{21}}{\partial q_2} \right) \mu^2 + M\mu + N \right) d\mu \wedge dp_1 \wedge dp_2 \wedge dq_1 \wedge dq_2, \end{aligned}$$

for functions  $K, L, M$  and  $N$  independent of  $\mu$ . Thus we have

$$\frac{\partial P_{12}}{\partial q_1} - \frac{\partial P_{11}}{\partial q_2} = 0, \quad \frac{\partial P_{22}}{\partial p_1} - \frac{\partial P_{21}}{\partial p_2} = 0.$$

Hence we can define new coordinates  $(x_1, x_2, s, t, \mu)$  by the following equations:

$$\begin{aligned} \frac{\partial x_1}{\partial q_i} &= -P_{1i}, \quad \frac{\partial x_2}{\partial q_i} = P_{2i}, \quad (i = 1, 2) \\ s &= p_2, \quad t = -p_1, \end{aligned}$$

The differential forms  $\theta, \sigma_1$  and  $\sigma_2$  are written in these coordinates as follows:

$$\begin{aligned} \theta &= d\mu + \mu^2(E_1 e_2 + E_2 e_3) + \mu(F_1 e_2 + F_2 e_3) + \sum_j J_j e_j, \\ \sigma_1 &= \mu ds + (dx_1 + A_1 ds + A_2 dt), \\ \sigma_2 &= \mu dt + (dx_2 + B_1 ds + B_2 dt), \end{aligned}$$

For functions  $A_j, B_j, E_j, F_j$  ( $j = 1, 2$ ), and  $J_j$  ( $j = 1, 2, 3, 4$ ) independent of  $\mu$ . It is easily verified that the following vectors  $v_1, v_2$  belong to  $A(\Omega)$ :

$$\begin{aligned} v_1 &= -\mu \frac{\partial}{\partial x_1} + \frac{\partial}{\partial s} - A_1 \frac{\partial}{\partial x_1} - B_1 \frac{\partial}{\partial x_2} + (E_1 \mu^3 + F_1 \mu^2 + J_2 \mu - J_4) \frac{\partial}{\partial \mu}, \\ v_2 &= -\mu \frac{\partial}{\partial x_2} + \frac{\partial}{\partial t} - A_2 \frac{\partial}{\partial x_1} - B_2 \frac{\partial}{\partial x_2} + (E_2 \mu^3 + F_2 \mu^2 + J_3 \mu - J_1) \frac{\partial}{\partial \mu}. \end{aligned}$$

We set the vector field

$$\begin{aligned} l_1 &= \frac{\partial}{\partial s} - A_1 \frac{\partial}{\partial x_1} - B_1 \frac{\partial}{\partial x_2}, \\ l_2 &= \frac{\partial}{\partial t} - A_2 \frac{\partial}{\partial x_1} - B_2 \frac{\partial}{\partial x_2}. \end{aligned}$$

The commutator  $[v_1, v_2]$  is written in the following form

$$\begin{aligned} [v_1, v_2] &= (E_2 \mu^3 + F_2 \mu^2) \frac{\partial}{\partial x_1} - (E_1 \mu^3 + F_1 \mu^2) \frac{\partial}{\partial x_2} \\ &\quad + \{ (E_2 F_1 - E_1 F_2) \mu^4 + (2E_2 J_1 - 2E_1 J_2 + l_1(E_2) - l_2(E_1)) \mu^3 \\ &\quad + (F_2 J_1 - F_1 J_2 + \frac{\partial J_2}{\partial x_2} - \frac{\partial J_3}{\partial x_1} + l_1(F_2) - l_2(F_1)) \mu^2 \} \frac{\partial}{\partial \mu} + \mu u_1 + u_0, \end{aligned} \tag{3.5}$$

where the coefficients of vectors  $u_1$  and  $u_0$  are independent of  $\mu$ .

By the integrability condition,  $[v_1, v_2]$  is a linear combination of  $v_1$  and  $v_2$  and since  $[v_1, v_2]$  does not contain  $\frac{\partial}{\partial s}$  nor  $\frac{\partial}{\partial t}$ ,  $v_1$  and  $v_2$  commute with each other. It follows from (3.5) that

$$E_j = 0, \quad F_j = 0 \quad (j = 1, 2), \quad \frac{\partial J_2}{\partial x_2} = \frac{\partial J_3}{\partial x_1}.$$

Thus there exists a function  $f = f(x_1, x_2, s, t)$  such that

$$\frac{\partial f}{\partial x_1} = J_2, \quad \frac{\partial f}{\partial x_2} = J_3.$$



We can take new coordinates  $(\lambda = \mu + f, x_1, x_2, s, t)$ . With these coordinates we have

$$\begin{aligned} v_1 &= \frac{\partial}{\partial s} - \lambda \frac{\partial}{\partial x_1} - (A_1 \frac{\partial}{\partial x_1} + B_1 \frac{\partial}{\partial x_2} + C_1 \frac{\partial}{\partial \lambda}), \\ v_2 &= \frac{\partial}{\partial t} - \lambda \frac{\partial}{\partial x_2} - (A_2 \frac{\partial}{\partial x_1} + B_2 \frac{\partial}{\partial x_2} + C_2 \frac{\partial}{\partial \lambda}), \end{aligned}$$

where  $C_1 = \frac{\partial f}{\partial s} - (J_4 + J_2 A_1 + J_3 B_1)$  and  $C_2 = \frac{\partial f}{\partial t} - (J_1 + J_2 A_2 + J_3 B_2)$ . Again taking the coefficients of  $\mu$  in  $[v_1, v_2]$ , we obtain that

$$\frac{\partial A_1}{\partial x_2} = \frac{\partial A_2}{\partial x_1}, \quad \frac{\partial B_1}{\partial x_2} = \frac{\partial B_2}{\partial x_1}, \quad \frac{\partial C_1}{\partial x_2} = \frac{\partial C_2}{\partial x_1}.$$

Thus there exist functions  $R, S$  and  $T$  which are independent of  $\lambda$  such that

$$\frac{\partial R}{\partial x_j} = A_j, \quad \frac{\partial S}{\partial x_j} = B_j, \quad \frac{\partial T}{\partial x_j} = C_j \quad \text{for } j = 1, 2.$$

This completes the proof of Proposition 3.2. □

*Remark.* By Theorem 3.1 and Proposition 3.2 the equations of self-dual metrics are equivalent to the compatibility condition  $[v_1, v_2] = 0$ :

$$\begin{aligned} & \frac{\partial^2 T}{\partial t \partial x_1} - \frac{\partial^2 T}{\partial s \partial x_2} - \frac{\partial R}{\partial x_2} \frac{\partial^2 T}{\partial x_1^2} \\ & \quad + \left( \frac{\partial R}{\partial x_1} - \frac{\partial S}{\partial x_2} \right) \frac{\partial^2 T}{\partial x_1 \partial x_2} + \frac{\partial S}{\partial x_1} \frac{\partial^2 T}{\partial x_2^2} = 0, \\ & \frac{\partial^2 R}{\partial t \partial x_1} - \frac{\partial^2 R}{\partial s \partial x_2} - \frac{\partial T}{\partial x_2} \frac{\partial^2 R}{\partial x_1^2} \\ & \quad + \left( \frac{\partial R}{\partial x_1} - \frac{\partial S}{\partial x_2} \right) \frac{\partial^2 R}{\partial x_1 \partial x_2} + \frac{\partial S}{\partial x_1} \frac{\partial^2 R}{\partial x_2^2} = 0, \\ & \frac{\partial^2 S}{\partial t \partial x_1} - \frac{\partial^2 S}{\partial s \partial x_2} + \frac{\partial T}{\partial x_1} \frac{\partial^2 S}{\partial x_1^2} \\ & \quad + \left( \frac{\partial R}{\partial x_1} - \frac{\partial S}{\partial x_2} \right) \frac{\partial^2 S}{\partial x_1 \partial x_2} + \frac{\partial S}{\partial x_1} \frac{\partial^2 S}{\partial x_2^2} = 0. \end{aligned} \tag{3.6}$$

In the equation (2.26) we take  $W \in \mathcal{W}_m$ . In this case  $w_1$  vanishes. Replacing  $g_{0,1}, g_{1,1}$  and  $g_{2,1}$  with  $R, S$  and  $T$ , respectively. The equations (2.27) reduces to (3.6). Therefore we have

**Theorem 3.4** The Lie algebra  $V_m$  acts on the space of self-dual metrics. This action is transitive.

Let  $\mathcal{W}_0$  be the space of  $W \in \mathcal{W}$  which commute with  $D_0$ . In the equation (2.26) we take  $W \in \mathcal{W}_0$ . In this case  $G_0$  vanishes. The equation (2.27) reduces to the composed system of the self-dual Yang-Mills equations and the self-dual Einstein equations (see Example 2.1 and 2.2). Let  $V_0$  be the Lie subalgebra of  $V$ :

$$V_0 = \left\{ \sum_{0 \leq j < r} F_j(x', D_0) D_j + E(x', D_0); \right. \\ \left. F_k(x', D_0) \in \mathcal{E} (0 \leq k < r), E(x', D_0) \in \mathcal{E} \right\}.$$

Then  $V_0$  acts on  $\mathcal{W}_0$ . Thus the self-dual Einstein equations are a specialization of our integrable system.

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