PAPER Special Section on Recent Progress in Electromagnetic Theory and its Application

# The Vacuum Impedance and Unit Systems

**Masao KITANO**[†,††a)], ***Member***

**SUMMARY** In the electromagnetic theory, the vacuum impedance $Z_0$ is a universal constant, which is as important as the velocity of light $c_0$ in vacuum. Unfortunately, however, its significance is not appreciated so well and sometimes the presence itself is ignored. It is partly because in the Gaussian system of units, which has widely been used for long time, $Z_0$ is a dimensionless constant and of unit magnitude. In this paper, we clarify that $Z_0$ is a fundamental parameter in electromagnetism and plays major roles in the following scenes: reorganizing the structure of the electromagnetic formula in reference to the relativity; renormalizing the quantities toward natural unit systems starting from the SI unit system; and defining the magnitudes of electromagnetic units.

***key words:*** *vacuum impedance, wave impedance, Maxwell's equation, SI unit system, Gaussian unit system, metamaterial*

## 1. Introduction

The notion of vacuum impedance was introduced in late 1930's by Schelkunoff [1] in the study of wave propagation. It is defined as the amplitude ratio of the electric and magnetic fields of plane waves in vacuum, $Z_0 = E/H$, which has the dimension of electrical resistance. It is also called the characteristic impedance of vacuum or the wave resistance of vacuum. Due to the historical reasons, it has been recognized as a special parameter for engineers rather than a universal physical constant. Compared with the famous formula [2] for the velocity of light $c_0$ in terms of the vacuum permittivity $\varepsilon_0$ and the vacuum permeability $\mu_0$,

$$c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}, \tag{1}$$

the expression for the vacuum impedance

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}, \tag{2}$$

is used far less often. It is obvious when we look up index pages of textbooks on electromagnetism. A possible reason is perhaps that the Gaussian system of units, in which $Z_0$ is a dimensionless constant and of unit magnitude, has been used for long time.

As we will see, the pair of constants $(c_0, Z_0)$ can be conveniently used in stead of the pair $(\varepsilon_0, \mu_0)$ for many cases. However conventionally the asymmetric pairs $(c_0, \mu_0)$ or $(c_0, \varepsilon_0)$ are used and SI equations become less memorable.

In this paper, we reexamine the structure of electromagnetism in view of the SI (The International System of Units) system and find that $Z_0$ plays very important roles as a universal constant. In this process we also find that a wide-spread belief that the Gaussian system of units is akin to natural unit systems and suitable for theoretical studies is just a myth.

The recent development of a new type of media called metamaterials [3] demands the reconsideration of wave impedance. In metamaterials, both permittivity $\epsilon$ and permeability $\mu$ can be varied from values for vacuum and thereby the phase velocity $v_{\mathrm{ph}} = 1/\sqrt{\epsilon\mu}$ and wave impedance $Z = \sqrt{\mu/\epsilon}$ can be adjusted independently. With the control of wave impedance the reflection can be reduced or suppressed [5].

## 2. Relativistic Pairs of Variables

In relativity, the space variable $\boldsymbol{x}$ and the time variable $t$ are combined to form a 4-vector $(c_0 t, \boldsymbol{x})$. The constant $c_0$, which has the dimension of velocity, serves as a factor matching the dimensions of time and space. When we introduce a normalized variable $\tau \equiv c_0 t$, the 4-vector is simplified as $(\tau, \boldsymbol{x})$. In this form, space and time are represented with the same dimension.

Normally this is done offhandedly by setting $c_0 = 1$. However, this procedure is irreversible; $c_0$ becomes dimensionless and the dimensional distinction between space and time is lost. It is desirable to introduce normalized quantities such as $\tau$ when we compare the different systems of units.

Here we introduce notation for dimensional analysis. When the ratio of two quantities $X$ and $Y$ is dimensionless (just a pure number), we write $X \overset{\mathrm{D}}{\sim} Y$ and read "$X$ and $Y$ are dimensionally equivalent." For example, we have $c_0 t \overset{\mathrm{D}}{\sim} \boldsymbol{x}$. If a quantity $X$ can be measured in a unit u, we can write $X \overset{\mathrm{D}}{\sim} \mathrm{u}$. For example, for $l = 2.5$ m we have $l \overset{\mathrm{D}}{\sim} \mathrm{m}$.

With this notation, we can repeat the above discussion. For $c_0 t \overset{\mathrm{D}}{\sim} x$, instead of recasting $t \overset{\mathrm{D}}{\sim} x$ by forcibly setting $c_0 = 1$, we introduce a new normalized quantity $\tau = c_0 t$ and have $\tau \overset{\mathrm{D}}{\sim} x$. Accordingly velocity $v$ and $c_0$ is normalized as $\tilde{v} = v/c_0$ and $\tilde{c_0} = c_0/c_0 = 1$, respectively.

From the relativistic point of view, the scalar potential $\phi$ and the vector potential $\boldsymbol{A}$ are respectively a part of a uni-

Manuscript received April 18, 2008.

[†]The author is with the Department of Electronic Science and Engineering, Kyoto University, Kyoto-shi, 615-8510 Japan.

[††]The author is with CREST, Japan Science and Technology Agency, Kawaguchi-shi, 332-0012 Japan.

a) E-mail: kitano@kuee.kyoto-u.ac.jp

DOI: 10.1587/transele.E92.C.3

$$
\begin{array}{ccc}
\underline{\text{F series}} & & \underline{\text{S series}} \\
(\phi, c_0\boldsymbol{A}) & & \\
\downarrow \mathsf{d} & Y_0 & \\
(\boldsymbol{E}, c_0\boldsymbol{B}) & \leftarrow * \rightarrow & (\boldsymbol{H}, c_0\boldsymbol{D}) \\
\downarrow \mathsf{d} & Z_0 & \downarrow \mathsf{d} \\
0 & & (\boldsymbol{J}, c_0\varrho) \\
 & & \downarrow \mathsf{d} \\
 & & 0
\end{array}
$$

**Fig. 1** The relativistic pairs of quantities are arranged as a diagram, the rows of which correspond to the orders of tensors ($n = 1, 2, 3, 4$). In the left column, the quantities related to the electromagnetic forces (the F series), and in the right column, the quantities related to the electromagnetic sources (the S series) are listed. The exterior derivative "d" connects a pair of quantities by increasing the tensor order by one. These differential relations correspond to the definition of (scalar and vector) potentials, the Maxwell's four equations, and the charge conservation. Hodge's star operator "$*$" connects $(\boldsymbol{E}, c_0\boldsymbol{B})$ and $(\boldsymbol{H}, c_0\boldsymbol{D})$ pairs. This corresponds to the constitutive relations for vacuum and here appears the vacuum impedance $Z_0 = 1/Y_0$ as the proportional factor.

fied quantity. Considering $\phi \overset{\mathrm{D}}{\sim} \mathrm{V}$, $\boldsymbol{A} \overset{\mathrm{D}}{\sim} \mathrm{Vs/m}$, we introduce a pair of quantities:

$$(\phi, c_0\boldsymbol{A}) \overset{\mathrm{D}}{\sim} \mathrm{V}. \tag{3}$$

Similarly, we introduce other pairs:

$$
\begin{aligned}
&(\boldsymbol{E}, c_0\boldsymbol{B}) \overset{\mathrm{D}}{\sim} \mathrm{V/m},\\
&(\boldsymbol{H}, c_0\boldsymbol{D}) \overset{\mathrm{D}}{\sim} \mathrm{A/m},\\
&(\boldsymbol{J}, c_0\varrho) \overset{\mathrm{D}}{\sim} \mathrm{A/m^2},
\end{aligned} \tag{4}
$$

where $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{H}$, $\boldsymbol{D}$, $\boldsymbol{J}$, and $\varrho$ are the electric field, magnetic flux density, magnetic field strength, electric flux density, current density, and charge density, respectively. More precisely those pairs are anti-symmetric tensors in the 4-dimensional space as defined in Appendix.

We have seen that the constant $c_0$ appears when we form a relativistic tensor from a pair of non-relativistic electromagnetic quantities. In Fig. 1, such relativistic pairs are listed according to their tensor orders. We will find that this table is very helpful to overview the structure of electromagnetism [6], [7].

## 3. Roles of the Vacuum Impedance

In this section we show some examples for which $Z_0$ plays important roles. The impedance (resistance) is a physical quantity by which voltage and current are related. In the SI system, the unit for voltage is V(= J/C) (volt) and the unit for current is A(= C/s) (ampere). We should note that the latter is proportional to and the former is inversely proportional to the unit of charge, C (coulomb). We also note from Fig. 1 and Eqs. (3) and (4) that the units for quantities related to the electromagnetic forces (left column, labeled F series) are proportional to the volts. On the other hand, the units for quantities related to the electromagnetic sources (right column, labeled S series) are proportional to the ampere. As we will see below, the vacuum impedance $Z_0$ (or the vacuum admittance $Y_0 = 1/Z_0$) plays the role to connect the quantities in the F and S series via the constitutive relations.

### 3.1 Constitutive Relation

The constitutive relations for vacuum, $\boldsymbol{D} = \varepsilon_0\boldsymbol{E}$ and $\boldsymbol{H} = \mu_0^{-1}\boldsymbol{B}$, can be simplified by using the relativistic pairs of variables as

$$\begin{bmatrix} \boldsymbol{E} \\ c_0\boldsymbol{B} \end{bmatrix} = Z_0 \begin{bmatrix} c_0\boldsymbol{D} \\ \boldsymbol{H} \end{bmatrix}. \tag{5}$$

The electric relation and magnetic relation are united under the sole parameter $Z_0$. More precisely, with Hodge's star operator "$*$" (see Appendix) [8], [9], it can be written as

$$(\boldsymbol{E}, c_0\boldsymbol{B}) = *Z_0(\boldsymbol{H}, c_0\boldsymbol{D}). \tag{6}$$

### 3.2 Source-Field Relation

We know that the scalar potential $\Delta\phi$ induced by a charge $\Delta q = \varrho\Delta v$ is

$$\Delta\phi = \frac{1}{4\pi\varepsilon_0}\frac{\varrho\Delta v}{r}, \tag{7}$$

where $r$ is the distance between the source and the point of observation. The charge is presented as a product of charge density $\varrho$ and a small volume $\Delta v$. Similarly a current moment (current times length) $\boldsymbol{J}\Delta v$ generates the vector potential

$$\Delta\boldsymbol{A} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{J}\Delta v}{r}. \tag{8}$$

The relations (3) and (4) are united as

$$\begin{bmatrix} \Delta\phi \\ \Delta(c_0\boldsymbol{A}) \end{bmatrix} = \frac{Z_0}{4\pi r}\begin{bmatrix} c_0\varrho \\ \boldsymbol{J} \end{bmatrix}\Delta v. \tag{9}$$

We see that the vacuum impedance $Z_0$ plays the role to relate the source $(\boldsymbol{J}, c_0\varrho)\Delta v$ and the resultant fields $\Delta(\phi, c_0\boldsymbol{A})$ in a unified manner.

### 3.3 Plane Waves

For linearly polarized plane waves propagating in one direction in vacuum, a simple relation $E = cB$ holds. If we introduce $H$ $(= \mu_0^{-1}B)$ instead of $B$, we have $E = Z_0H$. This relation was introduced by Schelkunoff [1] in 1938. The reason why $H$ is used instead of $B$ is as follows. The boundary conditions for magnetic fields at the interface of media 1 and 2 are $H_{1\mathrm{t}} = H_{2\mathrm{t}}$ (tangential) and $B_{1\mathrm{n}} = B_{2\mathrm{n}}$ (normal). For the case of normal incidence, which is most important practically, the latter condition becomes trivial and cannot be used. Therefore $H$ is used more conveniently. The mixed use of the quantities ($E$ and $H$) of the F and S series invite $Z_0$ unintentionally.

### 3.4 Magnetic Monopole and the Fine-Structure Constant

Let us compare the force between electric charges $q$ and that between magnetic monopoles $g$ ($\stackrel{\text{D}}{\sim}$ Vs = Wb). If these forces are the same for equal distances $r$, i.e., $q^2/(4\pi\varepsilon_0 r^2) = g^2/(4\pi\mu_0 r^2)$, we have the relation $g = Z_0 q$. With this relation in mind, the Dirac monopole $g_0$ [10], whose quantization condition is $g_0 e = h$, can be beautifully expressed in terms of the elementary charge $e$ as

$$g_0 = \frac{h}{e} = \frac{h}{Z_0 e^2}(Z_0 e) = \frac{Z_0 e}{2\alpha}, \tag{10}$$

where $h = 2\pi\hbar$ is Planck's constant. The dimensionless parameter $\alpha = Z_0 e^2/2h = e^2/4\pi\varepsilon_0\hbar c_0 \sim 1/137$ is called the fine-structure constant, whose value is independent of unit systems and characterizes the strength of the electromagnetic interaction. The fine-structure constant itself can be represented more simply with the use of $Z_0$. Further, by introducing the von Klitzing constant (the quantized Hall resistance) $R_\text{K} = h/e^2$, the fine-structure constant can be expressed as $\alpha = Z_0/2R_\text{K}$ [11]. We have learned that the use of $Z_0$ helps to keep SI-formulae in simple forms.

## 4. The Magnitude of the Unit of Resistance

Here we consider a hypothetical situation where we are allowed to redefine the magnitudes of units in electromagnetism.

The product of the unit of voltage and that of current should yield the unit of power, $1\,\text{V} \times 1\,\text{A} = 1\,\text{W} = 1\,\text{J/s}$, which is a fixed quantity determined mechanically, or outside of electromagnetism. Thus, a new volt V′ and a new ampere A′ must be defined so as to satisfy

$$\text{A}' = k\text{A}, \quad \text{V}' = k^{-1}\text{V}, \tag{11}$$

in terms of the currently used V and A, where $k$ ($\neq 0$) is a scaling factor. Accordingly a new ohm Ω′ must be redefined as

$$\Omega' = k^{-2}\Omega. \tag{12}$$

We denote the numerical value as $\{A\} \equiv A/\text{u}$, when we measure a physical quantity $A$ with a unit u. For example, for $l = 1.3$ m we write $\{l\} = l/\text{m} = 1.3$. We can have another numerical value $\{A\}' \equiv A/\text{u}'$, when we measure the same quantity $A$ in a different unit u′. Now we have the relation

$$A = \{A\}\text{u} = \{A\}'\text{u}'. \tag{13}$$

It should be stressed that the physical quantity $A$ itself is independent of the choice of units. What depends on the choice is the numerical value $\{A\}$.

In the SI system, from the definition of $c_0$ and $\mu_0$, the vacuum impedance is represented as $Z_0 = \sqrt{\mu_0/\varepsilon_0} = c_0\mu_0 = (299\,792\,458\ \text{m/s}) \times (4\pi \times 10^{-7}\,\text{H/m}) = (119.916\,983\,2 \times \pi)\,\Omega \sim 377\,\Omega$. In our new system, with (12) we have

$$Z_0 = \{Z_0\}\Omega = k^2\{Z_0\}\Omega' = \{Z_0\}'\Omega'. \tag{14}$$

The numerical value must be changed from $\{Z_0\} = 377$ to $\{Z_0\}' = 377k^2$. For example, we could choose a new ohm Ω′ so that $Z_0 = 1\,\Omega'$ is satisfied by setting $k \sim 1/\sqrt{377}$. Conversely, fixing $\{Z_0\}$ to a particular number implies the determination of the magnitude of units (Ω, V, A, and others) in electromagnetism.

Once $k$, or $\{Z_0\}$ is fixed, the numerical values for quantities in the F series are multiplied by $k$ and those in the S series are divided by $k$. The sole parameter $k$ or $\{Z_0\}$ determines the numerical relation between the F and S series.

Coulomb's law for charges $q_1$ and $q_2$ can be rewritten as

$$\begin{aligned} F &= \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} = q_2 E = \{q_2\}\text{A s} \times \{E\}\,\text{V/m} \\ &= \{q_2\}'\text{A}'\text{s} \times \{E\}'\text{V}'/\text{m}, \end{aligned} \tag{15}$$

where $E = (4\pi\varepsilon_0)^{-1}(q_1/r^2)$ is the electric field generated by $q_1$. We see

$$\{q_2\}' = k^{-1}\{q_2\}, \quad \{E\}' = k\{E\}, \tag{16}$$

and find that the numerical value for the charge $q$ and that for the electric field $E$ will be changed reciprocally. We also note $\{\varepsilon_0\}' = k^{-2}\{\varepsilon_0\}$ and $\{\mu_0\}' = k^2\{\mu_0\}$.

In the SI, the ampere is defined in terms of the force $F$ between the parallel two wires carrying the same amplitude of current $I$. We have $F/l = \mu_0 I^2/(2\pi r)$, where $r$ is the separation and $l$ is the length of wires. Substituting $F = 2\times10^{-7}$ N, $r = l = 1$ m, $I = 1$ A, we get $\mu_0 = 4\pi\times10^{-7}$ H/m. Thus the numerial value $\{\mu_0\}$ (or $\{Z_0\}$) is fixed.

We could determine $\{\varepsilon_0\}$ by the force between charges with the same magnitude. In Giorgi's unit system (1901), which is a predecessor of the MKSA unit system or the SI system, $k$ was fixed by determining the magnitude of the ohm. The way of determination $\{Z_0\}$ has been and will be changed according to the development of high precision measurement technique.

## 5. Toward Natural Unit Systems

We have seen that normalizing quantities we can reduce the number of base units or the dimension. As shown in Table 1(a), first we introduce a new set of normalized quantities, $\tilde{X} = c_0 X$, derived from SI quantities $X$. We can remove the unit for time and we only need three base units; the ampere, volt, and meter.

Further, as seen in Table 1(b), when we introduce a set of normalized quantities, $X^* = Z_0 X$, by multiplying $Z_0$, only the volt and meter are required. By normalizing the quantities with the universal constants, $c_0$ and $Z_0$, we have a simplified set of Maxwell's equations:

$$\begin{aligned} &\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{D}}^* = \tilde{\varrho}^*, \quad \boldsymbol{\nabla}\times\tilde{\boldsymbol{H}}^* = \frac{\partial\tilde{\boldsymbol{D}}^*}{\partial\tau} + \boldsymbol{J}^*, \\ &\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{B}} = 0, \quad \boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial\tilde{\boldsymbol{B}}}{\partial\tau}, \end{aligned} \tag{17}$$

**Table 1** Pairs of quantities in electromagnetism and their units (dimension). Quantities $X$ normalized with $c$ and $Z_0$ are accented as $\tilde{X} = c_0 X$ and $X^* = Z_0 X$, respectively. As seen in (a) and (b), the variety of units is reduced owing to the normalization. Gaussian unit system (c) and the modified Gaussian unit system (d) are presented with normalized quantities: $\hat{S} = S/\sqrt{\varepsilon_0}$, $\hat{F} = F\sqrt{\varepsilon_0}$, $\acute{S} = S/\sqrt{\mu_0}$, $\acute{F} = F\sqrt{\mu_0}$, where $S$ ($F$) represents a quantity in the S (F) series. We notice an irregularity in the fourth row of the Gaussian system.

| | |
|---|---|
| $(\phi, \tilde{\boldsymbol{A}})$ | V |
| $(\boldsymbol{E}, \tilde{\boldsymbol{B}})$ | V/m |
| $(\boldsymbol{H}, \tilde{\boldsymbol{D}})$ | A/m |
| $(\boldsymbol{J}, \tilde{\varrho})$ | A/m$^2$ |

(a) $c_0$-normalization

| | |
|---|---|
| $(\phi, \tilde{\boldsymbol{A}})$ | V |
| $(\boldsymbol{E}, \tilde{\boldsymbol{B}})$ | V/m |
| $(\boldsymbol{H}^*, \tilde{\boldsymbol{D}}^*)$ | V/m |
| $(\boldsymbol{J}^*, \tilde{\varrho}^*)$ | V/m$^2$ |

(b) $(c_0, Z_0)$-normalization

| | |
|---|---|
| $(\hat{\phi}, \acute{\boldsymbol{A}})$ | $\sqrt{\text{N}}$ |
| $(\hat{\boldsymbol{E}}, \acute{\boldsymbol{B}})$ | $\sqrt{\text{N}}$/m |
| $(\acute{\boldsymbol{H}}, \hat{\boldsymbol{D}})$ | $\sqrt{\text{N}}$/m |
| $(c^{-1}\hat{\boldsymbol{J}}, \hat{\varrho})$ | $\sqrt{\text{N}}$/m$^2$ |

(c) Gaussian

| | |
|---|---|
| $(\hat{\phi}, \acute{\boldsymbol{A}})$ | $\sqrt{\text{N}}$ |
| $(\hat{\boldsymbol{E}}, \acute{\boldsymbol{B}})$ | $\sqrt{\text{N}}$/m |
| $(\acute{\boldsymbol{H}}, \hat{\boldsymbol{D}})$ | $\sqrt{\text{N}}$/m |
| $(\acute{\boldsymbol{J}}, \hat{\varrho})$ | $\sqrt{\text{N}}$/m$^2$ |

(d) modified Gaussian

with the constitutive relation, $\tilde{\boldsymbol{D}}^* = \boldsymbol{E}$ and $\boldsymbol{H}^* = \tilde{\boldsymbol{B}}$. Considering $\tau = ct$, this set of equations resembles to the Maxwell equations in the Gaussian system of units except for the rationalizing factor $4\pi$ [See (29)]. However there is a significant difference; the factor $1/c_0$ is missing in the current density term. We will return to this point later. It should be stressed that a natural system of units can be reached from the SI system by normalizations without detouring via the Gaussian system.

The number of basic units has been reduced from four (m, kg, s, A) to two (m, V) by introducing the quantities normalized with $c_0$ and $Z_0$. For further reduction toward a natural unit system [12], [13], $\hbar$ and the gravitational constant $G$ can be used for example.

## 6. The Gauss and Modified Heaviside-Lorentz Systems of Units

The SI and the cgs (esu, emu, Gaussian) systems differ in three respects. First, in the cgs unit systems, no fundamental dimensions are supplemented to the three fundamental dimensions for mechanics; length, mass, and time. On the other hand in the SI (MKSA) system, a new fundamental dimension that for electric current is introduced. The cgs systems contain three basic units, while the SI system contains four.

Secondly, the cgs systems are irrational systems; the factor $(1/4\pi)$ is erased from Coulomb's law but the factor $4\pi$ appears in the source terms of Maxwell's equations instead. The SI is a rational system, which has the opposite appearance. The Heaviside-Lorentz system is a rationalized version of the Gaussian system.

Thirdly, the base mechanical system for the cgs systems is the cgs (centimeter, gram, and second) mechanical system. That for the SI system is the MKS (meter, kilogram, and second) system.

In order to focus all our attention on the first respect, i.e., the number of basic units, we will ignore the differences in the last two respects. *From now on, we pretend that all the cgs systems (esu, emu, and Gaussian) are constructed rationally on the MKS mechanical system.* Especially the Heaviside-Lorentz system thus modified or the MKS version of rationalized Gaussian system is a three-unit, rational system based on the MKS system.

To go from the SI system to the cgs systems, we have to reduce the number of basic units by normalization with a universal constant.

In the cgs electrostatic system of units (esu), Coulomb's law is expressed as

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} = \frac{1}{4\pi}\frac{\hat{q}_1\hat{q}_2}{r^2}. \tag{18}$$

Thus, the normalized charge

$$\hat{q} = \frac{q}{\sqrt{\varepsilon_0}} \overset{\text{D}}{\sim} \frac{\text{C}}{\sqrt{\text{F/m}}} = \sqrt{\text{J m}} = \sqrt{\text{N}}\,\text{m}, \tag{19}$$

is a quantity expressed by mechanical dimensions only.

We can confirm that the numerical values for the esu and Heaviside-Lorentz systems are related as

$$\{\hat{q}\}_{\text{esu}} = \sqrt{4\pi}\{\hat{q}\}_{\text{HL}}\frac{\sqrt{\text{N}}\,\text{m}}{\sqrt{\text{dyn}}\,\text{cm}} = \sqrt{4\pi \times 10^5}\{\hat{q}\}_{\text{HL}},$$

where dyn = g · c/ms$^2$ is the cgs unit of force.

The quantities in the S series, each of which is proportional to the coulomb, C, can be normalized by division with $\sqrt{\varepsilon_0}$. On the other hand, the quantities in the F series, each of which is inversely proportional to C, can be normalized by multiplication with $\sqrt{\varepsilon_0}$. For example, $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, and $\boldsymbol{H}$, are normalized as

$$\hat{\boldsymbol{E}} = \boldsymbol{E}\sqrt{\varepsilon_0} \overset{\text{D}}{\sim} \frac{\sqrt{\text{N}}}{\text{m}}, \quad \hat{\boldsymbol{D}} = \frac{\boldsymbol{D}}{\sqrt{\varepsilon_0}} \overset{\text{D}}{\sim} \frac{\sqrt{\text{N}}}{\text{m}},$$
$$\hat{\boldsymbol{B}} = \boldsymbol{B}\sqrt{\varepsilon_0} \overset{\text{D}}{\sim} \frac{\sqrt{\text{N}}\,\text{s}}{\text{m}^2}, \quad \hat{\boldsymbol{H}} = \frac{\boldsymbol{H}}{\sqrt{\varepsilon_0}} \overset{\text{D}}{\sim} \frac{\sqrt{\text{N}}}{\text{s}}, \tag{20}$$

respectively. We have the constitutive relation

$$\hat{\boldsymbol{D}} = \hat{\boldsymbol{E}}, \quad \hat{\boldsymbol{H}} = c_0^2\hat{\boldsymbol{B}}, \tag{21}$$

and the normalized permittivity $\hat{\varepsilon}_0 = 1$ and permeability $\hat{\mu}_0 = 1/c_0^2$. The normalized vacuum impedance is

$$\hat{Z}_0 = \sqrt{\frac{\hat{\mu}_0}{\hat{\varepsilon}_0}} = \frac{1}{c_0} \overset{\text{D}}{\sim} \frac{\text{s}}{\text{m}}. \tag{22}$$

For the cgs electromagnetic system of units (emu), S-series quantities are multiplied by $\sqrt{\mu_0}$ and F-series quantities are divided by $\sqrt{\mu_0}$. With this normalization, $\mu_0$ is eliminated from the magnetic Coulomb law or the law of magnetic force between currents. The fields are normalized as

$$\acute{\boldsymbol{B}} = \frac{\boldsymbol{B}}{\sqrt{\mu_0}} \overset{\text{D}}{\sim} \frac{\sqrt{\text{N}}}{\text{m}}, \quad \acute{\boldsymbol{H}} = \boldsymbol{H}\sqrt{\mu_0} \overset{\text{D}}{\sim} \frac{\sqrt{\text{N}}}{\text{m}},$$
$$\acute{\boldsymbol{E}} = \frac{\boldsymbol{E}}{\sqrt{\mu_0}} \overset{\text{D}}{\sim} \frac{\sqrt{\text{N}}}{\text{s}}, \quad \acute{\boldsymbol{D}} = \boldsymbol{D}\sqrt{\mu_0} \overset{\text{D}}{\sim} \frac{\sqrt{\text{N}}\,\text{s}}{\text{m}^2}. \tag{23}$$

The constitutive relations are

$$\acute{\boldsymbol{H}} = \acute{\boldsymbol{B}}, \quad \acute{\boldsymbol{D}} = c_0^{-2}\acute{\boldsymbol{E}}, \tag{24}$$

and we have the normalized permeability $\acute{\mu}_0 = 1$ and permittivity $\acute{\varepsilon}_0 = 1/c_0^2$. The normalized vacuum impedance is

$$\acute{Z}_0 = \sqrt{\frac{\acute{\mu}_0}{\acute{\varepsilon}_0}} = c_0 \overset{\mathrm{D}}{\sim} \frac{\mathrm{m}}{\mathrm{s}}. \tag{25}$$

The Gaussian system of units is a combination of the esu and emu systems. For electrical quantities, the esu normalization is used and for magnetic quantities, the emu normalization is used. Namely, we use $\hat{\boldsymbol{E}}$, $\hat{\boldsymbol{D}}$, $\acute{\boldsymbol{B}}$, and $\acute{\boldsymbol{H}}$, all of which have the dimension $\sqrt{\mathrm{N}}/\mathrm{m}$. The constitutive relations are simplified as

$$\hat{\boldsymbol{D}} = \hat{\boldsymbol{E}}, \quad \acute{\boldsymbol{H}} = \acute{\boldsymbol{B}}, \tag{26}$$

and we have $\hat{\varepsilon}_0 = 1$ and $\acute{\mu}_0 = 1$. So far it looks nice because electric and magnetic quantities are treated symmetrically. Due to this apparent simplicity the Gaussian system has been used so widely. However, there is an overlooked problem in the normalization of current density. It is normalized as $\hat{\boldsymbol{J}} = \boldsymbol{J}/\sqrt{\varepsilon_0}$ in the Gaussian system. The current density is the quantity primarily connected to magnetic fields and therefore it should be normalized as $\acute{\boldsymbol{J}} = \boldsymbol{J}\sqrt{\mu_0}$ as for the emu system. Because of this miscasting, we have an irregularity in the fourth row of Table 1(c). The Gaussian normalization happens to make the pairs of quantities relativistic except for the $(\boldsymbol{J}, \varrho)$ pair.

The relativistic expression for the conservation of charge should be

$$\frac{\partial \tilde{\varrho}^*}{\partial(c_0 t)} + \nabla\cdot\boldsymbol{J}^* = 0, \tag{27}$$

as for the cases of $c_0$- or $(c_0, Z_0)$-normalization. In the Gaussian system, however, the non-relativistic expression

$$\frac{\partial \hat{\varrho}}{\partial t} + \nabla\cdot\hat{\boldsymbol{J}} = 0, \tag{28}$$

is adopted. As a practical system of units, it is a reasonable (and perhaps unique) choice.

This quirk can clearly be seen, when we compare the Maxwell's equations (17) in the natural system of units and that for the Gaussian system:

$$\begin{aligned} \nabla\cdot\hat{\boldsymbol{D}} &= \hat{\varrho}, \quad \nabla\times\acute{\boldsymbol{H}} = \frac{1}{c_0}\frac{\partial\hat{\boldsymbol{D}}}{\partial t} + \frac{1}{c_0}\hat{\boldsymbol{J}}, \\ \nabla\cdot\acute{\boldsymbol{B}} &= 0, \quad \nabla\times\hat{\boldsymbol{E}} = -\frac{1}{c_0}\frac{\partial\acute{\boldsymbol{B}}}{\partial t}. \end{aligned} \tag{29}$$

The factor $1/c_0$ in the current density term is a seam introduced when the esu and emu systems are joined into the Gaussian system.

The common belief that the Gaussian system is superior to the SI system because of the similarity to a natural unit system or because of the compatibility with relativity is almost pointless. We should remember that the Gaussian unit system was established in 1870s, when the relativity or the Lorentz transformation were not known yet.

The *modified* Gaussian system [14], in which $\acute{\boldsymbol{J}}$ is adopted and the above seam is eliminated, has been proposed but is rarely used. Table 1(d) contains quantities in the modified Gaussian system. They differ uniformly by a factor $\sqrt{\varepsilon_0}$ from the $(c_0, Z_0)$-normalized quantities in column (b).

## 7. Summary and Discussion

The important expression (1) for the velocity of light also holds for the esu and emu systems:

$$c_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{\hat{\mu}_0\hat{\varepsilon}_0}} = \frac{1}{\sqrt{\acute{\mu}_0\acute{\varepsilon}_0}}, \tag{30}$$

but not for the Gaussian system; $1/\sqrt{\hat{\varepsilon}_0\acute{\mu}_0} = 1 \neq c_0$.

The expression (2) for the vacuum impedance, is rewritten as $\hat{Z}_0 = 1/c_0$ and $\acute{Z}_0 = c_0$ for the esu and emu systems, respectively. Maxwell himself worked with the emu system when he found that light is an electromagnetic disturbance propagated according to electromagnetic laws [2]. For the emu system the dimensions of resistance and velocity are degenerate. For the Gaussian system, the vacuum impedance reduces to unity, $\sqrt{\acute{\mu}_0/\hat{\varepsilon}_0} = 1$.

Thus there is no room for the vacuum impedance in the cgs systems, which contains only three base units. However when we move to a unit system with four base units, the vacuum impedance $Z_0$ should be recognized as a fundamental constant as important as the velocity of light, $c_0$.

As has been pointed out by Sommerfeld [6], the Gaussian system tends to veil the significance of $D$ and $H$ in vacuum. Sometimes it is told that in vacuum only $E$ and $B$ have their significance and $D$ and $H$ lose their meaning. This argument is strongly misled by the use of Gaussian system of units. Considering the tensorial nature of quantities as in (6), the constitutive relations for the Gaussian system are expressed as

$$(\hat{\boldsymbol{E}}, \acute{\boldsymbol{B}}) = *(\acute{\boldsymbol{H}}, \hat{\boldsymbol{D}}). \tag{31}$$

with Hodge's star operator, "$*$." This relation represents important geometrical relations [4] of electromagnetic fields, which can hardly be suggested by the simple vector relations (26).

Now we have understood that without the help of Gaussian system, we can reach natural systems of units directly from the SI system. We believe it's time to say goodbye to the Gaussian system of units. We will not miss its simpleness if we use the vacuum impedance $Z_0$ as a fundamental constant.

## Acknowledgements

We thank K. Shimoda, F.W. Hehl, M. Ueda, and V. Veselago for helpful discussions. This work is partly supported by the global COE program for "Photonics and electronics science and engineering."

**References**

[1] S.A. Schelkunoff, "The impedance concept and its application to problems of reflection, refraction, shielding, and power absorption," Bell Syst. Tech. J., vol.17, pp.17–48, 1938.
[2] J.C. Maxwell, "A dynamical theory of electromagnetic field," Royal Society Transactions, vol.155, pp.526–597, 1864.
[3] D.R. Smith, J.B. Pendry, and M.C.K. Wiltshire, "Metamaterials and negative refractive index," Science, vol.305, pp.788–792, 2004.
[4] U. Leonhardt, "Optical conformal mapping," Science, vol.312, pp.1777–1780, 2006.
[5] Y. Tamayama, T. Nakanishi, K. Sugiyama, and M. Kitano, "Observation of Brewster's effect for transverse-electric electromagnetic waves in metamaterials: Experiment and theory," Phys. Rev. B, vol.73, pp.193104(4), 2006.
[6] A. Sommerfeld, Electrodynamics — Lectures on Theoretical Physics, pp.45–54, pp.212–222, Academic Press, New York, 1952.
[7] G.A. Deschamps, "Electromagnetics and differential forms," Proc. IEEE, vol.69, pp.676–696, 1981.
[8] T. Frankel, The Geometry of Physics: An Introduction, 2nd ed., pp.118–123, Cambridge, 2004.
[9] F.W. Hehl and Y.N. Obukhov, Foundations of Classical Electrodynamics, Charge, Flux, and Metric, pp.143–162, Birkhäuser, Boston, 2003.
[10] J.J. Sakurai, Modern Quantum Mechanics, Addison-Wesley, Reading, Mass., 1995.
[11] F.W. Hehl, "Dimensions and units in electromagnetism," arXiv: physics/0407022, 2004.
[12] M.J. Duff, L.B. Okun, and G. Vemeziano, "Trialogue on the number of fundamental constants," J. High Energy Phys. PHEP03, 023 2002; arXiv:physics/0110060, 2002.
[13] F. Wilczek, "On absolute units, I: Choices," Phys. Today, vol.58, pp.12–13, 2005.
[14] J.D. Jackson, Classical Electrodynamics, 3rd ed., pp.782–784, See footnotes for Tables 3 and 4 in Appendix, John Wiley & Sons, New York, 1998.
[15] C.W. Misner, K.S. Thorne, and J.A. Wheeler, Gravitation, chapters 3 and 4, W.H. Freeman, New York, 1973.

## Appendix: Tensor Notations

In this appendix, we will explain mathematically the entities of Fig. 1. A more detailed description on the relativistic forms of electromagnetic equations are givin in [15]. With the basis vectors $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, a space-time vector can be represented as

$$(ct, \boldsymbol{x}) = (ct)\boldsymbol{e}_0 + \boldsymbol{x}, \quad \boldsymbol{x} = \sum_{i=1}^{3} x^i \boldsymbol{e}_i. \tag{A· 1}$$

The basis vectors satisfy $\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = g_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$. The nonzero elements are $g_{00} = -1$, $g_{11} = g_{22} = g_{33} = 1$. We also introduce the dual basis vectors: $\{\boldsymbol{e}^0, \boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3\}$, with $\boldsymbol{e}^0 = -\boldsymbol{e}_0$, $\boldsymbol{e}^i = \boldsymbol{e}_i$ $(i = 1, 2, 3)$.

The quantities in electromagnetism are expressed by antisymmetric tensors of rank $n$ ($n$-forms) in the four-dimensional space [7]–[9]. For scalar fields $\alpha$, $\beta$ and 3-dimensional vector fields $\boldsymbol{X}$, $\boldsymbol{Y}$, $n$-forms $(n = 1, 2, 3)$ are defined as

$$\begin{aligned}
(\alpha; \boldsymbol{X})_1 &= \alpha \boldsymbol{e}^0 + \boldsymbol{X}, \\
(\boldsymbol{X}; \boldsymbol{Y})_2 &= \boldsymbol{e}^0 \wedge \boldsymbol{X} + \mathsf{Y}, \\
(\boldsymbol{Y}; \beta)_3 &= \boldsymbol{e}^0 \wedge \mathsf{Y} + \beta\sigma,
\end{aligned} \tag{A· 2}$$

where $\sigma = \boldsymbol{e}^1 \wedge \boldsymbol{e}^2 \wedge \boldsymbol{e}^3$ is the 3-form representing the volume element and "$\wedge$" represents the antisymmetric tensor product. $\mathsf{Y} = \sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} Y_i \boldsymbol{e}_j \wedge \boldsymbol{e}_k$ is a 2-form (in three dimensional space) derived from $\boldsymbol{Y}$, where $\epsilon_{ijk}$ is the Levi-Civita symbol. With these, the pair quantities in Fig. 1 are defined as

$$\begin{aligned}
(\phi, c_0\boldsymbol{A}) &= (-\phi; c_0\boldsymbol{A})_1, \\
(\boldsymbol{E}, c_0\boldsymbol{B}) &= (-\boldsymbol{E}; c_0\boldsymbol{B})_2, \\
(\boldsymbol{H}, c_0\boldsymbol{D}) &= (\boldsymbol{H}; c_0\boldsymbol{D})_2, \\
(\boldsymbol{J}, c_0\varrho) &= (-\boldsymbol{J}; c_0\varrho)_3.
\end{aligned} \tag{A· 3}$$

The differential operator $\mathsf{d}$ is defined as

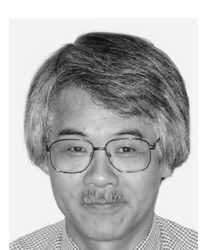
$$\mathsf{d} = \boldsymbol{e}^0 \wedge \frac{\partial}{\partial x^0} + \sum_{i=1}^{3} \boldsymbol{e}^i \wedge \frac{\partial}{\partial x^i}. \tag{A· 4}$$

and the application to an $n$-form results in an $(n+1)$-form. For example, we have $\mathsf{d}(\phi, c_0\boldsymbol{A}) = (\boldsymbol{E}, c_0\boldsymbol{B})$, which is equivalent to $\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \partial\boldsymbol{A}/\partial t$ and $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. The successive applications of $\mathsf{d}$ always yield zero ($\mathsf{dd} = 0$), therefore we have $\mathsf{d}(\boldsymbol{E}, c_0\boldsymbol{B}) = 0$, which corresponds to $\partial\boldsymbol{B}/\partial t + \boldsymbol{\nabla} \times \boldsymbol{E} = 0$ and $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$. Furthermore, $\mathsf{d}(\boldsymbol{H}, c_0\boldsymbol{D}) = (\boldsymbol{J}, c_0\varrho)$ yields $-\partial\boldsymbol{D}/\partial t + \boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \varrho$, and $\mathsf{d}(\boldsymbol{J}, c_0\varrho) = 0$ yields the conservation of charge: $\partial\varrho/\partial t + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$.

Another important notation is Hodge's star operator "$*$," which converts an $n$-form into a $(4-n)$-form;

$$\begin{aligned}
*\,(\alpha; \boldsymbol{X})_1 &= (\boldsymbol{X}; \alpha)_3, \\
*\,(\boldsymbol{X}; \boldsymbol{Y})_2 &= (-\boldsymbol{Y}; \boldsymbol{X})_2, \\
*\,(\boldsymbol{Y}; \beta)_3 &= (\beta; \boldsymbol{Y})_1.
\end{aligned} \tag{A· 5}$$

From the second relation and (A· 3), the constitutive relations are represented as $(\boldsymbol{E}, c_0\boldsymbol{B}) = *Z_0(\boldsymbol{H}, c_0\boldsymbol{D})$.



**Masao Kitano** was born in Kyoto, Japan, on August 5, 1952. He received the B.S., M.S., and Ph.D. degrees in electronic engineering from Kyoto University, Kyoto, Japan, in 1975, 1977, and 1984, respectively. In 1977 he joined the Department of Electronics, Kyoto University and since 1999, he is a Professor at the Department of Electronic Science and Engineering, Kyoto University. He spent the academic years 1984–1986 on leave at the Department of Physics, Princeton University. His research interest includes quantum optics, nearfield optics, optical pumping, electromagnetism. He received the Matsuo Science Prize in 2007. Dr. Kitano is a member of American Physical Society, the Physical Society of Japan, the Laser Society of Japan, and the Japan Society of Applied Physics.