Singularly perturbed spectral problems with Neumann boundary conditions

A. Piatnitski

Narvik University College, Postboks 385, 8505 Narvik, Norway,

P. N. Lebedev Physical Institute of RAS, 53, Leninski pr., Moscow 119991, Russia,

e-mail: andrey@sci.lebedev.ru

A. Rybalko

Simon Kuznets Kharkiv National University of Economics,

9a Lenin ave., Kharkiv 61166, Ukraine,

e-mail: Antonina.Rybalko@m.hneu.edu.ua

V. Rybalko

Mathematical Department, B.Verkin Institute for Low Temperature Physics

and Engineering of the NASU, 47 Lenin ave., Kharkiv 61103, Ukraine,

e-mail: vrybalko@ilt.kharkov.ua

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The paper deals with the Neumann spectral problem for a singularly perturbed second order elliptic operator with bounded lower order terms. The main goal is to provide a refined description of the limit behaviour of the principal eigenvalue and eigenfunction. Using the logarithmic transformation we reduce the studied problem to an additive eigenvalue problem for a singularly perturbed Hamilton-Jacobi equation. Then assuming that the Aubry set of the Hamiltonian consists of a finite number of points or limit cycles situated in the domain or on its boundary, we find the limit of the eigenvalue and formulate the selection criterium that allows us to choose a solution of the limit Hamilton-Jacobi equation which gives the logarithmic asymptotics of the principal eigenfunction.

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1 Introduction

This paper is devoted to the asymptotic analysis of the first eigenpair for singularly perturbed spectral problem, depending on the small parameter $\varepsilon > 0$, for the elliptic equation

$$\varepsilon a_{ij}(x)\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j} + b_i(x)\frac{\partial u_\varepsilon}{\partial x_i} + c(x)u_\varepsilon = \lambda_\varepsilon u \tag{1.1}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ with the boundary condition

$$\frac{\partial u_\varepsilon}{\partial \nu} = 0, \tag{1.2}$$

on $\partial\Omega$, where $\frac{\partial}{\partial \nu}$ denotes derivative with respect to the external normal.

The bottom of the spectrum of elliptic operators plays a crucial role in many applications. In particular, the first eigenvalue and the corresponding eigenfunction of (1.1)–(1.2), are important in understanding the large-time behavior of the underlying non-stationary convection-diffusion model with reflecting boundary. Due to the Krein-Rutman theorem the first eigenvalue λ_ε of (1.1)–(1.2) (the eigenvalue with the maximal real part) is simple and real, the corresponding eigenfunction u_ε can be chosen to satisfy $u_\varepsilon(x) > 0$ in Ω.

The goal of this work is to study the asymptotic behavior of λ_ε and u_ε as $\varepsilon \to 0$. While in the case of constant function $c(x)$ in (1.1) the first eigenpair is (trivially) explicitly found, the asymptotic behavior of the first eigenpair is quite nontrivial when $c(x)$ is a nonconstant function, in particular, the eigenfunction might exhibit an exponential localization.

Boundary-value problems for singularly perturbed elliptic operators have been actively studied starting from 1950s. We mention here a pioneering work [23], where for a wide class of operators (so-called regularly degenerated operators) the asymptotics of solutions were obtained.

In the works [21], [22], [8] (see also [7]) the principal eigenvalue of singularly perturbed convection-diffusion equations with the Dirichlet boundary condition was investigated by means of large deviation techniques for diffusion processes with small diffusion. In [4] the estimates for the principal eigenvalue were obtained by comparison arguments and elliptic techniques.

The case when convection vector field has a finite number of hyperbolic equilibrium points and cycles was studied in [10] where methods of dynamical systems are combined with those of stochastic differential equations. These results were generalized in [5] to the case when the boundary of domain is invariant with respect to convection vector field. Similar problem in the presence of zero order term was considered in [12].

In [16] the viscosity solutions techniques for singularly perturbed Hamilton-Jacobi equation were used in order to study the principal eigenfunction of the adjoint Neumann convection-diffusion problem. The logarithmic asymptotics of the eigenfunction were constructed.

The work [17] deals with the principal eigenpair of operators with a large zero order term on a compact Riemannian manifold. The approach developed in this work is based on large deviation and variational techniques.

Dirichlet spectral problem for singularly perturbed operators with rapidly oscillating locally periodic coefficients was studied in [18] and [19]. In [18] with the help of viscosity solutions method the limit of the principal eigenvalue and the logarithmic asymptotics of the principal eigenfunction were found. These asymptotics were improved in [18] and [19] using the blow up analysis.

In the present work when studying problem (1.1)–(1.2), we make use of the standard viscosity solutions techniques in order to obtain the logarithmic asymptotics of the principal eigenfunction. However, the limit Hamilton-Jacobi equation in general is not uniquely solvable and does not give information about the limit behaviour of λ_ε. Therefore, we have to consider higher order approximations in (1.1)–(1.2). Under rather general assumptions on the structure of the Aubry set of the limit Hamiltonian, we find the limit of λ_ε and can choose the solution of the limit problem which determines the asymptotics of the principal eigenfunction. Notice that we did not succeed to make the blow up analysis work in the case under consideration. In this case, for components of the Aubry set located on the boundary, the natural rescaling still leads to singularly perturbed operators . Instead, we study a refined structure of solutions of the limit Hamilton-Jacobi equation in the vicinity of the Aubry set. This allows us to construct test functions that satisfy the perturbed equation up to higher order.

We also would like to remark that, with obvious modifications, the results of this work as well as the developed techniques remain valid for the boundary condition of the form

$$\frac{\partial u_\varepsilon}{\partial \beta} = 0,$$

where β is a C^2-smooth vector field on $\partial\Omega$ non-tangential at any point of $\partial\Omega$. In particular, conormal vector field $\beta_i = a_{ij}\nu_j$ can be considered.

2 Problem setup and results

We study problem (1.1)–(1.2) under the following assumptions on the operator coefficients and the domain:

(a1) Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a C^2 boundary;

(a2) all the coefficients are C^2-functions in $\overline{\Omega}$;

(a3) the matrix (a_{ij}) is symmetric and uniformly elliptic.

Further assumptions on the vector field b will be formulated later on.

Since $u_\varepsilon > 0$ in Ω we can represent u_ε in the form

$$u_\varepsilon = e^{-W_\varepsilon(x)/\varepsilon},$$

this results in the following nonlinear PDE

$$-a_{ij}(x)\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_j} + \frac{1}{\varepsilon}H(\nabla W_\varepsilon, x) + c(x) = \lambda_\varepsilon \qquad \text{in } \Omega \tag{2.1}$$

or

$$-\varepsilon a_{ij}(x)\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_j} + H(\nabla W_\varepsilon, x) + \varepsilon c(x) = \varepsilon\lambda_\varepsilon \qquad \text{in } \Omega \tag{2.2}$$

with the boundary condition

$$\frac{\partial W_\varepsilon}{\partial \nu} = 0 \qquad \text{on } \partial\Omega, \tag{2.3}$$

where

$$H(p, x) = a_{ij}(x)p_i p_j - b_i(x)p_i \tag{2.4}$$

is a function to be referred to as a Hamiltonian. Passing to the limit as $\varepsilon \to 0$ in (2.2), with the help of the standard approach based on the maximum principle, we can show that W_ε converges uniformly (up to extracting a subsequence) to a viscosity solution $W(x)$ of the Hamilton-Jacobi equation

$$H(\nabla W(x), x) = 0 \qquad \text{in } \Omega \tag{2.5}$$

with the boundary condition

$$\frac{\partial W}{\partial \nu} = 0 \qquad \text{on } \partial\Omega. \tag{2.6}$$

Recall that a function $W \in C(\overline{\Omega})$ is called a viscosity solution of equation (2.5) if for every test function $\Phi \in C^\infty(\overline{\Omega})$ the following holds

- if $W - \Phi$ attains a maximum at a point $\xi \in \Omega$ then $W(\nabla\Phi(\xi), \xi) \leq 0$;
- if $W - \Phi$ attains a minimum at $\xi \in \Omega$ then $W(\nabla\Phi(\xi), \xi) \geq 0$.

The boundary condition (2.6) is understood in the following sense, $\forall \Phi \in C^\infty(\overline{\Omega})$

- if $W - \Phi$ attains a maximum at $\xi \in \partial\Omega$ then $\min\Big\{H(\nabla\Phi(\xi), \xi), \frac{\partial\Phi}{\partial\nu}(\xi)\Big\} \leq 0$;
- if $W - \Phi$ attains a minimum at $\xi \in \partial\Omega$ then $\max\Big\{H(\nabla\Phi(\xi), \xi), \frac{\partial\Phi}{\partial\nu}(\xi)\Big\} \geq 0$.

It is known [9] that every solution of problem (2.5)–(2.6) has the representation

$$W(x) = \inf_{y \in \mathcal{A}_H} \Big\{d_H(x, y) + W(y)\Big\}, \tag{2.7}$$

where $\mathcal{A}_H$ is so-called Aubry set and $d_H(x, y)$ is a distance function. To define $\mathcal{A}_H$ and $d_H(x, y)$ consider solutions of the following Skorohod problem:

$$\begin{cases} \eta(t) \in \overline{\Omega}, t \geq 0 \\ \dot{\eta}(t) + \alpha(t)\nu(\eta(t)) = v(t) \qquad \text{with } \alpha(t) \geq 0 \text{ and } \alpha(t) = 0 \text{ when } \eta(t) \notin \partial\Omega \\ \eta(0) = x, \end{cases} \tag{2.8}$$

where $v \in L^1((0,\infty);\mathbb{R}^N)$ is a given vector field and $x \in \overline{\Omega}$ is a given initial point, while the curve $\eta \in W^{1,1}_{loc}((0,\infty);\mathbb{R}^N)$ and the function $\alpha \in L^1((0,\infty);\mathbb{R}_+)$ are unknowns. Under our standing assumptions on Ω ($\partial\Omega \in C^2$) the Skorohod problem (2.8) has a solution, see [9].

Consider now the Legendre transform $L(v,x) = \sup_{p\in\mathbb{R}^N}(v\cdot p - H(p,x))$ and define the distance function

$$d_H(x,y) = \inf\Big\{\int_0^t L(-v(s),\eta(s))\,\mathrm{d}s,\quad \eta \text{ solves (2.8)},\ \eta(0)=x,\ \eta(t)=y,\ t>0\Big\}. \tag{2.9}$$

Next we recall the variational definition of the Aubry set

$$x \in \mathcal{A}_H \iff \forall\delta>0\quad \inf\Big\{\int_0^t L(-v(s),\eta(s))\,\mathrm{d}s,\quad \eta \text{ solves (2.8)},\ \eta(0)=\eta(t)=x,\ t>\delta\Big\} = 0. \tag{2.10}$$

In this work we assume that the Aubry set has a finite number of connected components

$$\mathcal{A}_H = \bigcup_{\text{finite}} \mathcal{A}_k \textit{ and each } \mathcal{A}_k \textit{ is either an isolated point}$$
$$\textit{or a closed curve lying entirely either in } \Omega \textit{ or on } \partial\Omega. \tag{2.11}$$

Additionally we assume that

$$\textit{if } \mathcal{A}_k \subset \Omega \textit{ then } \mathcal{A}_k \textit{ is either a hyperbolic fixed point}$$
$$\textit{or a hyperbolic limit cycle of the ODE } \dot{x} = b(x); \tag{2.12}$$

$$\textit{if } \mathcal{A}_k \subset \partial\Omega \textit{ then the normal component } b_\nu(x) \textit{ of the field } b(x) \textit{ is strictly positive on } \mathcal{A}_k$$
$$\textit{and } \mathcal{A}_k \textit{ is either a hyperbolic fixed point or a hyperbolic limit cycle of the ODE}$$
$$\dot{x} = b_\tau(x) \textit{ on } \partial\Omega,\ \textit{where } b_\tau(x) \textit{ denotes the tangential component of } b(x) \textit{ on } \partial\Omega. \tag{2.13}$$

Remark 1. *Note that the Aubry set $\mathcal{A}_H$ does not depend on the coefficients $a_{ij}(x)$, it is completely defined by the drift field $b(x)$. This fact follows from the variational definition (2.10) of the Aubry set observing that the Lagrangian $L(v,x)$ is given by $L(v,x) = \frac{1}{4}a^{ij}(x)(v_i+b_i(x))(v_j+b_j(x))$, where $\big(a^{ij}(x)\big)_{i,j=\overline{1,N}}$ is the matrix inverse to $\big(a_{ij}(x)\big)_{i,j=\overline{1,N}}$. More specifically, $\mathcal{A}_H$ is determined by the dynamical system $\mathcal{S}$ corresponding to the Skorohod problem (2.8) with $v(t)=b(\eta(t))$ and conditions (2.11)–(2.13) require that the ω-limit set of $\mathcal{S}$ consists of a finite number of hyperbolic fixed points or limit cycles. One observes that the ω-limit set of $\mathcal{S}$ is always nonempty. Furthermore, in the case of general position the Aubry set $\mathcal{A}_H$ consists of a finite number of hyperbolic fixed points and limit cycles of $\mathcal{S}$.*

In order to state the main result of this work we assign to each component $\mathcal{A}_k$ of $\mathcal{A}_H$ a number $\sigma(\mathcal{A}_k)$ as follows. If $\mathcal{A}_k$ is a fixed point $\{\xi\}$ of the ODE $\dot{x} = b(x)$ and $\xi \in \Omega$, linearizing the ODE near ξ to get $\dot{z} = B(\xi)z$ we define $\sigma(\mathcal{A}_k)$ by

$$\sigma(\mathcal{A}_k) = -\sum_{\theta_i > 0} \theta_i + c(\xi), \tag{2.14}$$

where θ_i are the real parts of eigenvalues of the matrix $B(\xi)$. Note that the hyperbolicity of the fixed point means that the eigenvalues of $B(\xi)$ cannot have zero real part. If $\mathcal{A}_k = \{\xi\}$ and $\xi \in \partial\Omega$, consider the ODE $\dot{x} = b_\tau(x)$ on $\partial\Omega$ in a neighborhood of the point ξ. Passing to the linearized ODE $\dot{z} = B_\tau(\xi)z$ in the tangent plane to $\partial\Omega$ at the point ξ, we denote by $\tilde{\theta}_i$ the real parts of the eigenvalues of $B_\tau(\xi)$ and set

$$\sigma(\mathcal{A}_k) = -\sum_{\tilde{\theta}_i > 0} \tilde{\theta}_i + c(\xi), \tag{2.15}$$

Consider now the case when $\{\mathcal{A}_k\} \subset \Omega$ is a limit cycle of ODE $\dot{x} = b(x)$. Let $P > 0$ be the minimal period of the cycle and let Θ_i be the absolute values of eigenvalues of the linearized Poincaré map. (Recall that the limit cycle is said hyperbolic if there are no eigenvalues of linearized Poincaré map with absolute value equal to 1.) We define now $\sigma(\mathcal{A}_k)$ by setting

$$\sigma(\mathcal{A}_k) = -\frac{1}{P}\sum_{\Theta_i > 1} \log \Theta_i + \frac{1}{P}\int_0^P c(\xi(t))\mathrm{d}t, \tag{2.16}$$

where $\xi(t)$ solves $\dot{\xi} = b(\xi)$ and $\xi(t) \in \mathcal{A}_k$.

Finally, in the case when $b_\nu > 0$ on $\mathcal{A}_k$ and $\mathcal{A}_k$ is a limit cycle of the ODE $\dot{x} = b_\tau(x)$ on $\partial\Omega$, we set

$$\sigma(\mathcal{A}_k) = -\frac{1}{P}\sum_{\widetilde{\Theta}_i > 1} \log \widetilde{\Theta}_i + \frac{1}{P}\int_0^P c(\xi(t))\mathrm{d}t, \tag{2.17}$$

where $\dot{\xi} = b_\tau(\xi)$ and $\xi(t) \in \mathcal{A}_k$, P is the minimal period and $\widetilde{\Theta}_i$ are the absolute values of the eigenvalues of the linearized Poincaré map.

The main result of this work is

Theorem 2. *Let conditions (**a1**)–(**a3**) be fulfilled, and assume that the Aubry set $\mathcal{A}_H$ satisfies (2.11), (2.12) and (2.13). Then the first eigenvalue λ_ε of (1.1) converges as $\varepsilon \to 0$ to*

$$\lim_{\varepsilon \to 0} \lambda_\varepsilon = \max\Big\{\sigma(\mathcal{A}_k); \mathcal{A}_k \subset \mathcal{A}_H\Big\}, \tag{2.18}$$

where $\sigma(\mathcal{A}_k)$ is given by (2.14) or (2.15) if $\mathcal{A}_k$ is a fixed point in Ω or on $\partial\Omega$, and $\sigma(\mathcal{A}_k)$ is defined by (2.16) or (2.17) if $\mathcal{A}_k$ is a limit cycle in Ω or on $\partial\Omega$. Moreover, if the maximum in (2.18) is attained at exactly one component $\mathcal{M} := \mathcal{A}_{k_0}$, then the scaled logarithmic transform $w_\varepsilon = -\varepsilon \log u_\varepsilon$ of the first eigenfunction u_ε (normalized by max $u_\varepsilon = 1$) converges uniformly in $\overline{\Omega}$ to the maximal viscosity solution W of (2.5)-(2.6) vanishing on $\mathcal{M}$, i.e. $W(x) = d_H(x, \mathcal{M})$.

3 Passing to the limit by vanishing viscosity techniques

In this section we pass to the limit, as $\varepsilon \to 0$, in equation (2.2) and boundary condition (2.3) to get problem (2.5)–(2.6). We use the standard technique based on the maximum principle and the a priori uniform $W^{1,\infty}$ bounds for W_ε obtained by Berstein's method (originally proposed in [2], [3] and further developed in, e.g., [13], [20]).

First, considering (2.2) at the maximum and minimum points of $W_\varepsilon(x)$ we easily get

Lemma 3. *The first eigenvalue λ_ε satisfies the estimates* $\min c(x) \le \lambda_\varepsilon \le \max c(x)$.

Next we establish the $W^{1,\infty}$ bound for W_ε in

Lemma 4. *Let u_ε be normalized by $\max u_\varepsilon = 1$ (i.e. $\min W_\varepsilon = 0$). Then $\|W_\varepsilon\|_{W^{1,\infty}(\Omega)} \le C$ with a constant C independent of ε.*

Proof. Following [15] observe that the boundary condition $\frac{\partial W_\varepsilon}{\partial \nu} = 0$ yields the pointwise bound

$$\frac{\partial}{\partial \nu}|\nabla W_\varepsilon|^2 \le C|\nabla W_\varepsilon|^2 \qquad \text{on } \partial\Omega.$$

Therefore, for an appropriate positive function $\phi \in C^2(\overline{\Omega})$,

$$\frac{\partial}{\partial \nu}\Big(\phi|\nabla W_\varepsilon|^2\Big) \le -\Big(\phi|\nabla W_\varepsilon|^2\Big) \qquad \text{on } \partial\Omega. \tag{3.1}$$

Next we use Bernstein's method to obtain a uniform bound for $\omega_\varepsilon(x) = \phi(x)|\nabla W_\varepsilon(x)|^2$, following closely the line of [6], Lemma 1.2. In view of (3.1) either $|\nabla W_\varepsilon| \equiv 0$ and we have nothing to prove, or $\max w_\varepsilon$ is attained at a point $\xi \in \Omega$. In the latter case we have $\nabla \omega_\varepsilon(\xi) = 0$ and

$$a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\Big(\phi|\nabla W_\varepsilon|^2\Big) \le 0 \qquad \text{at } x = \xi.$$

Expanding the left hand side of this inequality we get

$$2\varepsilon\phi a_{ij}\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_k}\frac{\partial^2 W_\varepsilon}{\partial x_j \partial x_k} \le -2\varepsilon\phi a_{ij}\frac{\partial^3 W_\varepsilon}{\partial x_i \partial x_j \partial x_k}\frac{\partial W_\varepsilon}{\partial x_k} - 4\varepsilon a_{ij}\frac{\partial \phi}{\partial x_j}\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_k}\frac{\partial W_\varepsilon}{\partial x_k} - \varepsilon a_{ij}\frac{\partial^2 \phi}{\partial x_i \partial x_j}|\nabla W_\varepsilon|^2. \tag{3.2}$$

Using (2.2) we obtain

$$-\varepsilon\phi a_{ij}\frac{\partial^3 W_\varepsilon}{\partial x_i \partial x_j \partial x_k}\frac{\partial W_\varepsilon}{\partial x_k} \le \varepsilon\phi\frac{\partial a_{ij}}{\partial x_k}\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_j}\frac{\partial W_\varepsilon}{\partial x_k} + C\Big(\omega_\varepsilon^{3/2} + \omega_\varepsilon + \omega_\varepsilon^{1/2} + 1\Big), \qquad \text{at } x = \xi, \tag{3.3}$$

where we have also exploited the fact that $\nabla \omega_\varepsilon(\xi) = 0$. Substitute now (3.3) into (3.2) to derive

$$\varepsilon\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_k}\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_k} \le C\Big(\omega_\varepsilon^{3/2} + 1\Big), \qquad \text{at } x = \xi. \tag{3.4}$$

On the other hand it follows from (2.2) that

$$\omega_\varepsilon \le C\left(\varepsilon\sum\left|\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_j}\right| + 1\right). \tag{3.5}$$

Combining (3.4) and (3.5) we obtain $\omega_\varepsilon \le C$ and the required uniform bound follows. □

With a priori bounds from Lemma 3 and Lemma 4 it is quite standard to pass to the limit in (2.2). Indeed, up to extracting a subsequence, $W_\varepsilon \to W$ uniformly in $\overline{\Omega}$ and $\lambda_\varepsilon \to \lambda$. Consider a test function $\Phi \in C^\infty(\overline{\Omega})$ and assume that $W - \Phi$ attains strict maximum at a point ξ. Then $W_\varepsilon - \Phi$ attains local maximum at ξ_ε such that $\xi_\varepsilon \to \xi$ as $\varepsilon \to 0$. If $\xi_\varepsilon \in \Omega$ then $\nabla W_\varepsilon(\xi_\varepsilon) = \nabla \Phi(\xi_\varepsilon)$ and

$$a_{ij}\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_j}(\xi_\varepsilon) \leq a_{ij}\frac{\partial^2 \Phi}{\partial x_i \partial x_j}(\xi_\varepsilon) \ \text{ if } \xi_\varepsilon \in \Omega$$

and $\frac{\partial \Phi}{\partial \nu}(\xi_\varepsilon) \leq 0$ if $\xi_\varepsilon \in \partial\Omega$. Passing to the limit as $\varepsilon \to 0$ and using (2.2) and Lemma 3 we get

$$H(\nabla\Phi(\xi), \xi) \leq 0 \ \text{ if } \xi \in \Omega, \quad \text{ and } \min\left\{H(\nabla\Phi(\xi), \xi), \frac{\partial \Phi}{\partial \nu}(\xi)\right\} \leq 0 \ \text{ if } \xi \in \partial\Omega.$$

Arguing similarly in the case when ξ is a strict minimum of $W - \Phi$ we conclude that W is a viscosity solution of (2.5)–(2.6).

4 Matching lower and upper bounds for eigenvalues and selection of the solution of (2.5)–(2.6)

Due to the results of Section 3 we can assume, passing to a subsequence if necessary, that eigenvalues λ_ε converge to a finite limit λ and functions W_ε converge uniformly in $\overline{\Omega}$ to a solution W of problem (2.5)–(2.6) as $\varepsilon \to 0$. In the following four steps we prove that λ and $W(x)$ are described by Theorem 2.

Step I: Significant component(s) of $\mathcal{A}_H$. Recall the definition of the partial order relation $\preceq$ on $\mathcal{A}_H$ introduced in [18] as follows

$$\mathcal{A}' \preceq \mathcal{A}'' \iff W(\mathcal{A}'') = d_H(\mathcal{A}'', \mathcal{A}') + W(\mathcal{A}'). \tag{4.1}$$

Note that since W is a solution of (2.5)–(2.6) then it is constant on each connected component of $\mathcal{A}_{\mathcal{H}}$. That is why hereafter we write $W(\mathcal{A}) := W(\xi)$, $\xi \in \mathcal{A}$, for a connected component $\mathcal{A}$ of $\mathcal{A}_{\mathcal{H}}$. Since the distance function $d_H(x, y)$ satisfies the triangle inequality and $d_H(\mathcal{A}'', \mathcal{A}') + d_H(\mathcal{A}', \mathcal{A}'') > 0$ for different components $\mathcal{A}', \mathcal{A}''$ of the Aubry set $\mathcal{A}_H$, (4.1) indeed defines a partial order relation.

Condition (2.11) assumes that there are finitely many different components of the Aubry set. It follows that there exists at least one minimal component $\mathcal{M} := \mathcal{A}_{k_0}$ (such that, $\forall \mathcal{A}_k \neq \mathcal{M}$, either $\mathcal{M} \preceq \mathcal{A}_k$ or $\mathcal{M}$ and $\mathcal{A}_k$ are not comparable).

Now show that

$$W(x) = d_H(x, \mathcal{M}) + W(\mathcal{M}) \text{ in } U \cap \overline{\Omega}, \quad \text{where } U \text{ is a neighborhood of } \mathcal{M}. \tag{4.2}$$

Indeed, otherwise there is a sequence $x_i \to \mathcal{M}$ and a component $\mathcal{A}_k \neq \mathcal{M}$ such that $W(x_i) = d_H(x_i, \mathcal{A}_k) + W(\mathcal{A}_k)$. Then taking the limit we derive $W(\mathcal{M}) = d_H(\mathcal{M}, \mathcal{A}_k) + W(\mathcal{A}_k)$, that is $\mathcal{A}_k \preceq \mathcal{M}$ which contradicts the minimality of $\mathcal{M}$.

In what follows a component $\mathcal{M}$ such that (4.2) is satisfied will be called a significant component. We have shown that under condition (2.11) there is at least one significant component in the Aubry set $\mathcal{A}_H$.

Step II: Upper bound for eigenvalues. The crucial technical result in the proof of Theorem 2 is the following Lemma whose proof is presented in subsequent four Sections dealing separately with four possible cases of the structure of $\mathcal{M}$.

Lemma 5. *Let $\mathcal{M}$ be a significant component of the Aubry set $\mathcal{A}_H$ satisfying either (2.12) or (2.13). Then for sufficiently small $\delta > 0$ there are continuous functions $W^{\pm}_\delta(x)$, $W^{\pm}_{\delta,\varepsilon}(x)$ and neighborhoods U_δ of $\mathcal{M}$ such that*

$$W^{\pm}_\delta(x) = 0 \quad \text{on} \quad \mathcal{M}, \quad \text{and} \quad W^-_\delta(x) < W(x) - W(\mathcal{M}) < W^+_\delta(x) \quad \text{in} \quad U_\delta \cap \overline{\Omega} \setminus \mathcal{M}, \tag{4.3}$$

$W^{\pm}_{\delta,\varepsilon} \in C^2(U_\delta \cap \overline{\Omega})$, $W^{\pm}_{\delta,\varepsilon} \to W^{\pm}_\delta$ uniformly in $U_\delta \cap \overline{\Omega}$ as $\varepsilon \to 0$, and

$$\liminf_{\delta \to 0} \liminf_{\varepsilon \to 0,\, \xi_\varepsilon \to \mathcal{M}} \Big(-a_{ij}(\xi_\varepsilon) \frac{\partial^2 W^+_{\delta,\varepsilon}}{\partial x_i \partial x_j}(\xi_\varepsilon) + \frac{1}{\varepsilon} H(\nabla W^+_{\delta,\varepsilon}(\xi_\varepsilon), \xi_\varepsilon) + c(\xi_\varepsilon) \Big) \geq \sigma(\mathcal{M}). \tag{4.4}$$

$$\limsup_{\delta \to 0} \limsup_{\varepsilon \to 0,\, \xi_\varepsilon \to \mathcal{M}} \Big(-a_{ij}(\xi_\varepsilon) \frac{\partial^2 W^-_{\delta,\varepsilon}}{\partial x_i \partial x_j}(\xi_\varepsilon) + \frac{1}{\varepsilon} H(\nabla W^-_{\delta,\varepsilon}(\xi_\varepsilon), \xi_\varepsilon) + c(\xi_\varepsilon) \Big) \leq \sigma(\mathcal{M}). \tag{4.5}$$

Moreover, if $U_\delta \cap \partial\Omega \neq \emptyset$ then the functions $W^{\pm}_{\delta,\varepsilon}$ also satisfy $\frac{\partial W^+_{\delta,\varepsilon}}{\partial \nu} > 0$ on $U_\delta \cap \partial\Omega$, and $\frac{\partial W^-_{\delta,\varepsilon}}{\partial \nu} < 0$ on $U_\delta \cap \partial\Omega$.

Now, assuming that we know a minimal component $\mathcal{M}$ of the Aubry set $\mathcal{A}_H$, we can identify the limit λ of eigenvalues λ_ε. Consider the difference $W_\varepsilon - W^-_{\delta,\varepsilon}$, where $W^-_{\delta,\varepsilon}$ are test functions described in Lemma 5. By (4.3) the function $W - W^-_\delta - W(\mathcal{M})$ vanishes on $\mathcal{M}$ while it is strictly positive in a punctured neighborhood of $\mathcal{M}$. Then, since $W_\varepsilon - W^-_{\delta,\varepsilon}$ converge uniformly to $W - W^-_\delta$ as $\varepsilon \to 0$ in a neighborhood of $\mathcal{M}$, there exists a sequence of local minima ξ_ε of $W_\varepsilon - W^-_{\delta,\varepsilon}$ such that $\xi_\varepsilon \to \mathcal{M}$. Moreover, if $\mathcal{M} \cap \partial\Omega \neq \emptyset$ then $\frac{\partial W^-_{\delta,\varepsilon}}{\partial \nu} < \frac{\partial W_\varepsilon}{\partial \nu} = 0$ on $\partial\Omega$ (locally near $\mathcal{M}$) and therefore $\xi_\varepsilon \in \Omega$ for sufficiently small ε. For such ε we have

$$\nabla W_\varepsilon = \nabla W^-_{\delta,\varepsilon} \quad \text{and} \quad -a_{ij} \frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_j} \leq -a_{ij} \frac{\partial^2 W^-_{\delta,\varepsilon}}{\partial x_i \partial x_j} \quad \text{at} \quad x = \xi_\varepsilon.$$

Therefore,

$$\begin{aligned} \lambda_\varepsilon &= -a_{ij}(\xi_\varepsilon) \frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_j}(\xi_\varepsilon) + \frac{1}{\varepsilon} H(\nabla W_\varepsilon(\xi_\varepsilon), \xi_\varepsilon) + c(\xi_\varepsilon) \\ &\leq -a_{ij}(\xi_\varepsilon) \frac{\partial^2 W^-_{\delta,\varepsilon}}{\partial x_i \partial x_j}(\xi_\varepsilon) + \frac{1}{\varepsilon} H(\nabla W^-_{\delta,\varepsilon}(\xi_\varepsilon), \xi_\varepsilon) + c(\xi_\varepsilon). \end{aligned}$$

Thus we can use (4.5) here to pass first to the $\limsup$ as $\varepsilon \to 0$ and then as $\delta \to 0$, this yields $\limsup_{\varepsilon\to 0} \lambda_\varepsilon \le \sigma(\mathcal{M})$. Similarly one obtains the matching upper bound so that

$$\lim_{\varepsilon\to 0} \lambda_\varepsilon = \sigma(\mathcal{M}). \tag{4.6}$$

However, since at this point $\mathcal{M}$ is unknown (it depends on W and thus on the particular choice of a subsequence made in the beginning of the Section) equality (4.6) guarantees only the upper bound

$$\limsup_{\varepsilon\to 0} \lambda_\varepsilon \le \max\Big\{\sigma(\mathcal{A}_k); \mathcal{A}_k \subset \mathcal{A}_H\Big\}, \tag{4.7}$$

where the $\limsup_{\varepsilon\to 0}$ is taken over the whole family $\{\lambda_\varepsilon,\, \varepsilon > 0\}$.

Step III: Lower bound for eigenvalues. Consider a component $\mathcal{A}$ of the Aubry set $\mathcal{A}_H$ such that $\sigma(\mathcal{A}) = \max\{\sigma(\mathcal{A}_k);\, \mathcal{A}_k \subset \mathcal{A}_H\}$. Introduce a smooth function $\rho(x)$ such that

$$\rho(x) \ge 0 \text{ in } \overline{\Omega}, \quad \rho(x) = 0 \text{ in a neighborhood of } \ \mathcal{A}, \text{ and } \rho(x) > 0, \text{ when } x \in \mathcal{A}_H \setminus \mathcal{A}$$

and consider the first eigenvalue $\overline{\lambda}_\varepsilon$ of the following auxiliary eigenvalue problem:

$$\varepsilon a_{ij}(x)\frac{\partial^2 \overline{u}_\varepsilon}{\partial x_i \partial x_j} + b_i(x)\frac{\partial \overline{u}_\varepsilon}{\partial x_i} + \Big(c(x) - \frac{1}{\varepsilon}\rho(x)\Big)\overline{u}_\varepsilon = \overline{\lambda}_\varepsilon \overline{u}_\varepsilon \text{ in } \Omega, \tag{4.8}$$

with the Neumann condition $\frac{\partial \overline{u}_\varepsilon}{\partial \nu} = 0$ on $\partial\Omega$. By the Krein-Rutman theorem the eigenvalue $\overline{\lambda}_\varepsilon$ is real and of multiplicity one, and $\overline{u}_\varepsilon$ being normalized by $\max_\Omega \overline{u}_\varepsilon = 1$ satisfies $\overline{u}_\varepsilon > 0$ in Ω. Note that the adjoint problem also has a sign preserving eigenfunction. Then it follows that

$$\overline{\lambda}_\varepsilon \le \lambda_\varepsilon. \tag{4.9}$$

Indeed, otherwise we have

$$\varepsilon a_{ij}(x)\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j} + b_i(x)\frac{\partial u_\varepsilon}{\partial x_i} + \Big(c(x) - \frac{1}{\varepsilon}\rho(x)\Big)u_\varepsilon - \overline{\lambda}_\varepsilon u_\varepsilon = -\Big(\overline{\lambda}_\varepsilon - \lambda_\varepsilon + \frac{1}{\varepsilon}\rho(x)\Big)u_\varepsilon < 0 \quad \text{in } \Omega. \tag{4.10}$$

On the other hand, by Fredholm's theorem the right hand side in (4.10) must be orthogonal (in $L^2(\Omega)$) to any eigenfunction of the problem adjoint to (4.8). Since the latter problem has a sign preserving eigenfunction we arrive at a contradiction which proves (4.9).

Let $\overline{W}_\varepsilon := -\varepsilon \log \overline{u}_\varepsilon$ be the scaled logarithmic transform of $\overline{u}_\varepsilon$, i.e. $\overline{u}_\varepsilon = e^{-\overline{W}_\varepsilon/\varepsilon}$. Following the line of Section 3 one can show that, up to extracting a subsequence, functions $\overline{W}_\varepsilon$ converge (uniformly in $\overline{\Omega}$) to a viscosity solution $\overline{W}$of the problem

$$H(\nabla \overline{W}(x), x) - \rho(x) = \Lambda \quad \text{in } \Omega \tag{4.11}$$

with the boundary condition $\frac{\partial \overline{W}}{\partial \nu} = 0$, where $\Lambda = \lim_{\varepsilon\to 0} \varepsilon\overline{\lambda}_\varepsilon$. Note that the argument in Lemma 3 yields now bounds of the form $-\frac{C}{\varepsilon} \le \overline{\lambda}_\varepsilon \le C$ with some $C > 0$ independent of ε. Nevertheless these bounds are sufficient to derive problem (4.11) with the Neumann boundary condition. Moreover, since $\rho = 0$ in a neighborhood of $\mathcal{A}$ one can show that $\Lambda = 0$ using testing curves η from (2.10) in the variational representation for the additive eigenvalue Λ (see [9]),

$$\Lambda = -\lim_{T\to\infty} \inf\Big\{\frac{1}{T}\int_0^T \big(L(-v,\eta)+\rho(\eta)\big)\,\mathrm{d}t;\ \eta \text{ solves (2.8) with } \eta(0) = x \in \overline{\Omega}\Big\}.$$

This implies, in particular, that

$$\overline{W}(x) = d_H(x,\mathcal{A}) \text{ in a neighborhood of } \mathcal{A},$$

where $d_H(x,y)$ is the distance function given by (2.9). Then arguing as in second step we obtain

$$\overline{\lambda}_\varepsilon \to \sigma(\mathcal{A}).$$

Thanks to (4.9) this yields the lower bound $\liminf \lambda_\varepsilon \ge \max\big\{\sigma(\mathcal{A}_k);\ \mathcal{A}_k \in \mathcal{A}_H\big\}$ complementary to (4.7). Thus formula (2.18) is proved.

Step IV: Selection of the solution of (2.5)-(2.6). Let us assume now that the maximum in (2.18) is attained at exactly one component $\mathcal{M}$. Then comparing (2.18) with (4.6) we see that $\mathcal{M}$ is the unique significant component in $\mathcal{A}_H$, therefore it is the only minimal component of $\mathcal{A}_H$ with respect to the order relation $\preceq$. Thus $\mathcal{M}$ is the least component of $\mathcal{A}_H$. It follows that $W(\mathcal{A}_k) - W(\mathcal{M}) = d_H(\mathcal{A}_k,\mathcal{M})$ for every $\mathcal{A}_k \subset \mathcal{A}_H$. Then by (2.7) the representation $W(x) = d_H(x,\mathcal{M}) + W(\mathcal{M})$ holds in $\overline{\Omega}$. Finally, since $\min_{\overline{\Omega}} W(x) = \lim_{\varepsilon\to 0}\min_{\overline{\Omega}} W_\varepsilon(x) = 0$ we have $W(\mathcal{M}) = 0$, i.e. $W(x) = d_H(x,\mathcal{M})$.

Theorem 2 is proved. $\square$

5 Construction of test functions: case of fixed points in Ω

The central part in the proof of Theorem 2 is the construction of test functions satisfying the conditions of Lemma 5 for different types of components of the Aubry set $\mathcal{A}_H$. Consider first the case when a fixed point $\xi \in \Omega$ of the ODE $\dot{x} = b(x)$ is a significant component of $\mathcal{A}_H$. We can assume that $W(\xi) = 0$, subtracting an appropriate constant if necessary. Then $W(x)$ is given by

$$W(x) = d_H(x,\xi) \quad \text{in a neighborhood } U(\xi) \text{ of } \xi. \tag{5.1}$$

We begin by studying the local behavior of $W(x)$ near ξ. Consider for sufficiently small z the ansatz

$$W(z+\xi) = \Gamma_{ij} z_i z_j + o(|z|^2) \tag{5.2}$$

with a symmetric $N \times N$ matrix Γ. After substituting (5.2) into (2.5) and collecting quadratic terms in the resulting relation we are led to the Riccati matrix equation

$$4\Gamma Q \Gamma - \Gamma B - B^* \Gamma = 0, \tag{5.3}$$

where $Q = \Big(a_{ij}(\xi)\Big)_{i,j=\overline{1,N}}$, $B = \Big(\frac{\partial b_i}{\partial x_j}(\xi)\Big)_{i,j=\overline{1,N}}$.

Next we show that (5.2) holds with Γ being the maximal symmetric solution of (5.3); for existence of such a solution see, e.g., [14] or [1] . To this end consider the solution D of the Lyapunov matrix equation

$$D(4\Gamma Q - B) + (4\Gamma Q - B)^* D = 2I \tag{5.4}$$

given by

$$D = 2\int_{-\infty}^{0} e^{(4\Gamma Q - B)^* t} e^{(4Q\Gamma - B)t} \mathrm{d}t. \tag{5.5}$$

By Theorem 9.1.3 in [14] all the eigenvalues of the matrix $4Q\Gamma - B$ have positive real parts, so that the integral in (5.5) does converge. Set

$$\Gamma_\delta^\pm = \Gamma \pm \delta D.$$

Then Γ_δ^- satisfies

$$4\Gamma_\delta^- Q \Gamma_\delta^- - \Gamma_\delta^- B - B^* \Gamma_\delta^- \leq -\delta I \tag{5.6}$$

for sufficiently small $\delta > 0$.

Introduce the quadratic function $W_\delta^-(x) := \Gamma_\delta^-(x-\xi)\cdot(x-\xi)$. Thanks to (5.6) this function satisfies

$$H(\nabla W_\delta^-(x), x) \leq -\frac{\delta}{2}|x-\xi|^2 \quad \text{in a neighborhood of } \xi. \tag{5.7}$$

This yields the following result whose proof is identical to the proof of Lemma 16 in [18] (see also the arguments in the proof of Lemma 8 below).

Lemma 6. *The strict pointwise inequality $W_\delta^-(x) < W(x)$ holds in a punctured neighborhood of ξ for sufficiently small $\delta > 0$.*

Next consider the function $W_\delta^+(x) := \Gamma_\delta^+(x-\xi)\cdot(x-\xi)$.

Lemma 7. *The strict pointwise inequality $W_\delta^+(x) > W(x)$ holds in a punctured neighborhood of ξ for sufficiently small $\delta > 0$.*

Proof. According to (5.1), the following inequality holds

$$W(x) \leq \int_0^t L(-v(\tau), \xi + \eta(\tau))\mathrm{d}\tau$$

for every control $v(\tau)$ such that the solution of the ODE

$$\dot\eta(\tau) = v(\tau), \quad \eta(0) = z := x - \xi$$

vanishes at the final time t and remains in a small neighborhood of 0 for any $0 \leq \tau \leq t$. We can take the final time $t = +\infty$ and construct $v(\tau)$ by setting $v(\tau) = -(4Q\Gamma - B)\eta(\tau)$, where $\eta(\tau)$ in the solution of the ODE

$$\dot\eta = -(4Q\Gamma - B)\eta, \quad \eta(0) = z.$$

As already mentioned (see Theorem 9.1.3. in [14]) all the eigenvalues of the matrix $4Q\Gamma - B$ have positive real parts, therefore $|\eta(\tau)| \le C|z|$ and $\eta(\tau) \to 0$ as $\tau \to +\infty$. Moreover, the latter convergence is exponentially fast.

Thus we have

$$L(-v(\tau), \eta(\tau) + \xi) = \frac{1}{4} a^{ij}(\xi + \eta)(-\dot{\eta}_i + b_i(\eta))(-\dot{\eta}_i + b_i(\eta)) = \\ = \frac{1}{4} a^{ij}(\xi)(-\dot{\eta}_i + B_{ik}\eta_k)(-\dot{\eta}_j + B_{jl}\eta_l) + O(|\eta|^3),$$

where $\big(a^{ij}(x)\big)_{i,j=\overline{1,N}}$ denotes the matrix inverse to $\big(a_{ij}(x)\big)_{i,j=\overline{1,N}}$. Next recall that $Q_{ij} = a_{ij}(\xi)$ and that Γ solves (5.3). Taking this into account we obtain

$$\int_0^\infty L(\eta + \xi, -v(\tau)) = \frac{1}{4} \int_0^\infty a^{ij}(\xi)(-\dot{\eta}_i + B_{ik}\eta_k)(-\dot{\eta}_j + B_{jl}\eta_l) + O(|z|^3) = \\ = -2 \int_0^\infty \Gamma\eta \cdot \dot{\eta} \mathrm{d}\tau + \int_0^\infty \Gamma\eta \cdot (\dot{\eta} + B\eta)\mathrm{d}\tau + O(|z|^3) = \\ = \Gamma z \cdot z + \int_0^\infty \eta \cdot (-4\Gamma Q\Gamma + \Gamma B + B^*\Gamma)\eta \mathrm{d}\tau + O(|z|^3) = \\ = \Gamma_{ij} z_i z_j + O(|z|^3).$$

Finally, since $\Gamma^+_\delta = \Gamma + \delta D$ with $D > 0$, then for sufficiently small $z \ne 0$ we have

$$W(z + \xi) \le \Gamma z \cdot z + O(|z|^3) < \Gamma^+_\delta z \cdot z.$$

□

Lemmas 6 and 7 show that functions $W^\pm_\delta$ do satisfy conditions of Lemma 5. To complete the proof of Lemma 5 in the case of $\mathcal{M}$ being a fixed point in Ω we define functions $W^\pm_{\delta,\varepsilon}$ simply by setting $W^\pm_{\delta,\varepsilon} := W^\pm_\delta$. Thanks to (5.7) we have

$$\limsup_{\varepsilon \to 0} \Big(a_{ij}(\xi_\varepsilon) \frac{\partial^2 W^-_{\delta,\varepsilon}}{\partial x_i \partial x_j}(\xi_\varepsilon) + \frac{1}{\varepsilon} H(\nabla W^-_{\delta,\varepsilon}(\xi_\varepsilon), \xi_\varepsilon) + c(\xi_\varepsilon) \Big) \le -2a_{ij}(\xi)(\Gamma_{ij} - \delta D_{ij}) + c(\xi), \quad (5.8)$$

as soon as $\xi_\varepsilon \to \xi$ when $\varepsilon \to 0$. According to Proposition 20 in [18], $-2a_{ij}(\xi)\Gamma_{ij} + c(\xi) = \sigma(\{\xi\})$, thus (5.8) yields (4.5). Similarly one verifies that $W^+_{\delta,\varepsilon}$ satisfies (4.4).

6 Construction of test functions: case of fixed points on $\partial\Omega$

Consider now the case of significant component of the Aubry set $\mathcal{A}_H$ being a hyperbolic fixed point ξ of the ODE $\dot{x} = b_\tau(x)$ on $\partial\Omega$, where $b_\tau(x)$ denotes the tangential component of the vector field $b(x)$ on $\partial\Omega$. As above, without loss of generality, we assume that $W(\xi) = 0$.

It is convenient to introduce local coordinates near $\partial\Omega$ so that $x = X(z_1, \dots, z_N)$ with $z_N = z_N(x)$ being the distance from x to $\partial\Omega$ ($z_N(x) > 0$ if $x \in \Omega$) and $z' = (z_1, ..., z_{N-1})$

representing coordinates on $\partial\Omega$ in a neighborhood of the point ξ. The latter coordinates are chosen so that the map $X(z', z_N)$ is C^2-smooth and $z'(\xi) = 0$. Moreover, the matrix $\left(\frac{\partial X_i}{\partial z_j}\right)_{i,j=\overline{1,N}}$ is orthogonal when $z' = 0$ and $z_N = 0$ (at the point ξ). In these new variables equations (2.5) and (2.2) read

$$S(\nabla_z W, z) = 0 \tag{6.1}$$

and

$$-\varepsilon a_{ij}\left(X(z)\right)\mathcal{T}_{ki}^{-1}(z)\frac{\partial}{\partial z_k}\left(\mathcal{T}_{lj}^{-1}(z)\frac{\partial W_\varepsilon}{\partial z_l}\right) + S(\nabla_z W_\varepsilon, z) = \varepsilon\big(\lambda_\varepsilon - c(X(z))\big), \tag{6.2}$$

where

$$S(p, z) = a_{ij}(X(z))\mathcal{T}_{ki}^{-1}(z)\mathcal{T}_{lj}^{-1}(z)p_k p_l - b_i(X(z))\mathcal{T}_{ki}^{-1}(z)p_k$$

and $\left(\mathcal{T}_{ij}^{-1}(z)\right)_{i,j=\overline{1,N}}$ is the inverse matrix to $\left(\frac{\partial X_i}{\partial z_j}(z)\right)_{i,j=\overline{1,N}}$. Note that according to hypothesis (2.13)

$$b_i(X(z))\mathcal{T}_{Ni}^{-1}(z) < 0 \ \text{ for sufficiently small } |z|. \tag{6.3}$$

Like in Section 5 we construct the leading term of the asymptotic expansion of W near the fixed point ξ in the form of a quadratic function. Taking into account the boundary condition $\frac{\partial W}{\partial z_N} = 0$ (that is (2.6) rewritten in aforementioned local coordinates) we write down the following ansatz

$$W(X(z', z_N)) = \tilde{\Gamma}_{ij} z_i' z_j' + o(|z|^2 + z_N^2).$$

with a symmetric $(N-1)\times(N-1)$ matrix $\widetilde{\Gamma}$ satisfying the Riccati equation

$$4\tilde{\Gamma}\tilde{Q}\tilde{\Gamma} - \tilde{\Gamma}\tilde{B} - \tilde{B}^*\tilde{\Gamma} = 0, \tag{6.4}$$

where $\tilde{Q} = \left(a_{ij}(\xi)\mathcal{T}_{ki}^{-1}(0)\mathcal{T}_{lj}^{-1}(0)\right)_{k,l=\overline{1,N-1}}$ and $\tilde{B} = \left(\mathcal{T}_{ki}^{-1}(0)\frac{\partial b_i}{\partial x_j}(\xi)\frac{\partial X_j}{\partial z_l}(0)\right)_{k,l=\overline{1,N-1}}$. Note that $\tilde{B}$ is nothing but the matrix in the ODE $\dot{z}' = \tilde{B}z'$ obtained by linearizing the ODE $\dot{x} = b_\tau(x)$ near ξ in the local coordinates $z' = (z_1', \ldots, z_{N-1}')$ on $\partial\Omega$.

Let $\tilde{\Gamma}$ be the maximal symmetric solution of (6.4), and let $\tilde{D}$ be a solution of the Lyapunov matrix equation

$$\tilde{D}(4\tilde{\Gamma}\tilde{Q} - \tilde{B}) + (4\tilde{\Gamma}\tilde{Q} - \tilde{B})^*\tilde{D} = 2I. \tag{6.5}$$

By Theorem 9.1.3 in [14] all the eigenvalues of the matrix $4\tilde{\Gamma}\tilde{Q} - \tilde{B}$ have positive real parts, therefore (6.5) has the unique solution $\tilde{D}$ given by

$$\tilde{D} = 2\int_{-\infty}^{0} e^{(4\tilde{\Gamma}\tilde{Q}-\tilde{B})^* t} e^{(4\tilde{\Gamma}\tilde{Q}-\tilde{B})t}\mathrm{d}t,$$

which is a symmetric positive definite matrix. Now introduce functions

$$W_\delta^{\pm}(z', z_N) = (\tilde{\Gamma} \pm \delta\tilde{D})_{ij} z_i' z_j' \pm \delta z_N^2 \tag{6.6}$$

depending on the parameter $\delta > 0$.

Lemma 8. *Let $\delta > 0$ be sufficiently small. Then, for small $|z| \neq 0$ such that $X(z) \in \overline{\Omega}$, we have*

$$W_\delta^-(z) < W(X(z)) < W_\delta^+(z). \tag{6.7}$$

Proof. By virtue of the definition of $W_\delta^\pm$ it suffices to prove (6.7) with non strict inequalities in place of strict ones an then pass to slightly bigger δ.

The proof of the inequality $W_\delta^- \leq W$ is based on the following two facts. First, we use the fact that $W(x) = d_H(x,\xi)$ in a neighborhood of ξ. Moreover, for a given $\delta' > 0$ there exists $\delta > 0$ such that if $|x - \xi| < \delta$ then the minimization in (2.9) is actually restricted to testing curves $\eta(\tau)$ which do not leave the set $\{|\eta - \xi| < \delta'\}$ (otherwise arguing as in [18, Lemma 19] one can show that ξ is not an isolated point of the Aubry set $\mathcal{A}_H$, contradicting (2.11)). Second, considering, with a little abuse of notation, $W_\delta^-(x) = W_\delta^-(X^{-1}(x))$ we have for sufficiently small $\delta > 0$

$$H(\nabla W_\delta^-, x) \leq -\delta|x-\xi|^2 \text{ in } \Omega, \text{ and } \frac{\partial W_\delta^-}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{6.8}$$

when $|x - \xi| < \delta'$ with some $\delta' > 0$ independent of δ. This follows from the construction (6.6) of $W_\delta^\pm$ and (6.4), (6.5), also taking into account (6.3).

Assume that $|x - \xi| < \delta$, and let $\eta(\tau)$ be a solution of (2.8) satisfying $\eta(0) = x$, $\eta(t) = \xi$ with a control $v(\tau)$ such that $|\eta(\tau) - \xi| < \delta'$ for all $0 \leq \tau \leq t$. Then

$$W_\delta^-(x) = -\int_0^t \nabla W_\delta^-(\eta) \cdot \dot\eta \, \mathrm{d}\tau = \int_0^t \nabla W_\delta^-(\eta) \cdot (-v(\tau)) \, \mathrm{d}\tau,$$

where we have used the fact that $\frac{\partial W_\delta^-}{\partial \nu} = 0$ on $\partial\Omega$. It follows by Fenchel's inequality $p \cdot (-v) \leq L(-v,\eta) + H(p,\eta)$ that

$$W_\delta^-(x) \leq \int_0^t L(-v,\eta)\, \mathrm{d}\tau + \int_0^t H(\nabla W_\delta^-, \eta) \mathrm{d}\tau \leq \int_0^t L(-v,\eta)\, \mathrm{d}\tau.$$

Therefore by (2.9) we obtain $W_\delta^-(x) \leq W(x)$.

In order to prove the second inequality in (6.7) for a given $x = X(z', z_N)$ we construct a test curve $\eta(\tau)$ first on a small interval $(0, \Delta t)$ by setting $\eta(\tau) = X(z', \zeta_N(\tau))$, $\zeta_N(\tau)$ being the solution of ODE $\dot\zeta_N(\tau) = b_i(X(z',\zeta_N))\mathcal{T}_{Ni}^{-1}(z',\zeta_N)$ with the initial condition $\zeta_N(0) = z_N$, and choosing Δt from the conditions $\zeta_N(\Delta t) = 0$, $\zeta_N(\tau) > 0$ for $\tau < \Delta t$. Thanks to (6.3) we have $\Delta t = O(z_N)$. Then, since

$$\begin{aligned}\dot\eta_i &= \frac{\partial X_i}{\partial z_N}(z',\zeta_N) b_k(\eta) \mathcal{T}_{Nk}^{-1}(z',\zeta_N) \\ &= \frac{\partial X_i}{\partial z_j}(z',\zeta_N) b_k(\eta) \mathcal{T}_{jk}^{-1}(z',\zeta_N) - \frac{\partial X_i}{\partial z_j'}(z',\zeta_N) b_k(\eta) \mathcal{T}_{jk}^{-1}(z',\zeta_N) = b_i(\eta) + O(|z|)\end{aligned}$$

(recall that the tangential component b_τ on $\partial\Omega$ vanishes at the point ξ) we obtain

$$\int_0^{\Delta t} L(-\dot\eta, \eta) \mathrm{d}\tau = O(|z|^3). \tag{6.9}$$

Next we construct $\eta(\tau)$ for $\tau > \Delta t$ which connects point $X(z',0)$ to ξ. Following closely the line of Lemma 7 we introduce $\zeta'(\tau)$ by solving the equation $\dot{\zeta}'(\tau) = -(4\tilde{Q}\tilde{\Gamma} - \tilde{B})\zeta'$ with the initial condition $\zeta'(\Delta t) = z'$ and set $\eta(\tau) = X(\zeta'(\tau), 0)$. Then $\eta(\tau)$ solves (2.8) for $\tau > \Delta t$ with

$$v(\tau) := \dot{\eta}(\tau) + \nu(\eta) b_\nu(\eta) - 4\frac{\partial X}{\partial z_N}(\zeta', 0)\mathcal{T}^{-1}_{Ni}(0) a_{ij}(\xi)\mathcal{T}^{-1}_{lj}(0)\tilde{\Gamma}_{lm}\zeta'_m$$

(note that $\frac{\partial X}{\partial z_N}(\zeta', 0) = -\nu(\eta)$ and $b_\nu(\eta) + 4\mathcal{T}^{-1}_{Ni}(0)a_{ij}(\xi)\mathcal{T}^{-1}_{lj}(0)\Gamma_{lm}\zeta'_m > 0$ as soon as $|z'|$ is sufficiently small) and using (6.4) we obtain

$$\begin{aligned}
\int_{\Delta t}^{\infty} L(-v(\tau), \eta(\tau))\, \mathrm{d}\tau &= \frac{1}{4}\int_{\Delta t}^{\infty} a^{ij}(\xi)(-v_i + b_i)(-v_j + b_j)\, \mathrm{d}\tau + O(|z|^3) \\
&= 4\int_{\Delta t}^{\infty} a^{ij}(\xi)\Big(\frac{\partial X_i}{\partial z'_k}(0)\tilde{Q}_{kl}\tilde{\Gamma}_{lm}\zeta'_m + \frac{\partial X_i}{\partial z_N}(0)\mathcal{T}^{-1}_{Nk}(0)a_{kl}(\xi)\mathcal{T}^{-1}_{ml}(0)\tilde{\Gamma}_{mn}\zeta'_n\Big) \\
&\qquad \times \Big(\frac{\partial X_j}{\partial z'_k}(0)\tilde{Q}_{kl}\tilde{\Gamma}_{lm}\zeta'_m + \frac{\partial X_j}{\partial z_N}(0)\mathcal{T}^{-1}_{Nk}(0)a_{kl}(\xi)\mathcal{T}^{-1}_{ml}(0)\tilde{\Gamma}_{mn}\zeta'_n\Big)\, \mathrm{d}\tau + O(|z|^3) \\
&= 4\int_{\Delta t}^{\infty} \tilde{\Gamma}\zeta' \cdot \tilde{Q}\tilde{\Gamma}\zeta'\, \mathrm{d}\tau + O(|z|^3) \\
&= -2\int_{\Delta t}^{\infty} \tilde{\Gamma}\zeta' \cdot \dot{\zeta}'\, \mathrm{d}\tau + \int_{\Delta t}^{\infty} \tilde{\Gamma}\zeta' \cdot \big(\dot{\zeta}' + \tilde{B}\zeta'\big)\, \mathrm{d}\tau + O(|z|^3) = \tilde{\Gamma}_{ij}z'_i z'_j + O(|z|^3).
\end{aligned} \tag{6.10}$$

The required upper bound $W \le W^+_\delta$ now follows from (6.9) and (6.10). □

Thus functions $W^\pm_\delta$ satisfy conditions of Lemma 5, moreover it follows from (6.6) in conjunction with (6.4), (6.5), taking also into account (6.3), that

$$S(\nabla_z W^+_\delta(z), z) \ge 0 \quad \text{and} \quad S(\nabla_z W^-_\delta(z), z) \le 0 \quad \text{when } |z| \text{ is sufficiently small.}$$

Then we set

$$W^\pm_{\delta,\varepsilon}(z', z_N) = W^\pm_\delta(z', z_N) \mp \varepsilon^2 z_N,$$

and verify (similarly to the case of interior fixed points) that conditions (4.4) and (4.5) are satisfied. Additionally we have $\mp\frac{\partial W^\pm_{\delta,\varepsilon}}{\partial z_N}(z', 0) > 0$, i.e. $\pm\frac{\partial W^\pm_{\delta,\varepsilon}}{\partial \nu} > 0$ on $\partial\Omega$.

7 Construction of test functions: case of limit cycles in Ω

We proceed with the case when a significant component of the Aubry set $\mathcal{A}_H$ is a limit cycle, assuming first that it is situated entirely inside Ω. Namely, let $\xi(t)$ be a periodic solution of the ODE $\dot{\xi} = b(\xi)$ whose minimal period is $P > 0$. We assume that $\mathcal{C} = \{\xi(t) : t \in [0, P)\} \subset \Omega$, $b(x) \ne 0$ on $\mathcal{C}$ and $\mathcal{C}$ is a hyperbolic limit cycle, i.e. the linearized Poincaré map associated to this cycle has no eigenvalues on the unit circle. In order to study the local behavior of W near the cycle $\mathcal{C}$, perform a C^2-smooth change of coordinates $x = X(z_1, \dots, z_{N-1}, z_N)$ with z_N

representing the arc length along the cycle and $z' = (z_1, \dots, z_{N-1})$ being some fixed Cartesian coordinates in the hyperplanes orthogonal to the cycle. Also we assume that $\mathcal{C}$ is oriented by the tangent vector $b(\xi)/|b(\xi)|$, and $z' = 0$ on $\mathcal{C}$. With this change of coordinates equations (2.5) and (2.2) take the form similar to (6.1) and (6.2). Assuming as above that $W(\mathcal{C}) = 0$, we postulate in the vicinity of the cycle (for sufficiently small $|z'|$) the following ansatz for W:

$$W(X(z', z_N)) = \overline{\Gamma}_{ij}(t) z_i' z_j' + o(|z'|^2), \tag{7.1}$$

where t refers to the parametrization of the cycle determined by the equation $\dot{\xi} = b(\xi)$, $t \in [0, P)$. Substitute W in (6.1) to find, after collecting quadratic terms and neglecting higher order terms,

$$\dot{\overline{\Gamma}} = 4\overline{\Gamma}\,\overline{Q}\,\overline{\Gamma} - \overline{\Gamma}\,\overline{B} - \overline{B}^*\overline{\Gamma}, \tag{7.2}$$

where $\overline{Q}(t)$ and $\overline{B}(t)$ are P-periodic $(N-1) \times (N-1)$ matrices whose entries are given by

$$\overline{Q}_{kl}(t) = a_{ij}(\xi(t))\, \mathcal{T}_{ki}^{-1}\big(0, z_N(\xi(t))\big)\, \mathcal{T}_{lj}^{-1}\big(0, z_N(\xi(t))\big), \tag{7.3}$$

$$\overline{B}_{kl}(t) = \mathcal{T}_{ki}^{-1}\big(0, z_N(\xi(t))\big) \frac{\partial b_i}{\partial x_j}\big(\xi(t)\big) \frac{\partial X_j}{\partial z_l}\big(0, z_N(\xi(t))\big) - \mathcal{T}_{kj}^{-1}\big(0, z_N(\xi(t))\big) \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial X_j}{\partial z_l}\big(0, z_N(\xi(t))\big)\right). \tag{7.4}$$

Recall that $\mathcal{T}_{ij}^{-1}\big(z', z_N\big)$ denote the entries of the matrix inverse to $\big(\frac{\partial X_i}{\partial z_j}(z', z_N)\big)_{i,j=\overline{1,N}}$ and for brevity, abusing slightly the notation, we set

$$\mathcal{T}_{ij}^{-1}(t) := \mathcal{T}_{ij}^{-1}(0, z_N(\xi(t)), \quad \mathcal{T}_{ij}(t) = \frac{\partial X_i}{\partial z_j}\big(0, z_N(\xi(t))\big). \tag{7.5}$$

The matrix $\overline{Q}(t)$ being positive definite, it is known [1] that Riccati equation (7.2) has a maximal symmetric P-periodic solution $\overline{\Gamma}(t)$. We next show that (7.1) does hold with the mentioned maximal solution $\overline{\Gamma}(t)$ under our standing hyperbolicity assumption on $\mathcal{C}$. Note that the ODE $\dot{z}' = \overline{B}z'$ corresponds to the linearization of $\dot{x} = b(x)$ on the cycle $\mathcal{C}$ written in local coordinates; thus assuming the hyperbolicity of $\mathcal{C}$ we require that the fundamental solution of the ODE

$$\frac{\partial \overline{\Phi}}{\partial t}(t, \tau) = \overline{B}(t)\overline{\Phi}(t, \tau), \qquad \overline{\Phi}(\tau, \tau) = I,$$

evaluated at $t = \tau + P$, has no eigenvalues with absolute value equal to 1.

Lemma 9. *The following bound holds uniformly in $t \in [0, P)$ for sufficiently small $|z'|$,*

$$W(X(z', z_N(\xi(t))) \le \overline{\Gamma}_{ij}(t) z_i' z_j' + C|z'|^3 \log \frac{1}{|z'|}. \tag{7.6}$$

Proof. We make use of variational representation (2.9). A natural guess about optimal test curve in (2.9) is that its first $N-1$ local coordinates are given by (cf. Sections 5 and 6)

$$\dot{\zeta}'(\tau) = (\overline{B} - 4\overline{Q}\,\overline{\Gamma})\zeta'(\tau), \quad \tau > t, \quad \zeta'(t) = z'. \tag{7.7}$$

Thanks to Theorem 5.4.15 in [1] the solutions of (7.7) are exponentially stable, i. e. $|\zeta'(\tau)| \le Ce^{-\delta\tau}|z'|$ for some $\delta > 0$. The choice of the last local coordinate is a little bit involved. We set $\eta(\tau) := X(\zeta'(\tau), z_N(\xi(\tau)) + \zeta_N(\tau))$ and want to choose $\zeta_N(\tau)$ in such a way that $|\zeta_N(\tau)| < C|z'|$, and

$$b_i(\eta(\tau)) - \dot\eta_i(\tau) = 4a_{ij}(\xi(\tau))\mathcal{T}^{-1}_{lj}\big(0, \xi(\tau)\big)\overline{\Gamma}_{lm}(\tau)\zeta'_m(\tau) + O(|z'|^2). \tag{7.8}$$

We skip for a moment the proof of the existence of such ζ_N. It will be given later on.

Considering (7.8) we obtain

$$\begin{aligned}\int_t^T L(-\dot\eta, \eta)\mathrm{d}\tau &= \frac{1}{4}\int_t^T a^{ij}(\eta(\tau))(-\dot\eta_i(\tau) + b_i(\eta(\tau)))(-\dot\eta_j(\tau) + b_j(\eta(\tau)))\mathrm{d}\tau \\ &\le 4\int_t^T (\overline{\Gamma}(\tau)\zeta'(\tau)) \cdot (\overline{Q}(\tau)\overline{\Gamma}(\tau)\zeta'(\tau))\mathrm{d}\tau + CT|z'|^3\end{aligned}$$

In view of (7.7) and (7.2) we have

$$\begin{aligned}\int_t^T L(-\dot\eta, \eta)\mathrm{d}\tau &\le \int_t^T (\overline{\Gamma}\zeta') \cdot (\overline{B}\zeta' - \dot\zeta')\mathrm{d}\tau + CT|z'|^3 \\ &= -\int_t^T \frac{\mathrm{d}}{\mathrm{d}\tau}(\overline{\Gamma}\zeta' \cdot \zeta')\mathrm{d}\tau + \int_t^T (\overline{\Gamma}\zeta' \cdot (\overline{B}\zeta' + \dot\zeta') + \dot{\overline{\Gamma}}\zeta' \cdot \zeta')\mathrm{d}\tau + CT|z'|^3 \\ &\le \overline{\Gamma}(t)z' \cdot z' + \int_t^T (\dot{\overline{\Gamma}} - 4\overline{\Gamma}\,\overline{Q}\,\overline{\Gamma} + \overline{\Gamma}\,\overline{B} + \overline{B}^*\,\overline{\Gamma})\zeta' \cdot \zeta')\mathrm{d}\tau + CT|z'|^3 \\ &= \overline{\Gamma}(t)z' \cdot z' + CT|z'|^3.\end{aligned}$$

If we choose $T := \frac{1}{\delta}\log\frac{1}{|z'|}$, then $\mathrm{dist}(\eta(T), \mathcal{C}) = O(|z'|^2)$ and

$$\int_t^T L(-\dot\eta, \eta)\mathrm{d}\tau \le \overline{\Gamma}(t)z' \cdot z' + C|z'|^3 \log\frac{1}{|z'|}. \tag{7.9}$$

For constructing ζ_N we will need the following facts. From the definition of $\mathcal{T}_{ij}$ in (7.5) it follows that $\mathcal{T}_{iN}(\tau) = \dot\xi_i(\tau)/|\dot\xi(\tau)|$. Then, since $\ddot\xi_i(\tau) = \frac{\partial b_i}{\partial x_j}(\xi(\tau))\dot\xi_j(\tau)$, we have

$$\frac{\partial b_i}{\partial x_j}(\xi(\tau))\mathcal{T}_{jN}(\tau) - \dot{\mathcal{T}}_{iN}(\tau) = \frac{\dot\xi_i(\tau)\dot\xi_j(\tau)\ddot\xi_j(\tau)}{|\dot\xi(\tau)|^3} = \mathcal{T}_{iN}(\tau)\frac{\dot\xi_j(\tau)\ddot\xi_j(\tau)}{|\dot\xi(\tau)|^2}, \quad \forall i = 1, \dots, N.$$

Multiplying this by $\mathcal{T}^{-1}(\tau)$, we conclude that

$$\mathcal{T}^{-1}_{ki}(\tau)\Big(\frac{\partial b_i}{\partial x_j}(\xi(\tau))\mathcal{T}_{jN}(\tau) - \dot{\mathcal{T}}_{iN}(\tau)\Big) = \begin{cases}\dot\xi_j(\tau)\ddot\xi_j(\tau)/|\dot\xi(\tau)|^2, & \text{if } k = N, \\ 0 \quad \text{if } k < N.\end{cases} \tag{7.10}$$

We proceed with constructing ζ_N. By the definition of η we have

$$
\begin{aligned}
\dot\eta_i(\tau) - b_i(\eta(\tau)) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\xi_i(\tau) + \mathcal{T}_{ik}(\tau)\zeta'_k(\tau) + \mathcal{T}_{iN}(\tau)\zeta_N(\tau)\Big) - b_i(\eta(\tau)) + O(|z'|^2)\\
&= \Big(\dot{\mathcal{T}}_{ik}(\tau) - \frac{\partial b_i}{\partial x_j}(\xi(\tau))\mathcal{T}_{jk}(\tau)\Big)\zeta'_k(\tau) + \Big(\dot{\mathcal{T}}_{iN}(\tau) - \frac{\partial b_i}{\partial x_j}(\xi(\tau))\mathcal{T}_{jN}(\tau)\Big)\zeta_N(\tau)\\
&\quad + \mathcal{T}_{ik}(\tau)\dot\zeta'_k(\tau) + \mathcal{T}_{iN}(\tau)\dot\zeta_N(\tau) + O(|z'|^2)\\
&= \mathcal{T}_{ik}(\tau)\mathcal{T}^{-1}_{kr}(\tau)\Big\{\Big(\dot{\mathcal{T}}_{rk}(\tau) - \frac{\partial b_r}{\partial x_j}(\xi(\tau))\mathcal{T}_{jk}(\tau)\Big)\zeta'_k(\tau) + \Big(\dot{\mathcal{T}}_{rN}(\tau) - \frac{\partial b_r}{\partial x_j}(\xi(\tau))\mathcal{T}_{jN}(\tau)\Big)\zeta_N(\tau)\Big\}\\
&\quad + \mathcal{T}_{ik}(\tau)\dot\zeta'_k(\tau) + \mathcal{T}_{iN}(\tau)\dot\zeta_N(\tau) + O(|z'|^2).
\end{aligned}
\tag{7.11}
$$

Substituting for $\dot\zeta'$ the expression on the right-hand side of (7.7) and considering (7.10) yields

$$
b_i(\eta(\tau)) - \dot\eta_i(\tau) = 4a_{ij}(\xi(\tau))\mathcal{T}^{-1}_{lj}\big(0,\xi(\tau)\big)\overline{\Gamma}_{lm}(\tau)\zeta'_m(\tau) + O(|z'|^2) - \mathcal{T}_{iN}(\tau)\dot\zeta_N(\tau) + \mathcal{T}_{iN}(\tau)R(\tau),
$$

where

$$
\begin{aligned}
R(\tau) &= \mathcal{T}^{-1}_{Ni}(\tau)\frac{\partial b_i}{\partial x_j}(\xi(\tau))\bigg(\mathcal{T}_{jN}(\tau)\zeta_N(\tau) + \mathcal{T}_{jl}(\tau)\zeta'_l(\tau)\bigg)\\
&\quad - \mathcal{T}^{-1}_{Ni}(\tau)\bigg(\dot{\mathcal{T}}_{iN}(\tau)\zeta_N(\tau) + \dot{\mathcal{T}}_{il}(\tau)\zeta'_l(\tau)\bigg) - 4\mathcal{T}^{-1}_{Ni}(\tau)a_{ij}(\xi(\tau))\mathcal{T}^{-1}_{lj}(\tau)\overline{\Gamma}_{lm}(\tau)\zeta'_m(\tau)
\end{aligned}
$$

Thus, in order to make (7.8) hold, we choose $\zeta_N(\tau)$ as a solution of the following equation

$$
\dot\zeta_N(\tau) = R(\tau) \tag{7.12}
$$

with the initial condition $\zeta_N(t) = 0$.

From (7.10) it follows that $|\dot\xi(\tau)|$ solves

$$
\frac{\mathrm{d}}{\mathrm{d}t}|\dot\xi(\tau)| = \big(\mathcal{T}^{-1}_{Ni}(\tau)\frac{\partial b_i}{\partial x_j}(\xi(\tau))\mathcal{T}_{jN}(\tau) - \mathcal{T}^{-1}_{Ni}(\tau)\dot{\mathcal{T}}_{iN}(\tau)\big)|\dot\xi(\tau)|,
$$

and we can write the solution $\zeta_N(\tau)$ of (7.12) as

$$
\begin{aligned}
\zeta_N(\tau) &= |\dot\xi(\tau)|\int\limits_0^\tau \bigg(\mathcal{T}^{-1}_{Ni}(s)\frac{\partial b_i}{\partial x_j}(\xi(s))\mathcal{T}_{jl}(s)\zeta'_l(s) - \mathcal{T}^{-1}_{Ni}(s)\dot{\mathcal{T}}_{il}(s)\zeta'_l(s)\bigg)\frac{\mathrm{d}s}{|\dot\xi(s)|}\\
&\quad - 4|\dot\xi(\tau)|\int\limits_0^\tau \mathcal{T}^{-1}_{Ni}(s)a_{ij}(\xi(s))\mathcal{T}^{-1}_{lj}(s)\overline{\Gamma}_{lm}(s)\zeta'_m(s)\frac{\mathrm{d}s}{|\dot\xi(s)|}.
\end{aligned}
\tag{7.13}
$$

From (7.13) we derive the uniform bound $|\zeta_N(\tau)| \le C|z'|$.

It remains to construct $\eta(\cdot)$ for $\tau > T$ in such a way that it reaches the cycle in a finite time. To this end we set

$$
\eta(\tau) = X\big(\zeta'(T)(T+1-\tau), z_N(\xi(\tau)) + \zeta_N(T)|\dot\xi(\tau)|/|\dot\xi(T)|\big), \quad T \le \tau \le T+1. \tag{7.14}
$$

Then for every $i = 1, \ldots, N$

$$\begin{aligned}\dot\eta_i(\tau) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\big(\xi_i(\tau) + \mathcal{T}_{iN}(\tau)\zeta_N(T)|\dot\xi(\tau)|/|\dot\xi(T)|\big) + O(|z'|^2)\\ &= \dot\xi_i(\tau) + \ddot\xi_i(\tau)\zeta_N(T)/|\dot\xi(T)| + O(|z'|^2)\\ &= b_i(\xi(\tau)) + \frac{\partial b_i}{\partial x_j}(\xi(\tau))\dot\xi_j(\tau)\zeta_N(T)/|\dot\xi(T)| + O(|z'|^2) = b_i(\eta(\tau)) + O(|z'|^2).\end{aligned}$$

Therefore, the following bound holds

$$\int\limits_T^{T+1} a^{ij}(\eta(\tau))(-\dot\eta_i + b_i(\eta))(-\dot\eta_j + b_j(\eta))\mathrm{d}\tau \le C|z'|^4. \tag{7.15}$$

Combining the last relation with (7.9) yields (7.6). □

In order to construct a sub- and supersolution of (6.1) we consider the solution $\overline{D}(t)$ of the matrix equation

$$\dot{\overline{D}} + \overline{D}(\overline{B} - 4\overline{Q}\,\overline{\Gamma}) + (\overline{B} - 4\overline{Q}\,\overline{\Gamma})^*\overline{D} = -2I, \tag{7.16}$$

given by

$$\overline{D}(t) = 2\int\limits_t^{\infty} \overline{\Psi}^*(\tau,t)\overline{\Psi}(\tau,t)\mathrm{d}\tau, \tag{7.17}$$

where $\overline{\Psi}(\tau,t)$ is the fundamental matrix solution of

$$\frac{\partial\overline{\Psi}}{\partial\tau} = (\overline{B}(\tau) - 4\overline{Q}(\tau)\overline{\Gamma}(\tau))\overline{\Psi}, \quad \overline{\Psi}(t,t) = I.$$

As already mentioned in the proof of Lemma 9 this solution $\overline{\Psi}(\tau,t)$ decays exponentially as $\tau \to +\infty$ and therefore the integral in (7.17) converges. Then it defines a P-periodic positive symmetric solution of (7.16). It follows from (7.2) and (7.16) that $\overline{\Gamma}_\delta^\pm := \overline{\Gamma} \pm \delta\overline{D}$ satisfy for sufficiently small $\delta > 0$

$$4\overline{\Gamma}_\delta^+\,\overline{Q}\,\overline{\Gamma}_\delta^+ - \frac{\mathrm{d}}{\mathrm{d}t}\overline{\Gamma}_\delta^+ - \overline{\Gamma}_\delta^+\,\overline{B} - \overline{B}^*\overline{\Gamma}_\delta^+ \ge \delta I \quad \text{and} \quad 4\overline{\Gamma}_\delta^-\,\overline{Q}\,\overline{\Gamma}_\delta^- - \frac{\mathrm{d}}{\mathrm{d}t}\overline{\Gamma}_\delta^- - \overline{\Gamma}_\delta^+\,\overline{B} - \overline{B}^*\overline{\Gamma}_\delta^- \le -\delta I.$$

Now define the functions $W_\delta^\pm(z)$ by

$$W_\delta^\pm(z) = (\overline{\Gamma}_\delta^\pm(t))_{ij} z_i' z_j' \quad \text{where } z_N = z_N(\xi(t)),$$

these functions satisfy

$$S(\nabla_z W_\delta^-(z), z) \le -\frac{\delta}{2}|z'|^2 \quad \text{and} \quad S(\nabla_z W_\delta^+(z), z) \ge \frac{\delta}{2}|z'|^2 \quad \text{for sufficiently small } |z'|. \tag{7.18}$$

The latter inequalities follow directly form the definitions of $W_\delta^-(z)$ and $W_\delta^+(z)$.

Lemma 10. *For sufficiently small $\delta > 0$ the strict pointwise inequalities $W_\delta^-(z) < W(X(z)) < W_\delta^+(z)$ hold for z' from a punctured neighborhood of zero.*

Proof. The first inequality $W_\delta^-(z) < W(X(z))$ can be proved similarly to Lemma 19 in [18] (see also the proof of Lemma 8), using (7.18). The second inequality $W(X(z)) < W_\delta^+(z)$ follows immediately from Lemma 9. □

At this point we have constructed functions $W_\delta^\pm$ satisfying conditions of Lemma 5. Next we define the test functions $W_{\delta,\varepsilon}^\pm$ by

$$W_{\delta,\varepsilon}^\pm := W_\delta^\pm - \varepsilon \overline{\Phi}_\delta^\pm(t),$$

where z_N and t are related by $z_N = z_N(\xi(t))$, and $\overline{\Phi}_\delta^\pm(t)$ are periodic solutions of the ODEs

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\Phi}_\delta^\pm(t) = -2\mathrm{tr}(\overline{Q}(t)\overline{\Gamma}_\delta^\pm(t)) + c(\xi(t)) + \frac{2}{P}\int_0^P \mathrm{tr}(\overline{Q}(\tau)\overline{\Gamma}_\delta^\pm(\tau))\mathrm{d}\tau - \frac{1}{P}\int_0^P c(\xi(\tau))\mathrm{d}\tau. \tag{7.19}$$

The first two terms on the right-hand side here are introduced in order to compensate the discrepancy of order ε in equation (6.2). Indeed, the test functions $W_{\delta,\varepsilon}^\pm$ constructed in this way satisfy for sufficiently small $|z'|$

$$\pm a_{ij}\left(X(z)\right)\alpha_{ki}(z)\frac{\partial}{\partial z_k}\left(\alpha_{lj}(z)\frac{\partial W_{\delta,\varepsilon}^\pm}{\partial z_l}\right) \mp \frac{1}{\varepsilon}S(\nabla_z W_{\delta,\varepsilon}^\pm, z) \mp c(X(z))$$
$$\leq \frac{1}{P}\int_0^P c(\xi(\tau))\mathrm{d}\tau - \frac{2}{P}\int_0^P \mathrm{tr}(\overline{Q}(\tau)\overline{\Gamma}_\delta^\pm(\tau))\mathrm{d}\tau + O(\varepsilon + |z'|).$$

In order to complete the proof of the fact that $W_{\delta,\varepsilon}^\pm$ satisfy (4.4) and (4.5) it remains to observe that $\int_0^P \mathrm{tr}(\overline{Q}(\tau)\overline{\Gamma}_\delta^\pm(\tau))\mathrm{d}\tau \to \int_0^P \mathrm{tr}(\overline{Q}(\tau)\overline{\Gamma}(\tau))\mathrm{d}\tau$ as $\delta \to 0$ and use the identity

$$2\int_0^P \mathrm{tr}(\overline{Q}(\tau)\overline{\Gamma}(\tau))\mathrm{d}\tau = \sum_{\Theta_i > 1} \log \Theta_i$$

(see Proposition 5.1 in [19]), where Θ_i are absolute values of eigenvalues of the linearized Poincaré map (corresponding to the ODE $\dot{x} = b(x)$ near $\mathcal{C}$).

8 Construction of test functions: case of limit cycles on $\partial\Omega$

In the case when ODE $\dot{x} = b_\tau(x)$ on $\partial\Omega$ has a limit cycle $\mathcal{C}$ which is significant component of the Aubry set, the analysis combines the ideas of Section 6 and Section 7. We pass to the local coordinates in a neighborhood of $\mathcal{C}$ via a map $x = X(z_1, \ldots, z_{N-1}, z_N)$, where $z_N = z_N(x)$ is the distance from x to $\partial\Omega$ (positive for $x \in \Omega$) and $(z_1, \ldots, z_{N-1})$ are coordinates on $\partial\Omega$. The coordinate $z_{N-1}(x)$ represents the arc length parametrization on $\mathcal{C}$ and other coordinates $z' = (z_1, \ldots, z_{N-2})$ are chosen so that the map $X(z', z_{N-1}, z_N)$ is C^2-smooth, moreover $z' = 0$

when $x \in \mathcal{C}$, and $\big(\frac{\partial X_i}{\partial z_j}(z)\big)_{i,j=\overline{1,N}}$ is an orthogonal matrix when $z_N = 0$ and $z' = 0$ (on the cycle). This change of coordinates leads to equations of the form (6.1) and (6.2) for $W(X(z))$ and $W_\varepsilon(X(z))$.

We use the following ansatz for W,

$$W(X(z)) = \widehat{\Gamma}_{ij}(t) z_i' z_j' + o(|z'|^2), \tag{8.1}$$

where $\widehat{\Gamma}$ is now $(N-2)\times(N-2)$ symmetric P-periodic matrix (P being the period of the cycle $\mathcal{C}$), and t refers to the parametrization $t \to \xi(t)$ of $\mathcal{C}$ such that $\dot{\xi}(t) = b_\tau(\xi(t))$. Moreover, $\widehat{\Gamma}$ is chosen to be the maximal P-periodic solution of the Riccati matrix equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\Gamma} = 4\widehat{\Gamma}\widehat{Q}\widehat{\Gamma} - \widehat{\Gamma}\widehat{B} - \widehat{B}^*\widehat{\Gamma},$$

with $(N-2)\times(N-2)$ matrices $\widehat{Q}(t)$ and $\widehat{B}(t)$ whose entries are given by the same formulas as (7.3) and (7.4).

Lemma 11. *For sufficiently small $|z'|$ and $|z_N|$ the following bound holds uniformly in $t \in [0, P)$*

$$W(X(z', z_{N-1}(\xi(t)), z_N) \le \widehat{\Gamma}_{ij}(t) z_i' z_j' + C(|z'|^2 \log\frac{1}{|z'|} + |z_N||z'|^2 + |z_N|^3). \tag{8.2}$$

Proof. First consider the case $z_N = 0$. As in Lemma 9 we use representation (2.9) and consider the solution $\zeta'(\tau)$ of the ODE $\dot{\zeta}'(\tau) = (\widehat{B} - 4\widehat{Q}\widehat{\Gamma})\zeta(\tau)$ for $\tau > t$ with the initial condition $\zeta'(t) = z'$. It decays exponentially as $\tau \to \infty$, $|\zeta'| \le Ce^{-\delta\tau}|z'|$ for some $\delta > 0$. Next we introduce ζ_{N-1} analogously to ζ_N introduced in Lemma 9, i.e. ζ_{N-1} solves

$$\begin{aligned}\dot{\zeta}_{N-1}(\tau) = {}& \mathcal{T}^{-1}_{(N-1)i}\big(0, z_{N-1}(\xi(\tau), 0\big)\frac{\partial b_i}{\partial x_j}(\xi(\tau))\Big(\mathcal{T}_{j(N-1)}(\tau)\zeta_{N-1}(\tau) + \mathcal{T}_{jl}(\tau)\zeta_l'(\tau)\Big)\\ &- \mathcal{T}^{-1}_{(N-1)i}\big(0, z_{N-1}(\xi(\tau)), 0\big)\Big(\dot{\mathcal{T}}_{i(N-1)}(\tau)\zeta_{N-1}(\tau) + \dot{\mathcal{T}}_{il}(\tau)\zeta_l'(\tau)\Big)\\ &- 4\mathcal{T}^{-1}_{(N-1)i}\big(0, z_{N-1}(\xi(\tau)), 0\big)a_{ij}(\xi(\tau))\mathcal{T}^{-1}_{lj}\big(0, z_{N-1}(\xi(\tau)), 0\big)\widehat{\Gamma}_{lm}(\tau)\zeta_m'(\tau), \quad \tau > t,\end{aligned}$$

where $(\mathcal{T}^{-1}_{ij}(z))_{i,j=\overline{1,N}}$ is the matrix inverse to $(\frac{\partial X_i}{\partial z_j}(z))_{i,j=\overline{1,N}}$ and $\mathcal{T}_{ij}(\tau) = \frac{\partial X_i}{\partial z_j}(0, z_{N-1}(\xi(\tau)), 0)$. Finally we define $\eta(\tau)$ by

$$\eta(\tau) = \begin{cases} X(\zeta'(\tau), z_{N-1}(\xi(\tau)) + \zeta_{N-1}(\tau), 0), \ t \le \tau < T\\ X(\zeta'(T)(T+1-\tau), z_{N-1}(\xi(\tau)) + \zeta_{N-1}(T)|\dot{\xi}(\tau)|/|\dot{\xi}(T)|, 0), \ T \le \tau < T+1,\end{cases}$$

with $T := \frac{1}{\delta}\log\frac{1}{|z'|}$, and the control $v(\tau)$ by

$$v(\tau) = \begin{cases} \dot{\eta} + \nu(\eta)\big(b_\nu(\eta) + R_1(\tau)\big), \quad t \le \tau < T\\ \dot{\eta} + \nu(\eta) b_\nu(\eta), \quad T \le \tau \le T+1,\end{cases}$$

where $z_{N-1} = z_{N-1}(\xi(\tau))$, and

$$R_1(\tau) = -\mathcal{T}^{-1}_{(N-1)i}\big(0, z_{N-1}, 0\big)\Big(\dot{\mathcal{T}}_{i(N-1)}(\tau)\zeta_{N-1}(\tau) + \dot{\mathcal{T}}_{il}(\tau)\zeta_l'(\tau) + 4a_{ij}(\xi(\tau))\mathcal{T}^{-1}_{lj}(0, z_{N-1}, 0)\widehat{\Gamma}_{lm}\zeta_m'\Big).$$

Letting

$$\alpha(\tau) := \begin{cases} b_\nu(\eta) + R_1(\tau), & t \le \tau < T \\ b_\nu(\eta), & T \le \tau \le T+1, \end{cases}$$

observe that for this control $v(\tau)$ the pair $(\eta(\tau), \alpha(\tau))$ solves (2.8) on $(t, T+1)$ with the initial value $\eta(t) = x (= X(z', \xi(t), 0))$, as far as $\alpha(\tau) \ge 0$ for all $\tau \in (t, T+1)$. Since $b_\nu(\xi(\tau)) > 0$, the latter condition is satisfied, provided that $|z'|$ is sufficiently small. Then the proof of (8.2) follows exactly the line of Lemma 9.

In the case when $z_N(x) > 0$ we construct a curve $\eta(\tau)$ connecting x with a point y on $\partial\Omega$ by setting

$$\eta(\tau) = X(z', z_{N-1}(\xi(t)), z_N + b_\nu(\xi(t))(t-\tau)) \text{ for all } \tau \ge t \text{ such that } z_N + b_\nu(\xi(t))(t-\tau) \ge 0.$$

Let $t + \Delta t$ be the time when $\eta(\tau)$ reaches $\partial\Omega$ (at the point $y = \eta(t+\Delta t)$) then $\Delta t = O(z_N)$. It follows from the construction of $\eta(\tau)$ that

$$\int_t^{t+\Delta t} a^{ij}(\eta(\tau))(-\dot\eta_i + b_i(\tau))(-\dot\eta_j + b_j(\eta))\,\mathrm{d}\tau \le C(|z'|^2 + z_N^2)z_N.$$

Then extending $\eta(\tau)$ along $\partial\Omega$ as described above we complete the proof of the Lemma. □

Now we construct test functions $W_\delta^\pm(z', z_{N-1}(\xi(t)), z_N) := (\widehat\Gamma \pm \delta \widehat D)_{ij}(t) z_i' z_j' \pm \delta z_N^2$ for (sufficiently small) $\delta > 0$, where the P-periodic symmetric matrix $\hat D(t) > 0$ is defined analogously to (7.17). Then

$$S(\nabla_z W_\delta^-, z) \le -\delta(|z'|^2 + b_\nu(\xi(t))z_N)$$

for sufficiently small $|z'|$ and $z_N \ge 0$. This yields the following bound

$$W_\delta^- < W(X(z)) \text{ for sufficiently small } |z'| \text{ and } z_N \text{ (when } |z'| + z_N > 0)$$

whose proof is analogous to that of the lower bound in Lemma 8. Thus functions $W_\delta^\pm$ satisfy the conditions of Lemma 5. Finally, we define the test functions $W_{\delta,\varepsilon}^\pm$ by

$$W_{\delta,\varepsilon}^\pm := W_\delta^\pm \pm \varepsilon \widehat\Phi_\delta^\pm \mp \varepsilon^2 z_N, \text{ where } z_{N-1} = z_{N-1}(\xi(t)),$$

with $\widehat\Phi_\delta^\pm$ being solutions of

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat\Phi_\delta^\pm(t) = -2\mathrm{tr}(\widehat Q(t)\widehat\Gamma_\delta^\pm(t)) + c(\xi(t)) + \frac{2}{P}\int_0^P \mathrm{tr}(\widehat Q(\tau)\widehat\Gamma_\delta^\pm(\tau))\mathrm{d}\tau - \frac{1}{P}\int_0^P c(\xi(\tau))\mathrm{d}\tau.$$

These functions $W_{\delta,\varepsilon}^\pm$ satisfy the conditions of Lemma 5.

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References

[1] Abou-Kandil, H., Freiling, G., Ionescu, V., Jank, G. Matrix Riccati equations. In control and systems theory. Systems and Control: Foundations and Applications. Birkhäuser Verlag, Basel, 2003.

[2] Bernstein, S.N. Sur la généralisation du problème de Dirichlet. *Math. Ann.* **62** (2), 1906, 253–271.

[3] Bernstein, S.N. Sur la généralisation du problème de Dirichlet. *Math. Ann.* **69** (1), 1910, 82–136.

[4] Devinatz, A.; Ellis, R.; Friedman, A. The asymptotic behavior of the first real eigenvalue of second order elliptic operators with a small parameter in the highest derivatives. II. *Indiana Univ. Math. J.* **23** (1973–1974), 991–1011.

[5] Eizenberg, A.; Kifer, Yu. The asymptotic behavior of the principal eigenvalue in a singular perturbation problem with invariant boundaries. *Probab. Theory Related Fields*, **76**(4), (1987), 439–476.

[6] Evans, L. C.; Ishii, H. A PDE approach to some asymptotic problems concerning random differential equations with small noise intensities. *Ann. Inst. H. Poincare Anal. Non Lineaire* **2**(1) (1985), 1–20.

[7] Freidlin, M.I., Wentzell, A.D. Random perturbations of dynamical systems. Fundamental Principles of Mathematical Sciences, 260. Springer-Verlag, New York, 1984.

[8] Friedman, A. The asymptotic behavior of the first real eigenvalue of a second order elliptic operator with a small parameter in the highest derivatives. *Indiana Univ. Math. J.* **22** (1972/73), 1005–1015.

[9] Ishii, H. Weak KAM aspects of convex Hamilton-Jacobi equations with Neumann type boundary conditions. *J. Math. Pures Appl.* **95**(1) (2011), 99–135.

[10] Kifer, Yu. On the principal eigenvalue in a singular perturbation problem with hyperbolic limit points and circles. *J. Differential Equations*, **37**(1), (1980), 108–139.

[11] Kifer, Yu., Stochastic stability of the topological pressure. *J. Analyse Math.*, **38**, (1980), 255–286.

[12] Kifer, Yu. Principal eigenvalues, topological pressure, and stochastic stability of equilibrium states. *Israel J. Math.* **70**(1) (1990), 1–47.

[13] Ladyzenskaja, O. A.; Ural'ceva, N. N. Certain classes of nonuniformly elliptic equations. *Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **5** (1967), 186–191.

[14] Lancaster, L.; Rodman, L., Algebraic Riccati equations. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[15] Lions, P.-L. Resolution de problemes elliptiques quasilineaires. *Arch. Rational Mech. Anal.* **74**(4) (1980), 335–353.

[16] Perthame, B. Perturbed dynamical systems with an attracting singularity and weak viscosity limits in Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.* **317**(2) (1990), 723–748.

[17] Piatnitski, A. Asymptotic Behaviour of the Ground State of Singularly Perturbed Elliptic Equations. *Commun. Math. Phys.* **197** (1998), 527–551.

[18] Piatnitski, A.; Rybalko, V. On the first eigenpair of singularly perturbed operators with oscillating coefficients. Submitted to Comm. Part. Diff. Eq.
Preprint at arXiv:1206.3754

[19] Piatnitski, A.; Rybalko, A.; Rybalko, V. Ground states of singularly perturbed convection-diffusion equation with oscillating coefficients. *ESAIM: COCV* **20**(4) (2014) , 1059–1077.

[20] Serrin, J. The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables. *Philos. Trans. Roy. Soc. London Ser. A* **264** (1969), 413–496.

[21] Ventcel', A.D. The asymptotic behavior of the largest eigenvalue of a second order elliptic differential operator with a small parameter multiplying the highest derivatives. (Russian) *Dokl. Akad. Nauk SSSR* **202** (1972), 19–22.

[22] Ventcel', A.D. The asymptotic behavior of the first eigenvalue of a second order differential operator with a small parameter multiplying the highest derivatives. *Teor. Verojatnost. i Primenen.* **20**(3) (1975), 610–613.

[23] Vishik, M.I., Lyusternik, L.A. Regular degeneracy and boundary layer for linear differential equations with a small parameter, *Usp. Mat. Nauk* **12**(5) (1957), 3–122.