Geometric Spanner of Segments

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Abstract. Geometric spanner is a fundamental structure in computational geometry and plays an important role in many geometric networks design applications. In this paper, we consider a generalization of the classical geometric spanner problem (called segment spanner): Given a set S of disjoint 2-D segments, find a spanning network G with minimum size so that for any pair of points in S, there exists a path in G with length no more than t times their Euclidean distance. Based on a number of interesting techniques (such as weakly dominating set, strongly dominating set, and interval cover), we present an efficient algorithm to construct the segment spanner. Our approach first identifies a set of Steiner points in S and then construct a point spanner for the set of Steiner points. Our algorithm runs in $O(|Q|+n^2 \log n)$ time, where Q is the set of Steiner points. We show that Q is an $O(1)$-approxiamtion in terms of its size when S is relatively "well" separated by a constant. For arbitrary rectilinear segments and under L_1 distance, the approximation ratio improves to 2.

1 Introduction

In this paper, we consider the following generalization of the classical geometric spanner problem: Given a set O of n disjoint objects in Euclidean space and a constant $t > 1$, construct a graph G for O of minimum size so that for any pair of points $p_i \in o_i$ and $p_j \in o_j$, there exists a path $P(p_i, p_j)$ in G whose total length is at most $t \times d(p_i, p_j)$, where o_i and o_j are objects in O and $d(p_i, p_j)$ is the Euclidean distance between p_i and p_j. The path $P(p_i, p_j)$ consists of three parts, P_1, P_2 and P_3, where P_1 and P_3 are the portions of $P(p_i, p_j)$ inside o_i and o_j respectively. We assume that there implicitly exists an edge (or path) between any pair of points inside each object $o \in O$. Thus, the objective of minimizing the size of G is equivalent to minimizing the total number of vertices and edges between vertices in different objects. In this paper, we consider the case where all objects are disjoint 2-D line segments.

Spanner is a fundamental structure in computational geometry and finds applications in many different areas. Extensive researches have done on this structure and a number of interesting results have been obtained [1–11]. Almost all previous results consider the case in which the objects are points and seek to minimize the spanner's construction time, size, weight, maximum degree of vertex, diameter, or combination of them.

A common approach for constructing geometric spanner is the use of Θ-graph [1–4]. In [5], Arya *et al.* showed that a t-spanner with constant degree can be constructed in $O(n \log n)$ time. In [6, 7], they gave a randomized construction of a sparse t-spanner with expected spanner diameter $O(\log n)$. In [9, 10], Das *et al.* proposed an $O(n \log^2 n)$-time greedy algorithm for a t-spanner with $O(n)$ edges and $O(1)wt(MST)$ weight in 3-D space. Gudmundsson *et al.* showed in [11] that an $O(n)$ edges, and $O(1)wt(MST)$ weight t-spanner is possible to be constructed in $O(n \log n)$ time.

In graph settings, Chandar *et al.* [8] showed that for an arbitrary positive edge-weighted graph G and any $t > 1, \epsilon > 0$, a t-spanner of G with weight $O(n^{\frac{2+\epsilon}{t-1}})wt(MST)$ can be constructed in polynomial time. They also showed that $(\log^2 n)$-spanners of weight $O(1)wt(MST)$ can be constructed.

For geometric spanners of objects other than points, Asano *et al.* considered the problem of constructing a spanner graph for a set of axis-aligned rectangles using rectilinear bridges and under L_1 distance [12]. They showed that in general it is NP-hard to minimize the dilation, and when the spanner graph is restricted to be trees, the problem can be solved using a linear program. They also considered other simple graphs such as paths and sorted paths, and presented polynomial time solution for each of them.

The spanner of segments problem considered in this paper is motivated by several interesting applications. One of such applications is for constructing bridges between a set of buildings so that the path (traveling through bridges) between locations in different buildings is close to their Euclidean distance [12]. Another application appears in wireless mesh networks. In such networks, a set of wireless routers (or stations) are to be installed in objects, such as streets or highways, so that for any pair of wireless devices in those objects there exists a routing path for them with length close to their Euclidean distance. The rationale of such distance requirement is for minimizing the total energy used for routing messages between them, as the energy consumption is proportional to the path length.

To build a spanner of segments, we view the construction as a two-phase process. In the first phase, a set of points (called Steiner points) are selected from each segment, and in the second phase, a spanner is constructed for the set of Steiner points. Since the second phase can be completed by using existing spanner algorithms (for points), our focus in this paper is thus on the first phase. Furthermore, since most existing spanners are sparse graphs (i.e., consist of $O(n)$ edges), minimizing the size of the segment spanner is equivalent to minimizing the total number of Steiner points. Our objective is hence to obtain a spanner with a minimum number of Steiner points.

Minimizing the number of Steiner points is in general quite challenging. Part of the reason is that the position of a Steiner point on one segment affects not only the positions of the Steiner points on the same segment but also on other segments. To overcome this difficulty, we first introduce the concept of *weakly dominating set* to lower bound the number of Steiner points on one segment. By using some imaginary segments and a few other interesting techniques, we are able to find a set of strongly dominating set for each segment. We show that the size of the strongly dominating set is only a constant factor away from the optimal solution for segments relatively separated by a constant. This gives us a constant approximation of the minimum-sized spanner. Our algorithm can be easily implemented and runs in near quadratic time. We also show that for arbitrary rectilinear segments and under L_1 distance, the approximation ratio improves to 2.

Due to space limit, we omit a lot of details and proofs from this extended abstract.

2 Preliminaries

Let $S = \{s_1, s_2, \cdots, s_n\}$ be a set of n disjoint segments on a plane with each segment $s_i = \overline{a_i b_i}$, where a_i and b_i are the left and right endpoints of s_i respectively (For a vertical segment, a_i is the lower endpoint and b_i is the upper endpoint). A t-spanner G_S of S is a network which connects the segments in S and satisfies the following condition. For any two points p_i and p_j on segments s_i and s_j in S, there exists a path (called spanner path) of G_S between p_i and p_j and with length no more than $t \times |p_i p_j|$, where t is the *stretch factor* of the spanner and $|p_i p_j|$ is the Euclidean distance between p_i and p_j. The spanner G_S contains two types of line segments: input segments and segments connecting the input segments. We call the former *segments* and the latter *bridges* to distinguish them. The intersections of segments and bridges are called *Steiner* points.

As mentioned in previous section, our main objective for the spanner G_S is to minimize its size. The size of G_S is the sum of the number of vertices and edges. The vertices of G_S include all endpoints of the input segments and the Steiner points. The edges consist of bridges and subsegments fragmented by the Steiner points. For a segment s_i with k Steiner points, the number of edges on s_i is bounded by $O(k)$ (i.e. at most $k+1$). Thus, to minimize the size of G_S, it is sufficient to minimize the total number of Steiner points and bridges.

To simplify the optimization task, our main idea is to separate the tasks of minimizing the number of Steiner points and the number of edges (i.e. bridges). For this, we consider the following approach: (1) First compute a set Q of Steiner points with small size, and then (2) construct a spanner G_Q for Q to minimize the number of edges. The spanner G_Q together with the edges on S forms a spanner for the set S of segments. Thus, the key to building the spanner G_S is to identify a small set of "good" Steiner points.

For a pair of segments s_i, $s_j \in S$, the distance between them is defined as $d(s_i, s_j) = \min_{p_i \in s_i, p_j \in s_j} |p_i p_j|$. The distance from s_i to S is defined as $d_i = \min_{j \neq i, s_j \in S} d(s_i, s_j)$. Let l_i be the length of s_i. The *relative separation ratio* of s_i in S is defined as d_i/l_i and the relative separation ratio of S is $\min_{s_i \in S} d_i/l_i$. In this paper, we assume that the set S of segments are "well" separated in a sense that its relative separation ratio is no less than ϵ for some constant $\epsilon > 0$. The rationale of this assumption is that in wireless networks, if two segments are too close to each other, they can share a set of routers (or stations) and therefore can be viewed as one segment.

3 Minimizing the Number of Steiner Points

To ensure that the resulted network is a t-spanner for the set S of input segments, each segment in S has to be well sampled by the set Q of Steiner points. More specifically, for any pair of points p_1 and p_2 on some segments, there should be a pair of Steiner points q_1 and q_2 which are close enough to p_1 and p_2 respectively so that the spanner path between q_1 and q_2 in G_Q plus the edges (p_1, q_1) and (p_2, q_2) forms a spanner path for p_1 and p_2 (see Figure 1). One way to meet this requirement is to place as many Steiner points as possible along each segment. However, this could lead to a very large set of Q and results in a very large network G_S. Thus, to minimize the size of G_S, we should compute such a Q that contains only barely enough Steiner points to cover every pair of points on the input segments. Unfortunately, to compute such a set of Q is actually quite challenging. Part of the reason is that the position of a Steiner point on one segment could affect the positions of other Steiner points not only on the same segment but also on other segments. Thus, it is desirable to isolate the interference between Steiner points on different segments. This leads us to consider weakly dominating sets.

3.1 Weakly Dominating Set

To investigate how the Steiner points affect each other, we consider the problem of placing Steiner points on two disjoint segments, $s_1 = \overline{a_1 b_1}$ and $s_2 = \overline{a_2 b_2}$. Let p_1 and p_2 be two arbitrary points on s_1 and s_2 respectively. Let q_1 and q_2 be two Steiner points on the neighborhoods of p_1 and p_2 respectively such that the path $p_1 \to q_1 \to q_2 \to p_2$ is a spanner path for p_1 and p_2. In this case, we say that q_1 and q_2 *t-dominate* the pair of p_1 and p_2. Clearly, the positions of q_1 and q_2 are constrained by p_1 and p_2. If we fix p_1, p_2, and one Steiner point q_1, then all possible positions of the other Steiner point q_2 form a (possibly empty) interval $I_S(p_1, p_2, q_1)$ (which is a function of p_1, p_2, and q_1) on s_2 (see Figure 1). When q_1 moves along s_1, the interval changes accordingly (on its position and length). Similarly, if we fix the two Steiner points q_2, q_1, together with p_2, all points on s_1 t-dominated by q_2 and q_1, with respect to p_2, also form an interval $I_D(q_2, q_1, p_2)$ on s_1. Since the spanner G_S needs to guarantee that there exists a spanner path (or equivalently a t-dominating pair of Steiner points) from p_2 to every point on s_1, from p_2's point of view, it expects q_2 to be in some position such that s_1 can be covered by a minimum number of q_1's, i.e. the union of $I_D(q_2, q_1, p_2)$'s covers s_1.

Clearly, the total number of Steiner points needed to cover s_1 depends on the positions of the Steiner points on s_2, and vice versa. When the number of segments increases, the influence of the Steiner points on one segment will be propagated to all the others, making the optimization problem extremely difficult to solve exactly. Thus, our objective is to achieve a good approximation. To shed some light on the property of an optimal solution, we first consider lower bounding the problem, that is, finding a set of Steiner points with size no more than that of the optimal solution.

To lower bound the total number of Steiner points in an optimal solution, our main idea is to lower bound it for each pair of segments by isolating their Steiner point determination from that of the rest of the segments. As discussed previously, for an arbitrary pair of points $p_1 \in s_1$ and $p_2 \in s_2$, the positions of their t-dominating pair q_1 and q_2 are constrained by a function. To relax this constraint, when placing Steiner points on s_1, we assume that q_2 can be placed at any arbitrary position on s_2. Ideally, we assume that q_2 always overlaps with p_2. Thus, we only need to consider the relation between q_1 and p_1. We say that q_1 *t-weakly dominates* p_1 and p_2 if the length of the path $p_1 \to q_1 \to p_2$ is no more than $t \times |p_1p_2|$. If q_1 t-weakly dominates p_1 and p_2 for every possible choice of p_2 (while fixing p_1), then we say q_1 t-weakly dominates p_1. Our objective is thus to select a minimum number of points on s_1 so that every point on s_1 is t-weakly dominated by some selected Steiner point. We call such a set of points as a *t-weakly dominating set* of s_1.

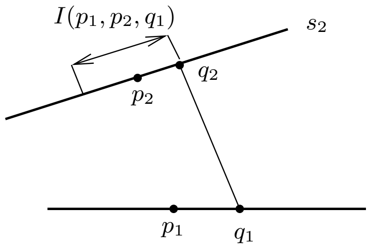

Fig. 1. Steiner points on two segments.

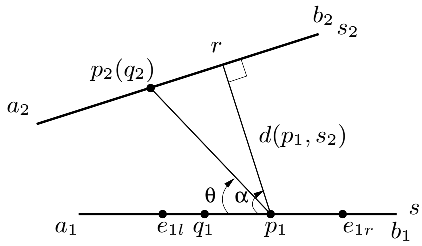

Fig. 2. The interval of $I(p_2, p_1)$.

3.2 Computing t-Weakly Dominating Set in a Brute-Force Manner

Let θ be the angle $\angle a_1p_1p_2$, and e_{1l} and e_{1r} be the two endpoints of the interval $I_S(p_2, p_1, p_2)$, i.e. the interval of all possible positions of q_1 while q_2 coincides with p_2 (see Figure 2). The range of $I(p_2, p_1, p_2)$ can be determined by the following lemma.

Lemma 1. *The two endpoints e_{1l} and e_{1r} locate on different sides of p_1 with $|p_1e_{1l}| = \min\{|p_1a_1|, \frac{t^2-1}{2(t-\cos\theta)}|p_1p_2|\}$ and $|p_1e_{1r}| = \min\{|p_1b_1|, \frac{t^2-1}{2(t+\cos\theta)}|p_1p_2|\ \}$.*

Proof. As shown in Figure 2, by the law of cosines, we know that $|q_1p_2|^2 = |q_1p_1|^2 + |p_1p_2|^2 - 2|q_1p_1||p_1p_2|\cos\theta$ if q_1 is on the left side of p_1, or $|q_1p_2|^2 = |q_1p_1|^2 + |p_1p_2|^2 + 2|q_1p_1||p_1p_2|\cos\theta$ if q_1 is on the right side of p_1. By the spanner property, we have $|p_1q_1| + |q_1p_2| \le t|p_1p_2|$. Solving the system of the two equations, we get either $|p_1q_1| \le \frac{t^2-1}{2(t-\cos\theta)}|p_1p_2|$ or $|p_1q_1| \le \frac{t^2-1}{2(t+\cos\theta)}|p_1p_2|$. The lemma follows from the fact that q_1 is on s_1. □

To find a point q_1 t-weakly dominating p_1, we need to make sure that q_1 t-weakly dominates p_1 and p_2 for all possible choices of p_2. That is, q_1 has to be in the common intersection, denoted as $I(p_1, s_2)$, of all $I_S(p_2, p_1, p_2)$'s, i.e. $q_1 \in \bigcap_{p_2 \in s_2} I_S(p_2, p_1, p_2)$. The following lemma shows some property of $I(p_1, s_2)$.

Lemma 2. *$|p_1e_{1l}|$ (or $|p_1e_{1r}|$) achieves its minimum either when e_{1l} coincides with a_1 (or e_{1r} coincides with b_1), or p_2 is at the endpoints of s_2, or θ is one of the two constants depending only on s_1, s_2 and t.*

Proof. We prove the lemma only for $|p_1e_{1l}|$, and the other part can be proved similarly.

Let $d(p_1, s_2)$ be the shortest Euclidean distance from p_1 to s_2's supporting line and r be the point that achieves the shortest distance. Let α be the angle $\angle a_1p_1r$. See Figure 2 for an example. It is easy to see that $|p_1p_2| = \frac{d(p_1,s_2)}{\cos(\alpha-\theta)}$. Thus $|p_1e_{1l}| = \min\{|p_1a_1|, \frac{(t^2-1)d(p_1,s_2)}{2(t-\cos\theta)\cos(\alpha-\theta)}\}$. When p_1, s_2 and s_1 are fixed, both $d(p_1, s_2)$ and α are fixed. Thus to achieve its minimum, $f_l(\theta) = (t - \cos\theta)\cos(\alpha - \theta)$ has to be maximized. Since $f_l(\theta)$

achieves its maximum either when θ is the root of $f_l'(\theta) = 0$ or θ is the minimum or maximum in its domain (i.e. when p_2 is at the endpoints of s_2), where $f_l'(\theta) = \sin(2\theta - \alpha) + t\sin(\alpha - \theta)$ is the derivative of $f_l(\theta)$. It is easy to see that the equation of $f_l'(\theta) = 0$ has at most four roots. Among the four roots, there exists at most two real roots that allow $f_l(\theta)$ to achieve the maximum for all p_1. W.l.o.g let θ_1 and θ_2 be the two roots, and $f_l(\theta_1) \geq f_l(\theta_2)$ (See Figure 3 for an illustration). The lemma follows from the fact that the roots depend only on t and α. □

From the above lemmas, we know that for each point $p_1 \in s_1$, its corresponding interval $I(p_1, s_2)$ can be obtained by evaluating the functions of $|p_1e_{1l}|$ and $|p_1e_{1r}|$ at $O(1)$ possible values and taking the minimum.

To obtain a size-minimized t-weakly dominating set for s_1, our basic idea is to let each t-weakly dominating point be shared by as many points on s_1 as possible. Thus, we consider the following greedy approach.

1. Mark all points on s_1 as non-dominated.
2. Starting from the first non-dominated point on s_1 (initially it is a_1), walk along s_1 until encounter the first point p_i, whose interval $I(p_i)$ overlaps at only one point, say q_i, with the common intersection of the intervals of all visited but non-dominated points.
3. Select q_i as a weakly dominating point, and mark all points visited in Step 2 as dominated points.
4. Keep walking along s_1 and marking points as dominated until the encountered point cannot be dominated by q_i.
5. Repeat Steps 2-4 until all points are dominated.

The following lemma shows that the above procedure minimizes the total number of points t-weakly dominating s_1.

Lemma 3. *The set of t-weakly dominating points selected by the above procedure has the minimum size among all sets of points t-weakly dominating s_1.*

Proof. We prove by contradiction. Let Q_g be the set of t-weakly dominating points chosen by the above greedy procedure. Suppose that there exists another set Q_o of t-weakly dominating points which has smaller size. We sort both sets along s_1. Let the set of sorted points in Q_g be $\{g_1, g_2, \cdots, g_k\}$ and the set of sorted points in Q_o be $\{o_1, o_2, \cdots, o_j\}$ with $j < k$. By the greedy choice, g_1 is the rightmost point that can t-weakly dominate the left endpoint of s_1. Thus o_1 either coincides g_1 or is to its left. Now consider the two intervals t-weakly dominated by o_1 and g_1, say I_{o_1} and I_{g_1}. Step 4 of the greedy approach guarantees that I_{o_1}'s right endpoint either coincides or is to the left of I_{g_1}'s right endpoint. Consequently, by a similar argument, o_i either coincides or is to the left of g_i for all $1 \leq i \leq j$. Since o_j is the rightmost t-weakly dominating point on s_1, it t-weakly dominates all points to its right. Therefore g_j can also t-weakly dominate all points to its right as it coincides or is to the right of o_j. Thus, the t-weakly dominating points $g_{j+1}, \cdots, g_k$ are not necessary. This contradicts the stop condition of the greedy procedure. □

Lemma 3 implies the following Corollary.

Corollary 1. *The set of t-weakly dominating points selected by the greedy procedure lower-bounds the number of Steiner points on s_1 in an optimal solution.*

3.3 Parameterization

The greedy procedure, although minimizes the size of the t-weakly dominating set, cannot be directly implemented as it requires to check an infinite number of points on s_1. To efficiently implement it, we consider the following parameterization.

Let m be the parameter of p_1 in its convex combination of the two endpoints of s_1, i.e. $p_1 = (1-m)a_1 + mb_1$, for $m \in [0, 1]$. Let $L_{1,2}(m)$ and $R_{1,2}(m)$ be the functions defining the positions of e_{1l} and e_{1r} (respectively) on s_1, i.e. $L_{1,2}(m) = m - |p_1e_{1l}|/|a_1b_1|$ and $R_{1,2}(m) = m + |p_1e_{1r}|/|a_1b_1|$.

Consider the two functions when m increases from 0 to 1, it is possible that the beginning part of $L_{1,2}(m)$ always has value 0 (because $e_{1l} = a_1$), and the ending part of $R_{1,2}(m)$ always has value 1 (because $e_{1r} = b_1$). To simplify the discussion in this section, from now on we focus on the remaining part of the two functions, and assume that $|p_1e_{1l}| = \frac{t^2-1}{2(t-\cos\theta)}|p_1p_2|$ and $|p_1e_{1r}| = \frac{t^2-1}{2(t+\cos\theta)}|p_1p_2|$.

By Lemma 2, we know that, for each fixed m, $L_{1,2}(m)$ (or $R_{1,2}(m)$) is the maximum (or minimum) of $O(1)$ values with each corresponding to the position of e_{1l} (or e_{1r}) at a fixed θ value. Let $\Theta = \{\theta_1, \theta_2\}$ be the real roots of $f_l'(\theta) = 0$ (or $f_r'(\theta) = 0$) that allow $f_l(\theta)$ (or $f_r(\theta)$) to achieve its maximum. Since Θ depends only on

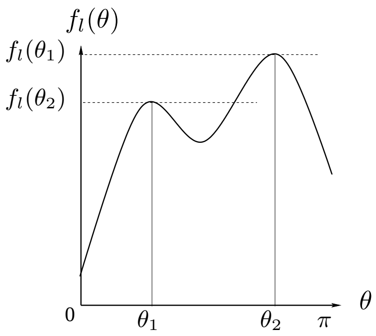

Fig. 3. An illustration of the function $(t - \cos\theta)\cos(\alpha - \theta)$.

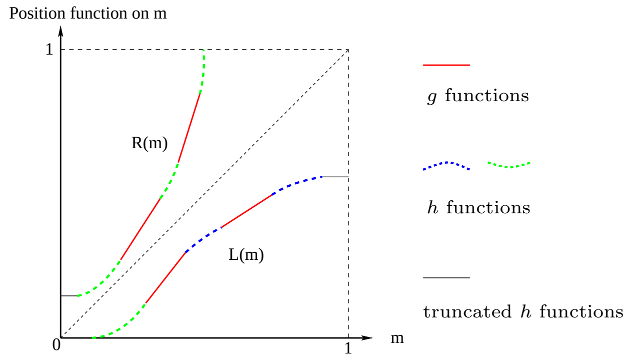

Fig. 4. An illustration of $L_{1,2}(m)$ and $R_{1,2}(m)$.

the input segments and t, it is the same for any $p_1 \in s_1$ and can be computed in advance. Let $\theta_a(m)$ and $\theta_b(m)$ be the two angles $\angle a_1p_1a_2$ and $\angle a_1p_1b_2$ respectively (i.e. when p_2 is at the two endpoints of s_2). Notice that, $\theta_b(m) \geq \theta_a(m)$ by our definition. Thus we have $\frac{t^2-1}{2(t-\cos\theta_b(m))} \leq \frac{t^2-1}{2(t-\cos\theta_a(m))}$ and $\frac{t^2-1}{2(t+\cos\theta_b(m))} \geq \frac{t^2-1}{2(t+\cos\theta_a(m))}$. Therefore, the position of e_{1l} depends on $\theta_b(m)$ and that of e_{1r} depends on $\theta_a(m)$.

$L_{1,2}(m)$ (or $R_{1,2}(m)$) can be viewed as the the upper (or lower) envelope of up to three functions, $g_i^l(m)$ (or $g_i^r(m)$), $1 \leq i \leq 2$, and $h^l(m)$ (or $h^r(m)$), where $g_i^l(m)$ (or $g_i^r(m)$) is the function of e_{1l} (or e_{1r}) when $\theta = \theta_i \in \Theta$, $h^l(m)$ is the function of e_{1l} when $\theta = \theta_b(m)$, and $h^r(m)$ is the function of e_{1r} when $\theta = \theta_a(m)$. The following lemma shows some property of $g_i^l(m)$ and $g_i^r(m)$.

Lemma 4. *Each $g_i^l(m)$ (or $g_i^r(m)$), $1 \leq i \leq 2$, is a linear function of m.*

Proof. From the previous discussion, we know that $|p_1e_{1l}| = \frac{(t^2-1)d(p_1,s_2)}{2(t-\cos\theta)\cos(\alpha-\theta)}$. When θ is fixed to be the real root $\theta_i \in \Theta$ that maximizes $f_l(\theta)$, $|p_1e_{1l}|$ is a linear function of $d(p_1, s_2)$. Since $d(p_1, s_2)$ is the distance from p_1 to s_2's supporting line, clearly it is a linear function of m. Thus $g_i^l(m)$ is a linear function of m. The case for $g_i^r(m)$ can be proved similarly. □

For $\theta_a(m)$ and $\theta_b(m)$, since their values depend on the position of p_1 on s_1, they are not constants and their corresponding functions, $h^r(m)$ and $h^l(m)$, are non-linear. The following lemma shows some property of them.

Lemma 5. *Each of $h^l(m)$ and $h^r(m)$) is either a monotone function or the concatenation of a monotonically increasing function and a monotonically decreasing function.*

Proof. We show this lemma only for $h^l(m)$, and $h^r(m)$ can be similarly proved. Let r_0 be the foot of perpendicular from b_2 to s_1, and m_0 be the parameter of r_0 in its affine combination of a_1 and b_1. Then we have $h^l(m) = m - \frac{t^2-1}{2(t-\cos\theta_b(m))}\frac{|p_1p_2|}{|a_1b_1|} = m - \frac{(t^2-1)(m-m_0)}{2(t-\cos\theta_b(m))\cos\theta_b(m)}$. Its derivative is

$$(h^l(m))' = \frac{2t^2\cos^2\theta_b(m) - (t^3+3t)\cos\theta_b(m) + t^2 + 1}{2(t-\cos\theta_b(m))^2}.$$

Let $x = \cos\theta_b(m)$. We get $\frac{2t^2x^2-(t^3+3t)x+t^2+1}{2(t-x)^2} = t^2 - \frac{3t^3-3t}{2(t-x)} + \frac{(t^2-1)^2}{2(t-x)^2} = \frac{1}{2}(\frac{t^2-1}{t-x} - t)(\frac{t^2-1}{t-x} - 2t)$. The function has two roots, $x_1 = \frac{1}{t}$ and $x_2 = \frac{1}{2}(t + \frac{1}{t})$. Since $0 < x_1 < 1$ and $x_2 > 1$, obviously x_2 is not feasible since $-1 \leq x = \cos\theta_b(m) \leq 1$. The derivative of $h^l(m)$ is decreasing on x when $x \leq \frac{1}{t}$; increasing otherwise. This shows that the function $h^l(m)$ can be partitioned into at most two pieces, with the first one monotonically increasing on $\theta_b(m)$ and the second monotonically decreasing on $\theta_b(m)$. □

Lemma 6. *$h^l(m)$ is an increasing function on $\theta_b(m)$ when $\theta_b(m) \geq \pi/2$; $h^r(m)$ is a decreasing function on $\theta_a(m)$ when $\theta_a(m) \leq \pi/2$. Further, the derivatives of $h^l(m)$ and $h^r(m)$ both have a minimum value of $\frac{1}{2}(1 + \frac{1}{t^2})$.*

3.4 Truncating the h Functions

Lemma 5 shows that each of $h^l(m)$ and $h^r(m)$ could be a bitonic function (i.e., an increasing function followed by a decreasing function). Taking $h^l(m)$ as an example, the geometric meaning of this lemma is that, as p_1

moves along s_1 from left to right, the left endpoint of $I(p_1, s_2)$ calculated by fixing p_2 at b_2 might move from right to left at some positions.

In such a scenario, let p_i be the first such point on s_1 that the left endpoint e_{il} of interval $I(p_i, s_2)$ starts to move "from right to left". By Lemma 5, every point p_j on s_1 that is to the right of p_i, will have its e_{jl} (calculated by the same function $h^l(m)$) located to the left of e_{il}. This means that if we replace the e_{jl} by e_{il} for every such p_j, we can make $h^l(m)$ a completely monotone function. Geometrically, this seems to truncate the tail of $h^l(m)$ (i.e., the decreasing piece) and replace it with a constant function.

Based on the above lemmas and analysis, we have the following Corollary for $L_{1,2}(m)$ and $R_{1,2}(m)$. See Figure 4 for an illustration.

Corollary 2. *$L_{1,2}(m)$ (or $R_{1,2}(m)$) is monotonically increasing after truncating $h^l(m)$ (or $h^r(m)$), and can be partitioned into $O(1)$ pieces with each being either a portion of the $h^l(m)$ (or $h^r(m)$) function or the $g_i^l(m)$ (or $g_i^r(m)$) function.*

3.5 Computing *t*-Weakly Dominating Set

Now, we are ready to discuss our approach for selecting the t-weakly dominating set. Clearly the two functions $L_{1,2}(m)$ and $R_{1,2}(m)$ together form a "band" $B_{1,2}$ (see Figure 4). A horizontal line segment located within the band represents the interval $I(p_1, s_2)$ of a point p_1 on s_1 determined by the corresponding e_{1l} and e_{1r} positions. An interval cover $IC_{1,2}$ for $B_{1,2}$ is a set of horizontal intervals inside $B_{1,2}$ so that the union of each interval's vertical projection covers the domain of m (i.e., covers the interval $[0, 1]$). It is easy to see that an interval cover for $B_{1,2}$ corresponds to a t-weakly dominating set for s_1. This is because each horizontal interval uniquely determines a point on s_1. Since every point on s_1 is covered by some interval, it is thus t-weakly dominated by the point corresponding to the interval. Thus, to find a t-weakly dominating set of minimum size for s_1, it is sufficient to find a size-minimized interval cover for $B_{1,2}$.

Let $L(m) = \{l_1, l_2, \ldots, l_{n_1}\}$ be the 2-D curve corresponding to the function of $L_{1,2}(m)$, where each segment (l_i, l_{i+1}) corresponding a piece of the h or g function in $L_{1,2}(m)$. Similarly we have $R(m) = \{r_1, r_2, \ldots, r_{n_2}\}$. The following algorithm gives an efficient solution to compute an interval cover.

1. Starting from r_1, shoot a horizontal ray to the right, and let v_1 be the intersection of the ray and $L(m)$.
2. Shoot a vertical ray upwards from v_1, and let v_2 be the intersection of the ray with $R(m)$.
3. Starting from v_2, repeat the above steps until all the $L(m)$ and $R(m)$ are horizontally covered.
4. Return the set of the horizontal segments as the cover.

Figure 5 shows an example of the minimum interval cover. The ray shooting operation in the above algorithm can be easily implemented by using the fact that the vertices of $L(m)$ and $R(m)$ are in sorted orders.

Lemma 7. *The above algorithm computes an interval cover of minimum size in $O(|IC| + n_1 + n_2)$ time, where $|IC|$ is the size of the interval cover and n_1 and n_2 are the number of vertices in $L(m)$ and $R(m)$ respectively.*

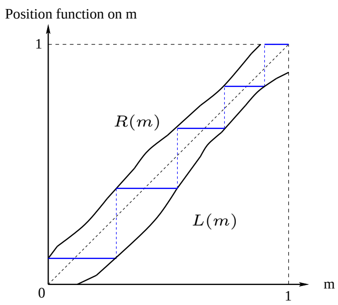

Fig. 5. An example of the interval cover.

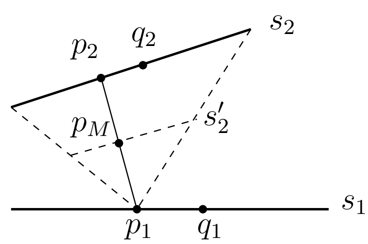

Fig. 6. Imaginary Steiner point p_M.

3.6 Imaginary Steiner Points

A t-weakly dominating set Q in general is not sufficient to construct a t-spanner for S. Actually, even a complete graph built on Q does not yield a t-spanner for S. This is because weak domination is based on the assumption that there are an infinite number of Steiner points on the other segment(s), when computing weakly dominating points for one segment. However, once the assumption is removed, points from two segments may no longer have a dominating pair.

Let p_1 and p_2 be two arbitrary points on segments s_1 and s_2 respectively, and q_1 and q_2 be their t-weakly dominating points. Ideally, the path $p_1 \to q_1 \to q_2 \to p_2$ should be a t-spanner path for p_1 and p_2, i.e. $|p_1q_1| + |q_1q_2| + |q_2p_2| \leq t|p_1p_2|$. Due to the weak domination, we only know $|p_1q_1| + |q_1q_2| \leq t|p_1q_2|$ and $|p_2q_2| + |q_2q_1| \leq t|p_2q_1|$. If we add the two together, we have $|p_1q_1| + |q_1q_2| + |q_2p_2| \leq t|p_1q_2| + t|p_2q_1| - |q_1q_2|$. Thus, to make q_1 and q_2 be a t-dominating pair for p_1 and p_2, we need to have $t|p_1q_2|+t|p_2q_1|-|q_1q_2| \leq t|p_1p_2|$.

Our main idea for solving this problem is to introduce an *imaginary* Steiner point p_M, which is the median of $\overline{p_1p_2}$, and use this imaginary Steiner point to help determining the dominating points for p_1 and p_2 (see Figure 6). More specifically, when computing the interval $I(p_1, s_2)$, we assume that there exists a Steiner point p_M at the median of $\overline{p_1p_2}$ for every possible choice of p_2 on s_2. These imaginary Steiner points form an imaginary "Steiner" segment s_2' for every p_1 (See Figure 6 for an example). Thus instead of computing $I(p_1, s_2)$ directly, we can calculate $I(p_1, s_2') = \cap_{p_M \in s_2'} I_S(p_M, p_1, p_M)$. ($I(p_2, s_1')$ can be defined similarly.) Steiner point q_1 in such $I(p_1, s_2')$ is therefore a t-weakly dominating point for p_1 and $\forall p_M \in s_2'$.

The following lemma shows that by using the imaginary Steiner point, we are able to obtain a t-dominating pair for p_1 and p_2.

Lemma 8. *Let $q_1 \in I(p_1, s_2')$ and $q_2 \in I(p_2, s_1')$ be t-weakly dominating Steiner points for p_1 and p_2 with respect to their imaginary Steiner segments respectively. Then, q_1 and q_2 are a t-dominating pair for p_1 and p_2.*

Proof. Let p_M be the median of line segment $\overline{p_1p_2}$. By the definition of imaginary Steiner segment, p_M is contained in both s_2' and s_1'. Since q_1 t-weakly dominates p_1 with respect to s_2', it t-weakly dominates p_1 and p_M. Thus $|p_1q_1| + |q_1p_M| \leq t|p_1p_M|$. Similarly, we have $|p_2q_2| + |q_2p_M| \leq t|p_2p_M|$. Adding the two together, we have $t|p_1p_2| = t(|p_1p_M| + |p_2p_M|) \geq |p_1q_1| + |p_2q_2| + |q_1p_M| + |q_2p_M|$. By triangle inequality, we have $|q_1p_M| + |q_2p_M| \geq |q_1q_2|$. Thus, we have $t|p_1p_2| \geq |p_1q_1| + |q_1q_2| + |q_2p_2|$. □

3.7 From Weakly Dominating Set to Dominating Set

The computation for $I(p_1, s_2')$ is almost the same as that for $I(p_1, s_2)$. The only difference is that $|p_1p_2|$ is replaced by $|p_1p_M|$ in the functions (given in Lemma 1) for e_{1l} and e_{1r}. For the new functions $\bar{L}_{1,2}(m)$ and $\bar{R}_{1,2}(m)$ for e_{1l} and e_{1r}, we have the following lemma.

Let m_1 be the parameter of p_1 when e_{1l} is at a_1 (i.e., $L_{1,2}(m_1) = 0$ and $\forall m_1 < m \leq 1$, $L_{1,2}(m) > 0$), and m_2 be the parameter of p_1 when e_{1r} coincides with b_1 (i.e. $R_{1,2}(m_2) = 1$ and $\forall 0 \leq m < m_2$, $R_{1,2}(m) < 1$).

Lemma 9. *$\bar{L}_{1,2}(m) = (m + L_{1,2}(m))/2$ for $m_1 \leq m \leq 1$; $\bar{R}_{1,2}(m) = (m + R_{1,2}(m))/2$ for $0 \leq m \leq m_2$.*

Proof. Since p_M is the median of $\overline{p_1p_2}$, $|p_1p_M| = |p_1p_2|/2$. The Lemma follows from the definition of the L and R functions. □

Let H_L be the longest maximal horizontal line segment within $B_{1,2}$, and H_S be the shortest maximal horizontal line segment within $B_{1,2}$.

Lemma 10. $|H_L|/|H_S| \leq \frac{1}{(t-1)\epsilon}$*, where ϵ is the relative separation ratio of S.*

Proof. A horizontal line segment within $B_{1,2}$ satisfying $R_{1,2}(\lambda_1) = L_{1,2}(\lambda_{1'})$ represents a t-weakly dominating point q determined by two points p_1 and $p_{1'}$ on s_1 (corresponding to parameter λ_1 and $\lambda_{1'}$ respectively, see Figure 7 for an example.). For p_1, assume that the point on the segment s_2 that allows e_{1r} to achieve its minimum is p_2. For $p_{1'}$, assume that the point on s_2 that allows $e_{1'l}$ to achieve its minimum is $p_{2'}$. Since q is the t-weakly dominating point determined by such minimum p_1e_{1r} and $p_{1'}e_{1'l}$, they (q, e_{1r} and $e_{1'l}$) coincide at one point. Then we have $|p_1e_{1r}|+|p_2e_{1r}| = |p_1q|+|p_2q| = t|p_1p_2|$ and $|p_{1'}e_{1'l}|+|p_{2'}e_{1'l}| = |p_{1'}q|+|p_{2'}q| = t|p_{1'}p_{2'}|$ (equalities are achieved because p_1e_{1r} and $p_{1'}e_{1'l}$ are minimum). By triangle inequality, $|p_2q| \leq |p_1q|+|p_1p_2|$, $|p_{2'}q| \leq |p_{1'}q|+|p_{1'}p_{2'}|$. Hence, $|p_1q|+|p_{1'}q| \geq (t-1)(|p_1p_2|+|p_{1'}p_{2'}|)/2$. As we described before, the segments in S are well separated with relative separation ratio ϵ, therefore $|p_1p_2| \geq \epsilon|a_1b_1|$ and $|p_{1'}p_{2'}| \geq \epsilon|a_1b_1|$. Thus we have $|H_S| = (|p_1q| + |p_{1'}q|)/|a_1b_1| \geq (t-1)\epsilon$. The lemma follows from the fact that $|H_L| \leq 1$. □

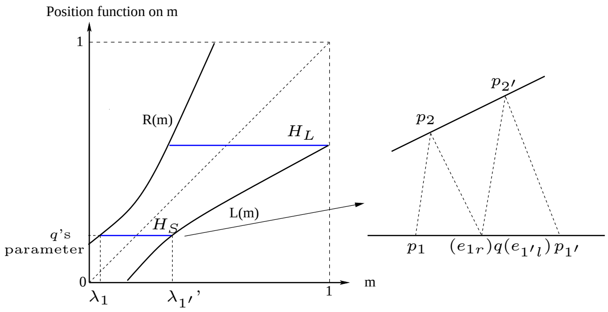

Fig. 7. An example of H_L and H_S.

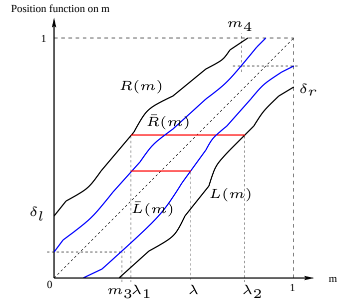

Fig. 8. An example of $B_{1,2}$ and $\bar{B}_{1,2}$.

$\bar{L}_{1,2}(m)$ and $\bar{R}_{1,2}(m)$ form a "shrunk" band $\bar{B}_{1,2}$. A minimum-sized interval cover $\overline{IC}_{1,2}$ can also be found within $\bar{B}_{1,2}$ by using the algorithm given in Section 3.5. We have the following lemma regarding the ratio $\beta = |\overline{IC}_{1,2}|/|IC_{1,2}|$.

Let $\delta_l = R_{1,2}(0)$ and $\delta_r = L_{1,2}(1)$. By Lemma 9, we have $\bar{R}_{1,2}(0) = \delta_l/2$ and $\bar{L}_{1,2}(1) = (1+\delta_r)/2$. Let m_3 and m_4 be the parameter of p_1 satisfying $\bar{L}_{1,2}(m_3) = \delta_l/2$ and $\bar{R}_{1,2}(m_4) = (1+\delta_r)/2$ respectively.

Lemma 11. *If $\delta_r \geq \delta_l$, then $\beta \leq \dfrac{1}{(t-1)\epsilon} \cdot \min\{1+1/(2\min_{m_3<m<1} \bar{L}'_{1,2}(m)-1), 1+1/(2\min_{0<m<m_4} \bar{R}'_{1,2}(m)-1)\}$, where $\bar{L}'_{1,2}(m)$ and $\bar{R}'_{1,2}(m)$ are the derivative of $\bar{L}_{1,2}(m)$ and $\bar{R}_{1,2}(m)$ respectively.*

Proof. We prove $\beta \leq 1 + 1/(2\min_{m_3<m<1} \bar{L}'_{1,2}(m) - 1)$ first. Let H be any maximal horizontal line segment within $B_{1,2}$ with endpoints $(\lambda_1, R_{1,2}(\lambda_1))$ and $(\lambda_2, L_{1,2}(\lambda_2))$. Obviously, $R_{1,2}(\lambda_1) = L_{1,2}(\lambda_2)$. Starting at point $(\lambda_1, \bar{R}_{1,2}(\lambda_1))$, there exists a maximal horizontal line segment $\bar{H}$ within $\bar{B}_{1,2}$ with right endpoint $(\lambda, \bar{L}_{1,2}(\lambda))$. Clearly,$\lambda \in [\lambda_1, \lambda_2]$ and $\bar{R}_{1,2}(\lambda_1) = \bar{L}_{1,2}(\lambda)$. Assume that we can extend $L_{1,2}(m)$ such that Lemma 9 can be applied to all $m \in [0,1]$ (i.e., allow e_{1l} to be placed to the left of a_1; Note that this will not affect bounding β). Hence by Lemma 9, $(\lambda_1 + R_{1,2}(\lambda_1))/2 = (\lambda + L_{1,2}(\lambda))/2$, i.e., $R_{1,2}(\lambda_1) - L_{1,2}(\lambda) = \lambda - \lambda_1$. Since $R_{1,2}(\lambda_1) = L_{1,2}(\lambda_2)$, we get $L_{1,2}(\lambda_2) - L_{1,2}(\lambda) = \lambda - \lambda_1$. By Mean Value Theorem (Note that the function $L_{1,2}(m)$ is smooth when $L_{1,2}(m) > 0$), we have $L'_{1,2}(\lambda')(\lambda_2 - \lambda) = \lambda - \lambda_1$, where $\lambda < \lambda' < \lambda_2$. Thus $|H|/|\bar{H}| = (\lambda_2 - \lambda_1)/(\lambda - \lambda_1) = 1 + (\lambda_2 - \lambda)/(\lambda - \lambda_1) = 1 + 1/L'_{1,2}(\lambda') = 1 + 1/(2\bar{L}'_{1,2}(\lambda') - 1)$. To bound β, consider each interval $\bar{H}_i \in \overline{IC}_{1,2}$. Corresponding to $\bar{H}_i$, there is a maximal horizontal segment H_i within $B_{1,2}$. If we consider all possible $|\bar{H}_i|$ and find the minimum, say $|\bar{H}_i| = \tilde{\lambda} - \tilde{\lambda}_1$ with $\tilde{\lambda}_1 < \tilde{\lambda} < \tilde{\lambda}_2$, we have $|\overline{IC}_{1,2}| \leq \frac{1}{\tilde{\lambda} - \tilde{\lambda}_1}$. For the corresponding $|H_i|$ within $B_{1,2}$, we have $|H_i| = \tilde{\lambda}_2 - \tilde{\lambda}_1 \geq |H_S|$. Consider each interval $H_j \in IC_{1,2}$, we have $|H_j| \leq |H_L|$ such that $|IC_{1,2}| \geq \frac{1}{|H_L|}$. Hence, $\beta = \dfrac{|\overline{IC}_{1,2}|}{|IC_{1,2}|} \leq \dfrac{|H_L|}{\tilde{\lambda} - \tilde{\lambda}_1} \leq \dfrac{\tilde{\lambda}_2 - \tilde{\lambda}_1}{\tilde{\lambda} - \tilde{\lambda}_1} \times \dfrac{|H_L|}{|H_S|}$. Therefore, by Lemma 10, we can bound the ratio as $\beta \leq \max_{\lambda_1<\lambda<\lambda_2} \dfrac{(\lambda_2 - \lambda_1)}{(\lambda - \lambda_1)} \times \dfrac{|H_L|}{|H_S|} \leq \dfrac{1}{(t-1)\epsilon}(1 + \dfrac{1}{2\min_{\lambda<\lambda'<\lambda_2} \bar{L}'_{1,2}(\lambda') - 1})$. Since $\bar{R}_{1,2}(0) = \bar{L}_{1,2}(m_3)$, $\beta \leq \dfrac{1}{(t-1)\epsilon}(1 + \dfrac{1}{2\min_{m_3<m<1} \bar{L}'_{1,2}(m) - 1})$.

The lemma follows as $\beta \leq \dfrac{1}{(t-1)\epsilon}(1 + \dfrac{1}{2\min_{0<m<m_4} \bar{R}'_{1,2}(m) - 1})$ can be similarly proved. □

Corollary 3. *If $\delta_r \geq \delta_l$, then $\beta \leq \dfrac{1}{(t-1)\epsilon} \times \min\{1 + 1/\min_{0<m<1} L'_{1,2}(m), 1 + 1/\min_{0<m<1} R'_{1,2}(m)\}$.*

Corollary 4. *Let $\beta_{[u,v]}$ be the ratio of $|\overline{IC}_{1,2}|$ over $|IC_{1,2}|$ in the interval $[u,v]$. Then, $\beta_{[u,v]} \leq \dfrac{1}{(t-1)\epsilon} \times \min\{1 + 1/\min_{u\leq m\leq v} L'_{1,2}(m), 1 + 1/\min_{u\leq m\leq v} R'_{1,2}(m)\}$, for all $0 \leq u < v \leq 1$.*

Now, we are ready to bound β in term of t and other constants.

Lemma 12. $\beta \leq \frac{1}{(t-1)\epsilon} \times \max\{1 + \frac{2f_l(\theta_2)}{2f_l(\theta_2) - (t^2-1)\cos\alpha}, 1 + \frac{2f_r(\theta_2)}{2f_r(\theta_2) - (t^2-1)\cos\alpha}, 3 - \frac{2}{t^2+1}\}$.

Proof. If $\delta_r \leq \delta_l$, one Steiner point is sufficient to t-weakly dominate s_1. Since the t-dominating interval of a point p_1 is always half of its t-weakly dominating interval, by choosing q, a_1 and b_1 as the Steiner points, they are sufficient to t-dominate s_1. In this case, we have $\beta \leq 3$.

Thus, from now on we assume $\delta_r > \delta_l$. We consider two cases based on the pieces of $L_{1,2}(m)$.

1. For the parts of $L_{1,2}(m)$ determined by the linear g_i functions, we have

$$\min L'_{1,2}(m) = \min_{\theta_i \in \Theta}\{1 - \frac{(t^2-1)\cos\alpha}{2(t-\cos\theta_i)\cos(\alpha-\theta_i)}\} = 1 - \frac{(t^2-1)\cos\alpha}{2f_l(\theta_2)}.$$

2. For the parts of $L_{1,2}(m)$ determined by $h^l(m)$, notice the fact that $\theta_b(m)$ decreases when m increases. Let m_b be the parameter of p_1 such that $\theta_b(m_b) = \pi/2$. Thus, $\theta_b(m) > \pi/2$, for $0 \le m < \min\{m_b, m_2\}$, and $\theta_b(m) < \pi/2$, for $\max\{0, m_b\} < m \le 1$. (Note that one of the two intervals in the above two inequalities might be empty.)
   (a) For $m \in [0, \min\{m_b, m_2\}]$, by Lemma 6 we have $\min L'_{1,2}(m) = \min \frac{dh^l}{dm} \ge \frac{1}{2}(1 + \frac{1}{t^2})$.
   (b) For $m \in [\max\{0, m_b\}, 1]$, we have two subcases.
      i. If $R_{1,2}(m)$ is determined by the linear g_i functions, we have $\min R'_{1,2}(m) = 1 - \frac{(t^2-1)\cos\alpha}{2f_r(\theta_2)}$.
      ii. If $R_{1,2}(m)$ is determined by $h^r(m)$, we have $\theta_a(m) \le \theta_b(m) \le \pi/2$. By Lemma 6, $\min R'_{1,2}(m) = \min \frac{dh^r}{dm} \ge \frac{1}{2}(1 + \frac{1}{t^2})$.

Combining all the cases, the lemma follows from Corollary 4. □

Notice that the ratio β in the above lemma is $O(1)$ since $f_l(\theta_2)$, $f_r(\theta_2)$, t and ϵ are all constants.

3.8 From Dominating Set To Strongly Dominating Set

By using the imaginary Steiner points mentioned above, we can find a minimum-sized t-dominating set for s_1 with respect to another input segment s_2. The idea of imaginary Steiner points can be easily extended to a set S of segments and generates the set of t-strongly dominating points on each input segment. The following lemma shows that to determine the t-strongly dominating points on s_1 we only need to consider a subset of the input segments.

For any two segments s_i and s_j in S, we say that s_i and s_j are weakly visible to each other if there exists a pair of points $p_i \in s_i$ and $p_j \in s_j$ such that p_i and p_j are visible to each other (i.e., the segment $\overline{p_i p_j}$ does not intersect the interior of any other input segment).

Lemma 13. *To compute a t-strongly dominating set of an arbitrary segment s_1, it is sufficient to consider only those segments weakly visible to it.*

The above lemma suggests us the following algorithm for computing a t-strongly dominating set for S.

1. For each segment $s_i \in S$, compute the set WV_i of segments which are weakly visible to s_i.
   (a) For each segment $s_j \in WV_i$, compute the g and h functions for s_i with respect to s_j.
   (b) Determine $L_{i,j}(m)$ and $R_{i,j}(m)$ by computing the upper and lower envelopes of the set of g and h functions.
2. Let $L_i(m)$ be the upper envelope of the set of $L_{i,j}(m)$ functions, and $R_i(m)$ be the lower envelope of the set of $R_{i,j}(m)$ functions.
3. Determine $\bar{L}_i(m)$ and $\bar{R}_i(m)$ with the help of the imaginary Steiner points.
4. Compute an interval cover $\overline{IC}_i$ for the band formed by $\bar{L}_i(m)$ and $\bar{R}_i(m)$.
5. For each interval in $\overline{IC}_i$, determine its corresponding t-strongly dominating Steiner point.

Firstly we show that the ratio β_i of $|\overline{IC}_i|$ to $|IC_i|$ is still bounded by a constant.

Consider s_i and the set of segments $WV_i = \{s_1, s_2, \ldots, s_{k_i}\}$ which are visible to s_i. For each pair of segments s_i and s_j, $s_j \in WV_i$, we have a pair of parameterized functions $L_i(m)$ and $R_i(m)$. It is not difficult to see that both $L_i(m)$ and $R_i(m)$ are piecewise smooth.

Lemma 14. *Let $\beta_{i[u,v]}$ be the ratio of $\overline{IC}_i$ to IC_i in the interval $[u, v]$. Then,*

$$\beta_{i[u,v]} \le \frac{1}{(t-1)\epsilon} \times \min\{1 + 1/\min_{0<m<1} L_i(m), 1 + 1/\min_{0<m<1} R_i(m)\}.$$

Proof. Notice that if $L_{i,j}(\lambda) < L_{i,k}(\lambda)$, then $\bar{L}_{i,j}(\lambda) < \bar{L}_{i,k}(\lambda)$. Same property holds for the $R_{i,j}(m)$ functions. This property ensures that the ideas used in the proof of Lemma 11 can still be applied on each smooth piece. □

Lemma 15. *Let* $K_1 = \max_{\forall s_i, s_j \in S} \dfrac{2f_l(\theta_2)}{2f_l(\theta_2) - (t^2-1)\cos\alpha}$ *and* $K_2 = \max_{\forall s_i, s_j \in S} \dfrac{2f_r(\theta_2)}{2f_r(\theta_2) - (t^2-1)\cos\alpha}$. *Then,* $\beta_i \leq \frac{1}{(t-1)\epsilon} \times \max\{1+K_1, 1+K_2, 3-\frac{2}{t^2+1}\}$.

Proof. The lemma follows from Lemma 14 and a similar argument in Lemma 12.

Lemma 16. *The above algorithm computes a t-strongly dominating set for S in* $O(|Q|+n^2\log n)$ *time, where* $|Q|$ *is the size of the computed t-strongly dominating set.*

Proof. In [13], it has shown that two segments s_i and s_j are weakly visible from each other if and only if one endpoint of s_i or s_j sees a point on the other segment or there is at least one edge in the extended visibility graph [14] of S which is intersected by both s_1 and s_2. To check the first case, for an endpoint of s_i and another segment s_j, it takes $O(n \log n)$ time by solving the point location problem. The second case can be checked for all segments in S when computing the extended visibility graph in $O(n^2)$ time [14]. Therefore, computing all pairs of the weakly visible segments for S takes $O(n^2 \log n)$ time. The envelops of $L_i(m)$ and $R_i(m)$ can be computed in $O(|WV_i| \log |WV_i|)$ time using a plane sweep algorithm. Thus the total time for this step is $O(n^2 \log n)$. And for each interval covering problem solved in the algorithm, the running time is linear in terms of the interval cover's size. The total time is hence $O(|Q|)$. □

With the previous lemmas, we have the following theorem.

Theorem 1. *For a set S of n disjoint 2-D segments with constant relative separation ratio, a set of t-strongly dominating Steiner points whose size is an* $O(1)$*-approximation of (the size of) the optimal solution can be computed in* $O(|Q| + n^2 \log n)$ *time, where* $|Q|$ *is the size of the set of Steiner points.*

4 Minimizing the Size of the Segment Spanner

Previous section shows how to obtain a small set Q of (t-strongly dominating) Steiner points for a set of segment S. The size of Q is no more than $\beta \times |OPT|$, where OPT is the set of optimal Steiner points and β is a function of the stretch factor t. To complete the construction of the segment spanner, bridges are added between the selected Steiner points using some existing spanner algorithms for points. The spanner of the Steiner points introduces another stretch factor, say t_2. Let the stretch factor in the Steiner point selection be t_1. Thus, to ensure the resulting segment spanner have a stretch factor t, we need to have $t = t_1 \times t_2$. An interesting question is how to select t_1 and t_2 so that the size of spanner is minimized.

To answer this question, consider an optimal solution $\mathcal{O}$. Let N be the number of Steiner points in $\mathcal{O}$, and M be the number of bridges in $\mathcal{O}$. It is easy to see that $\mathcal{O}$ contains: i) $2n$ endpoints and N Steiner points; ii) $N+n$ subsegments and M bridges. Therefore $|\mathcal{O}| = 3n + 2N + M$. Recently, [15] shows that given a set of n_0 points S, in the worst case, any graph with $n_0 - 1 + k$ edges on S has dilation at least $\frac{2n_0}{\pi(k+1)}$. That is to say, as a t-spanner for the input segment, $\mathcal{O}$ needs to contain at least $M = 2N/\pi t + N - 2$ edges. Therefore $|\mathcal{O}| \geq 3n + 3N + 2N/\pi t - 2$.

Consider a spanner $\mathcal{A}$ generated by our algorithm. Let M' be the number of bridges in $\mathcal{A}$. $\mathcal{A}$ contains: i) $2n$ endpoints and βN Steiner points; ii) $\beta N + n$ subsegments and M' bridges. Therefore $|\mathcal{A}| = 3n + 2\beta N + M'$. Again, in [15], the authors give an algorithm that for a set of n_0 points S, finds a spanner on S with at most $n_0 - 1 + k$ edges and dilation $O(\frac{n}{k+1})$. Thus if using the algorithm in [15], we have $t_2 = O(\frac{\beta N}{M' - \beta N + 2}) = \frac{c \cdot \beta N}{M' - \beta N + 2}$. Therefore $M' = c \cdot \beta N / t_2 + \beta N - 2$ and $|\mathcal{A}| = 3n + 3\beta N + c \cdot \beta N / t_2 - 2$. To achieve the best approximation ratio, we can minimize the ratio $\frac{|\mathcal{A}|}{|\mathcal{O}|} \leq \frac{3n + 3\beta N + c \cdot \beta N / t_2 - 2}{3n + 3N + 2N/\pi t - 2}$. Since β is a function of t_1 and $t_1 t_2 = t$, we can choose t_1 and t_2 to minimize $|\mathcal{A}|/|\mathcal{O}|$.

5 Constructing t-Spanner for Rectilinear Segments Under L_1 Distance

In this section we consider a special case of the segment spanner problem in which a better approximation can be obtained.

Assume that the input is a set S of rectilinear segments, and the distance function is based on the L_1 distance (i.e., the Manhattan distance). We have the following theorem. The details of the algorithm and the proof are left for the full paper.

Theorem 2. *Given a set of n rectilinear segments, a set of t-strongly dominating set of Steiner points with size no more than* $2 \times |OPT|$ *can be computed in* $O(|Q| + n^2 \log n)$ *time.*

Notice that in this theorem, the segments are not required to be well separated.

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