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# The Carleman type estimates and non-well-posed problems.

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## 1 Introduction

Let  $\Omega$  be a connected open set in  $\mathbb{R}^n$ , and let  $P = P(x, D)$  be a differential operator of order  $m$  in  $\Omega$  with principal symbol  $p$ . Let  $\phi : \bar{\Omega} \rightarrow \mathbb{R}$  be a  $C^\infty$  function, with  $\nabla\phi(x) \neq 0$ ,  $x \in \Omega$  and which is strongly pseudo convex (this is a convexity property relatively to  $p$ .) We say that the Carleman type estimate holds for  $P$  if there exists a constant  $K > 0$  such that

$$\sum_{|\alpha| < m} \tau^{2(m-|\alpha|)-1} \int_{\Omega} |D^\alpha u|^2 e^{2\tau\phi} dx \leq K \int_{\Omega} |P(x, D)u|^2 e^{2\tau\phi} dx \quad (1)$$

$$\forall u \in C_0^\infty(\Omega), \quad \tau > 0 \text{ large enough.}$$

Estimates of this form were first used by Carleman in work on unique continuation property for second order elliptic operators in  $\mathbb{R}^2$ . Here  $P$  is said to have the unique continuation property if the following holds: Suppose  $u$  solves  $P(x, D)u = 0$  on  $\Omega$  and  $u = 0$  on a empty open set in  $\Omega$ . Then,  $u$  vanishes identically in  $\Omega$ .

This property is equivalent to uniqueness in the Cauchy problem for any smooth hypersurface.

The Carleman type estimates are established under various assumptions on  $P(x, D)$  and have a large field of applications:

1. Unique continuation property and uniqueness of Cauchy problem. (see [3], [4], [6], [9], [10])

2. Spectral properties of Schrödinger operator.(see [12])
3. Generic properties of nonlinear elliptic equations. (see [13]).
4. Stability of (non-well-posed) Cauchy problem (see [1]).
5. Identifiability of spatially-varying coefficients in partial differential equations.  
(see [1], [2])

The aim of this paper is to present new results concerning the last two subjects. In section 2 we establish an abstract analogue of Carleman estimates, which is an extension of Bukhgeim's result ([1]). In section 3 we apply it to the uniqueness question and identifiability of coefficients for the initial-boundary value problems for some (nonlinear) partial differential equations.

## 2 Stability estimates

Let  $H$  be a complex (or real) Hilbert space, the scalar product and the norm in  $H$  being denoted by  $\langle \cdot, \cdot \rangle$  and  $\| \cdot \|$ , respectively. Let  $M(t)$  and  $A(t)$  be linear operators whose domains are dense subspaces in  $H$  and are possibly changeable in  $t$  for  $t \in [0, T]$ .

The subscript  $t$  denotes differentiation with respect to  $t$ .

In Theorem 1 stated below, we assume the following.

- (A1) For every  $t \in [0, T]$   $M(t)$  is a selfadjoint operator.
- (A2)  $M(t)$  and  $A(t)$  are strongly continuous and weakly differentiable with respect to  $t$ .
- (A3) Let

$$D(P) = \left\{ u : [0, T] \rightarrow H \mid \begin{aligned} &u(t), u_t(t) \in D(M(t)), \\ &u(t) \in D(A(t)) \text{ for each } t \in [0, T], \\ &M(\cdot)u(\cdot) \in C^1([0, T]; H) \text{ and } A(\cdot)u(\cdot) \in C([0, T]; H) \end{aligned} \right\}.$$

and

$$Z = \left\{ u : [0, T] \rightarrow H \mid \begin{aligned} &u(t) \in D(A(t) + A^*(t)), \text{ for each } t \in [0, T] \\ &\{A(\cdot) + A^*(\cdot)\}u(\cdot) \in C([0, T]; H) \end{aligned} \right\}$$

There exists a linear subspace  $\mathbf{D}$  dense in  $D(P) \cap Z$  such that, setting  $D(t) = \mathbf{D} \cap (\{t\} \times H) \subset ([0, T] \times H)$ ,

(a) There exists a positive constant  $C_1$  such that

$$\|M_t(t)v\| \leq C_1 \|M(t)v\|, \quad \forall v \in D(t).$$

(b)  $M(t)$  and  $A(t) + A^*(t)$  commute each other on  $D(t)$ , that is, for  $v \in D(t)$   $(A(t) + A^*(t))v \in D(M(t))$  and  $M(t)v \in D(A(t) + A^*(t))$ , we have

$$M(t)(A(t) + A^*(t))v = (A(t) + A^*(t))M(t)v.$$

(c) There exist positive constants  $C_j$  ( $j = 2, 3, 4$ ) such that

$$\|(A(t) - A^*(t))v\| \leq C_2 \|M(t)v\| \quad \forall v \in D(t),$$

$$\|A^*(t)v\|^2 - \|A(t)v\|^2 \leq C_3 \|M(t)v\|^2 \quad \forall v \in D(t),$$

and

$$\|(A_t(t) + A_t^*(t))v\| \leq C_4 \|Mv\| \quad \forall v \in D(t).$$

We define the operator

$$P(t)u(t) = M(t)u_t - A(t)u(t) \quad \text{for } \forall u \in D(P).$$

For brevity we write

$$\|u\|_T = \|u\|_{L^2(0,T;H)} \quad \|u\|_{s,T} = \|e^{s\phi}u\|_T$$

where  $\phi = \phi(t)$  is a real-valued continuous function defined on  $[0, T]$  and  $s$  is an arbitrary nonnegative number.

The following theorem is an extension of abstract versions of Carleman's estimates for the Cauchy problems. (see Nirenberg [11], Bukhgeim [1]).

**Theorem 1** *Suppose that the assumptions (A1)-(A3) hold. Suppose that  $\phi \in C^2([0, T])$  satisfies*

$$\phi_t(t) \leq 0 \quad \forall t \in [0, T],$$

and

$$\phi_{tt}(t) + (C_1 + C_2)\phi_t(t) \geq \delta > 0 \quad \forall t \in [0, T].$$

Then, there exist positive constants  $s_0$  and  $C_5$  such that for all  $s \geq s_0$  and  $u \in D(P) \cap Z$

$$\begin{aligned} & s \|Mu\|_{s,T}^2 + \frac{1}{1 + s|\phi_t(0)|^2} (\|(A + A^*)v\|_{s,T}^2 + \|Mu_t\|_{s,T}^2) \\ & \leq C_5 (\|Pu\|_{s,T}^2 + [s\phi_t(t)e^{2s\phi(t)}\|M(t)u(t)\|^2 \\ & \quad + e^{2s\phi(t)}\langle (A(t) + A^*(t))u(t), M(t)u(t) \rangle] \Big|_0^T) \end{aligned} \quad (2)$$

Using Theorem 1, we can establish stability estimates as follows.

**Theorem 2** *Suppose that all the assumptions stated in Theorem 1 are satisfied. Let  $f \in C([0, T]; H)$ . Suppose that there exists a subset  $U \subset (D(P) \cap Z)$  such that  $\forall u \in U$*

$$\begin{aligned} & \|P(t)u(t)\| \\ & \leq C_6 \int_0^t (\|(A(\tau) + A^*(\tau))u(\tau)\| + \|M(\tau)u_t(\tau)\| + \|M(\tau)u(\tau)\|) d\tau \\ & \quad + C_7 \|M(t)u(t)\| + C_8 \|f(t)\| \end{aligned} \quad (3)$$

where  $C_j$  ( $j = 6, 7, 8$ ) are positive constants independent of  $t$ . Then, there exists positive constants  $s_0$ ,  $C_9$  and  $C_{10}$ , independent of  $u$ ,  $f$  and  $t$ , such that for  $\forall u \in U$  and  $\forall s \geq s_0$

$$\begin{aligned} \|Mu\|_T & \leq C_9 \left[ \frac{1}{\sqrt{s}} \|(A(T) + A^*(T))u(T)\| \right. \\ & \quad \left. + \exp(sC_{10}) (\|M(0)u(0)\| + \|(A(0) + A^*(0))u(0)\| + \|f\|_T) \right]. \end{aligned} \quad (4)$$

Furthermore, if  $\langle M(T)u(T), (A(T) + A^*(T))u(T) \rangle \leq C \|M(T)u(T)\|^2$ , then

$$\|Mu\|_T \leq C_9 \frac{\exp(sC_8)}{\sqrt{s}} (\|M(0)u(0)\| + \|(A(0) + A^*(0))u(0)\| + \|f\|_T). \quad (5)$$

*Proof of Theorem 1.* Let  $u \in D$ ,  $v = e^{s\phi}u$  and

$$\begin{aligned} P_\phi(t)v & = e^{s\phi(t)}P(t)(e^{-s\phi(t)}v) \\ & = -s\phi_t(t)M(t)v + M(t)v_t - A(t)v. \end{aligned}$$

Then we have

$$P_\phi^*(t)v = -s\phi_t(t)M(t)v - M_t(t)v - M(t)v_t - A^*v.$$

Define  $P_\phi^s$  and  $P_\phi^a$  by

$$\begin{aligned} P_\phi^s &= \frac{1}{2} (P_\phi + P_\phi^*) v \\ &= -s\phi_t Mv - \frac{1}{2} M_t v - \frac{1}{2} (A + A^*) v \end{aligned} \quad (6)$$

and

$$\begin{aligned} P_\phi^a &= \frac{1}{2} (P_\phi - P_\phi^*) v \\ &= \frac{1}{2} M_t v + Mv_t - \frac{1}{2} (A - A^*) v, \end{aligned} \quad (7)$$

respectively.

We see that

$$\begin{aligned} \|Pu\|_{s,T} &= \int_0^T \left\{ \|P_\phi^s(\tau)v(\tau)\|^2 + \|P_\phi^a(\tau)v(\tau)\|^2 \right. \\ &\quad \left. + 2\operatorname{Re}\langle P_\phi^s(\tau)v(\tau), P_\phi^a(\tau)v(\tau) \rangle \right\} d\tau. \end{aligned} \quad (8)$$

Making use of the assumptions (A1)-(A3), we have

$$\begin{aligned} 2\operatorname{Re}\langle P_\phi^s v, P_\phi^a v \rangle &= -s\phi_t \{ \operatorname{Re}\langle Mv, M_t v \rangle + 2\operatorname{Re}\langle Mv, Mv_t \rangle \} \\ &\quad + s\phi_t \operatorname{Re}\langle Mv, (A - A^*)v \rangle \\ &\quad - \left\{ \frac{1}{2} \|M_t v\|^2 + \operatorname{Re}\langle M_t v, Mv_t \rangle \right\} \\ &\quad + \frac{1}{2} \operatorname{Re}\langle M_t v, (A - A^*)v \rangle - \frac{1}{2} \{ \operatorname{Re}\langle (A + A^*)v, M_t v \rangle \\ &\quad + 2\operatorname{Re}\langle (A + A^*)v, Mv_t \rangle + (\|Av\|^2 - \|A^*v\|^2) \} \\ &\geq -\frac{d}{dt} \left\{ s\phi_t \|Mv\|^2 + \frac{1}{2} \langle (A + A^*)v, Mv \rangle \right\} \\ &\quad + s\phi_{tt} \|Mv\|^2 + s\phi_t (C_1 + C_2) \|Mv\|^2 \\ &\quad - \frac{1}{2} \{ 3C_1^2 + C_1 C_2 + 2C_3 + C_4 \} \|Mv\|^2 - \frac{1}{4} \|Mv_t\|^2. \end{aligned}$$

We also have

$$\begin{aligned} \|P_\phi^a v\|^2 &\geq \frac{1}{2} (\|Mv_t\|^2 - \|M_t v\|^2 - \|(A - A^*)v\|^2) \\ &\geq \frac{1}{2} \|Mv_t\|^2 - \frac{1}{2} (C_1^2 + C_2^2) \|Mv\|^2. \end{aligned}$$

Hence, if we take  $s$  so large that

$$s \geq \frac{1}{\delta} (4C_1^2 + C_1C_2 + C_2^2 + 2C_3 + C_4),$$

we obtain

$$\begin{aligned} & \frac{s\delta}{2} \|Mv\|_T^2 + \|P_\phi^s v\|_T^2 + \frac{1}{2} \|Mv_t\|_T^2 \\ & \leq \|Pu\|_{s,T}^2 + \left\{ s\phi_t \|Mv\|^2 + \frac{1}{2} \langle (A + A^*)v, Mv \rangle \right\} \Big|_0^T \equiv I. \end{aligned} \quad (9)$$

We have

$$\|P_\phi^s v\|^2 \geq \frac{1}{2} s\phi_t \operatorname{Re} \langle (A + A^*)v, Mv \rangle + \frac{1}{8} \|(A + A^*)v\|^2 - \frac{1}{4} C_2^2 \|Mv\|^2,$$

from which it follows that

$$\begin{aligned} & \frac{1}{8} \|(A + A^*)v\|_T^2 \\ & \leq \|P_\phi^s v\|_T^2 + \frac{s}{2} |\phi_t(0)| \int_0^T \|Mv\| \|(A + A^*)v\| dt + \frac{1}{4} C_2^2 \|Mv\|_T^2 \end{aligned}$$

Making use of (9), we have

$$\begin{aligned} & \frac{1}{8} \|(A + A^*)v\|_T^2 \\ & \leq I + \left( \frac{s}{2\delta} \right)^{1/2} |\phi_t(0)| I^{1/2} \left( \int_0^T \|(A + A^*)v\|^2 dt \right)^{1/2} + \frac{1}{2s\delta} C_2^2 I \\ & \leq \left( 1 + \frac{s}{\delta} |\phi_t(0)|^2 + \frac{2}{s\delta} C_2^2 \right) I + \frac{1}{16} \|(A + A^*)v\|_T^2. \end{aligned}$$

from which we deduce

$$\int_0^T \|(A + A^*)v\|^2 dt \leq C(1 + s|\phi_t(0)|^2) I \quad (10)$$

where and in the sequel by  $C$  we denote various positive constants which do not depend on  $t$  and  $u$  and are changeable from line to line. From (9) and (10), we have

$$\frac{s\delta}{2} \|Mv\|_T^2 + \frac{1}{1 + s|\phi_t(0)|^2} \|(A + A^*)v\|_T^2 + \frac{1}{2} \|Mv_t\|_T^2 \leq C I.$$

Noting that  $Mv_t = s\phi_t e^{s\phi} Mu + e^{s\phi} Mu_t$ , we get

$$\begin{aligned} \|Mu_t\|_{s,T}^2 &\leq \|Mv_t\|_T^2 + 2s|\phi_t(0)|\|Mv\|_T\|Mu_t\|_{s,T} \\ &\leq I + 2\sqrt{2}s^{1/2}|\phi_t(0)|I^{1/2}\|Mu_t\|_{s,T} \\ &\leq I + 4s|\phi_t(0)|^2I + \frac{1}{2}\|Mu_t\|_{s,T}^2 \end{aligned}$$

from which it follows that

$$\|Mu_t\|_{s,T}^2 \leq 2(1 + 4s|\phi_t(0)|^2)I.$$

Hence, we finally obtain

$$\frac{s\delta}{2}\|Mu\|_{s,T}^2 + \frac{1}{1 + s|\phi_t(0)|^2} (\|(A + A^*)u\|_{s,T}^2 + \|Mu_t\|_{s,T}^2) \leq C_5I. \quad (11)$$

Since  $\mathbf{D}$  is dense in  $D(P) \cap Z$ , the estimate (11) holds for any  $u \in D(P) \cap Z$ . This completes the proof of Theorem 1.

In order to establish Theorem 2, we need

**Lemma 1** *Suppose that  $\phi(t)$  is a real-valued  $C^1$ -function defined on  $[0, T]$  satisfying  $\phi_t < 0 \forall t \in [0, T]$ . Then we have for any  $f \in C([0, T]; H)$*

$$s \left\| \int_0^t f(\tau) d\tau \right\|_{s,T} \leq \frac{1}{\min_{t \in [0, T]} |\phi_t(t)|} \|f\|_{s,T} \quad (12)$$

*Proof.* Note that

$$\phi(t) - \phi(\tau) = \phi_t(\xi)(t - \tau) \leq L(t - \tau)$$

where  $L = \max_{t \in [0, T]} \phi_t(t)$ . Set  $g = e^{s\phi} f$  and  $F = e^{s\phi} \int_0^t f(\tau) d\tau$ . Then

$$\begin{aligned} \|F(t)\| &\leq \int_0^t e^{s(\phi(t) - \phi(\tau))} \|g(\tau)\| d\tau \\ &\leq \int_0^t e^{sL(t - \tau)} \|g(\tau)\| d\tau. \end{aligned}$$

Hence, we have

$$\|F(t)\|_T \leq \|e^{sLt}\|_{L^1(0, T)} \|g\|_T \leq -\frac{1}{L} \|g\|_T$$



which implies (12). ■

*Proof of Theorem 2.* From the assumptions and Lemma 1, we see that

$$\begin{aligned} \|Pu\|_{s,T}^2 \leq & \frac{2C_6^2}{s^2|\phi_t(T)|^2} \left\{ \|(A + A^*)u\|_{s,T}^2 + \|Mu_t\|_{s,T}^2 + \|Mu\|_{s,T}^2 \right\} \\ & + 2C_7^2\|Mu\|_{s,T}^2 + 2C_8^2\|f\|_{s,T}^2. \end{aligned}$$

We take  $s_0$  so large that for any  $s \geq s_0$

$$|\phi_t(T)|^2 \geq \frac{2C_5C_6^2}{s^2}(1 + s|\phi_t(0)|^2)$$

and

$$\frac{s}{2} \geq \frac{2C_5C_6^2}{s^2} + 2C_7.$$

Then, Theorem 1 yields that

$$\begin{aligned} s\|Mu\|_{s,T}^2 \leq & 2C_5 \left\{ s\phi_t(t)e^{2s\phi(t)}\|M(t)u(t)\|^2 \right. \\ & \left. + e^{2s\phi(t)}\langle (A(t) + A^*(t))u(t), M(t)u(t) \rangle \right\} \Big|_0^T + 2C_5C_8^2\|f\|_{s,T}^2 \end{aligned} \quad (13)$$

from which it follows that

$$\begin{aligned} & se^{2s\phi(T)}\|Mu\|_T^2 - 2C_5s\phi_t(T)e^{2s\phi(T)}\|M(T)u(T)\|^2 \\ & \leq -2C_5s\phi_t(0)e^{2s\phi(0)}\|M(0)u(0)\|^2 + \frac{1}{2}e^{2s\phi(T)}\|(A(T) + A^*(T))u(T)\|^2 \\ & \quad + \frac{1}{2}e^{2s\phi(T)}\|M(T)u(T)\|^2 + \frac{1}{2}e^{2s\phi(0)}\|(A(0) + A^*(0))u(0)\|^2 \\ & \quad + \frac{1}{2}e^{2s\phi(0)}\|M(0)u(0)\|^2 + 2C_5C_8^2\|f\|_{s,T}^2. \end{aligned}$$

Hence, taking  $s_0$  so large that

$$s_0 \geq -\frac{1}{4C_5\phi_t(T)},$$

we conclude that (4) holds.

If  $\langle M(t)u, (A(t) + A^*(t))u \rangle \leq C\|M(t)u\|^2$  for all  $t \in [0, T]$ , from (13) we see that (5) follows. This completes the proof of Theorem 2.

### 3 Applications

In this section we discuss the uniqueness of Cauchy problems for semilinear evolution equations and identifiability of coefficients of evolution equations.

#### 3.1 Uniqueness

Let  $M(t)$  and  $A(t)$  be the same as in section 2. We consider the Cauchy problem for semilinear evolution equation of the form

$$M(t)u_t = A(t)u + \int_0^t f(t, s, u(s))ds + g_1(t, u) + g_2(t), \quad t \in [0, T], \quad (14)$$

$$u(0) = u_0. \quad (15)$$

For brevity we introduce

$$\|u(t)\|_t = \|(A(t) + A^*(t))u(t)\| + \|M(t)u_t(t)\| + \|M(t)u(t)\|.$$

**Theorem 3** *Suppose that  $M(t)$  and  $A(t)$  satisfy (A1)-(A3). Moreover we assume that  $M(t)$  or  $A(t) + A^*(t)$  is injective for each  $t \in [0, T]$ . Suppose that*

$$\begin{aligned} & \left\| \int_0^t (f(\tau, x, u(\tau)) - f(\tau, x, v(\tau))) d\tau \right\| \\ & \leq C \int_0^t K_1(\|u(\tau)\|_\tau + \|v(\tau)\|_\tau) (\|u(\tau) - v(\tau)\|_\tau) d\tau \end{aligned} \quad (16)$$

and

$$\begin{aligned} & \|g_1(t, u(t)) - g_2(t, u(t))\| \\ & \leq K_2(\|M(t)u\| + \|M(t)v\|) \|M(t)(u - v)\| \end{aligned} \quad (17)$$

where  $K_1$  and  $K_2$  are non decreasing continuous functions defined on  $[0, \infty)$ . Then, for every  $u_0 \in D(M(0)) \cap D(A(0) + A^*(0))$  and  $g_2 \in C([0, T]; H)$ , the problem (14)-(15) has at most one solution.

*Proof.* We take the set  $U$  in Theorem 2 as

$$U = \left\{ u \in (D(P) \cap Z) \mid \sup_{0 \leq t \leq T} (\|u\|_t + \|M(t)u(t)\|) \leq R \right\}$$

for some  $R > 0$ . Let  $u(t)$  and  $v(t)$  be two solutions of (14)-(15). Put  $w = u - v$ . Then

$$M(t)w_t = A(t)w + \int_0^t (f(t, \tau, u(\tau)) - f(t, \tau, v(\tau))) d\tau + (g_1(t, u) - g_1(t, v)), \quad t \in [0, T], \quad (18)$$

$$u(0) = 0. \quad (19)$$

The assumptions yield that

$$\left\| \int_0^t (f(\tau, x, u(\tau)) - f(\tau, x, v(\tau))) d\tau \right\| \leq C \int_0^t K_1(2R) \|w(\tau)\|_\tau d\tau \quad (20)$$

and

$$\|g_1(t, u(t)) - g_2(t, v(t))\| \leq K_2(2R) \|M(t)w\|. \quad (21)$$

Hence, from Theorem 2 we see that

$$\|Mw\|_T \leq \frac{C_9}{\sqrt{s}} \|(A(T) + A^*(T))u(T)\|$$

By letting  $s \rightarrow \infty$ , we conclude that

$$\|Mw\|_T = 0$$

which implies

$$M(t)w(t) = 0 \quad \forall t \in [0, T]. \quad (22)$$

If  $M(t)$  is injective for each  $t$ , then

$$w(t) = 0 \quad \forall t \in [0, T]. \quad (23)$$

Assume that  $A(t) + A^*$  is injective. In much the same way as in the proof of Theorem 2, using (22), we see that

$$\|(A + A^*)w\|_{s,T} \leq 0$$

provided that  $s$  is taken large enough. Hence we easily see that (23) holds for this case. This completes the proof.

**Remark 1** *Since our assumptions does not require positivity or accretiveness of the operators  $M(t)$ ,  $A(t)$ , Theorem 3 covers very wide class of uniqueness question for the Cauchy problem. For instance we can show the backward uniqueness for the heat equation and for pseudo-parabolic equations (see below).*

### Examples

Let  $\Omega$  be a domain in  $\mathbb{R}^N$ . Let

$$M(t, x, D)u = \sum_{0 \leq |\alpha|, |\beta| \leq p} (-1)^\alpha D^\alpha (m_{\alpha\beta}(t, x) D^\beta u),$$

and

$$A(t, x, D)u = \sum_{0 \leq |\alpha|, |\beta| \leq q} (-1)^\alpha D^\alpha (a_{\alpha\beta}(t, x) D^\beta u)$$

be linear differential operators of order  $2p$  and  $2q$ , respectively with complex-valued smooth coefficients defined on  $[0, T] \times \Omega$ . Let  $H = L^2(\Omega)$  and define

$$D(M(t)) = \{u : \Omega \rightarrow \mathbb{C} \mid u \in H^{2p}(\Omega) \cap H_0^p(\Omega)\}$$

and for any  $u \in D(M(t))$

$$(M(t)u)(x) = M(t, x, D)u(t, x), \quad (t, x) \in [0, T] \times \Omega.$$

We assume that  $M(t, x, D)$  is formally symmetric, that is,

$$m_{\alpha\beta} = \overline{m_{\beta\alpha}}, \quad \forall \alpha, \beta.$$

Then, under some suitable assumptions on the coefficients  $m_{\alpha\beta}$ , we can see that for each  $t$   $M(t)$  is selfadjoint in  $H$  and  $D(t) = C_0^\infty(\Omega)$  is the core of  $M(t)$ .

Let

$$D(A(t)) = \{u : \Omega \rightarrow \mathbb{C} \mid u \in H^{2q}(\Omega) \cap H_0^q(\Omega)\}$$

and define for any  $u \in D(A(t))$

$$(A(t)u)(x) = A(t, x, D)u(t, x), \quad (t, x) \in [0, T] \times \Omega.$$

In this case the Cauchy problem (14)-(15) is as follows:

$$\begin{aligned} M(t, x, D)u_t &= A(t, x, D)u + \int_0^t f(t, s, x, u(s))ds \\ &+ g_1(t, x, u) + g_2(t, x) \quad (t, x) \in [0, T] \times \Omega, \end{aligned} \quad (24)$$

with

$$u(0, x) = u_0(x), \quad x \in \Omega, \quad (25)$$

$$D^\alpha u(t, x) = 0, \quad (t, x) \in [0, T] \times \partial\Omega, \quad |\alpha| \leq q, \quad (26)$$

$$D^\alpha u_t(t, x) = 0, \quad (t, x) \in [0, T] \times \partial\Omega, \quad |\alpha| \leq p. \quad (27)$$

If the coefficients  $m_{\alpha\beta}(t, x)$  and  $a_{\alpha\beta}(t, x)$  are many-times boundedly differentiable in  $(t, x)$  on  $(0, T) \times \Omega$ , we easily see that the assumption holds valid.

We can impose additional conditions on  $M(t)$ ,  $A(t)$  so as to satisfy (A3). We list up below some of them:

- (Ex.1)  $M(t, x, D)$  and  $A(t, x, D)$  are of constant coefficients and  $A(t, x, D)$  is formally symmetric.
- (Ex.2)  $M(t, x, D)$  or  $-M(t, x, D)$  is a uniformly elliptic operator for each  $t$ , and  $m_{\alpha\beta}(t, x)$  and  $a_{\alpha\beta}(t, x)$  are independent of  $x$ , and  $p \geq q$ .
- (Ex.3)  $M(t, x, D) = m(t) \neq 0$  for  $t \in [0, T]$  and  $A(t, x, D)$  is independent of  $t$  and formally symmetric or anti-symmetric with many-times boundedly differentiable coefficients.

**Remark 2** *The form of Eq. (24) contains pseudo-parabolic equations. Concerning the well-posedness of the initial-boundary value problem for them we refer to the book of Carroll and Showalter [5].*

## 3.2 Identifiability

Consider the initial-periodic boundary value problem

$$u_t = u_{xx} + a(t)f(x, u), \quad 0 < x < 1, \quad t > 0 \quad (28)$$

$$u(0, t) = u(1, t) \quad t > 0 \quad (29)$$

$$u_x(0, t) = u_x(1, t) \quad t > 0 \quad (30)$$

$$u(x, 0) = u_0(x), \quad 0 < x < 1, \quad (31)$$

where  $f(x, u)$  is a known function of  $u$  and  $u_0$  is a given function.

Our problem is to recover the coefficient  $a(t)$  when we know some observation of the state. Here we are interested in the case when our observation is given by

$$u(x_0, t) = u_{ob}(t) \quad 0 < t < T \quad (32)$$

for some point  $x_0 \in [0, 1]$ . We establish identifiability of the coefficients for the problem, that is, to show that the coefficient  $a(t)$  is uniquely determined by the data and the observation (32).

**Theorem 4** *Suppose that  $a(t), u_{ob} \in C(0, T]$  and  $u_0 \in C([0, 1])$ . Assume that for given  $a(t)$  and  $u_0$  there exists a unique solution  $u \in C^2([0, 1] \times [0, T])$  of (28)-(31), which satisfies*

$$u_{xx}(0, t) = u_{xx}(1, t)$$

and

$$f(x, u(x, t)) > 0 \quad \forall t \in [0, 1] \times [0, T]. \quad (33)$$

Then,  $(u, a)$  is uniquely determined by the initial condition (31) and the observation (32).

**Remark 3** *The assumption (33) is satisfied by, for example,*

$$f(x, u) = q(x)e^u$$

where  $q(x)$  is a known positive function. or, if we consider positive solutions, it is satisfied by

$$f(x, u) = q(x)|u|^{p-1}u.$$

*Proof.* Let  $(u_1, a_1)$  and  $(u_2, a_2)$  be two solutions. Then, putting  $w = u_1 - u_2$  and  $a = a_1 - a_2$ , we have

$$w_t = w_{xx} + a_1(t) \int_0^1 f'(\theta u_1 + (1 - \theta)u_2) d\theta w + a(t)f(u_2), \quad 0 < x < 1, \quad t > 0 \quad (34)$$

$$w(x_0, t) = 0, \quad t > 0, \quad (35)$$

$$\frac{\partial^k}{\partial x^k} w(0, t) = \frac{\partial^k}{\partial x^k} w(1, t) \quad (k = 0, 1, 2) \quad t > 0, \quad (36)$$

$$w(x, 0) = 0, \quad 0 < x < 1. \quad (37)$$

Define

$$Q = \partial_x - (\log f(u_2))_x$$

and

$$\tilde{P}(t) = \partial_t - \partial_{xx} - a_1(t)G(x, t),$$

where

$$G(x, t) = \int_0^1 f'(\theta u_1 + (1 - \theta)u_2) d\theta.$$

We easily see that

$$Q(a(t)f(u_2)) = 0 \quad \forall (x, t) \in [0, 1] \times [0, T].$$

Applying  $Q$  to (34), we have

$$Q\tilde{P}w = 0 \quad \forall (x, t) \in [0, 1] \times [0, T].$$

Hence, we have

$$\begin{aligned} \tilde{P}Qw &= [\tilde{P}, Q]w \\ &= H(x, t)w + 2(\log(f(u_2)))_{xx}w_x \end{aligned} \quad (38)$$

where

$$H(x, t) = -(\log f(u_2))_{xt} + (\log f(u_2))_{xxx} + a_1(t)G_x. \quad (39)$$

Put  $v = Qw$ . Since  $w(x_0, t) = 0$ , we get

$$Q^{-1}v = \int_{x_0}^x \frac{f(u_2(x, t))}{f(u_2(\xi, t))} v(\xi, t) d\xi \quad (40)$$

Hence, we can rewrite (38) as

$$\tilde{P}v = [\tilde{P}, Q]Q^{-1}v \quad (41)$$

with

$$v(0, t) = v(1, t), \quad v_x(0, t) = v_x(1, t), \quad \forall t > 0 \quad (42)$$

and

$$v(x, 0) = 0. \quad (43)$$

In view of (38)-(40) we easily see that

$$\|[\tilde{P}, Q]Q^{-1}v\|_{L^2([0,1])} \leq C\|u\|_{L^2([0,1])}. \quad (44)$$

Let  $H = L^2([0, 1])$  and  $A : H \rightarrow H$  defined by

$$Au = u_{xx} \quad u \in D(A)$$

with

$$D(A) = \left\{ u : [0, 1] \rightarrow \mathbb{R} \mid u \in H^2([0, 1]) \right. \\ \left. u \text{ satisfies (29), (30)} \right\}$$

Then, we can apply Theorem 2 to obtain

$$\|v\|_T \leq \frac{C}{\sqrt{s}} \|A(T)u(T)\|$$

for any  $s \geq s_0$  where  $C$  and  $s_0$  are positive constants independent of  $v$ . Letting  $s$  tend to infinity, we get

$$\|v\|_T \equiv 0$$

from which it follows that  $v \equiv 0$  on  $[0, T^*]$ . Then, we conclude that

$$w(t) \equiv 0 \quad t \in [0, T^*]$$

from which we deduce

$$0 = \tilde{P}(t)w = af(u_2) \quad t \in [0, T^*]$$

Noting (33), we see that

$$a = a_1 - a_2 = 0.$$

This completes the proof.

**Remark 4** *In much the same manner we can obtain analogous results for the initial periodic-boundary value problems in many space dimensions (or equivalently, initial value problems on multi-dimensional torus) not only for (nonlinear) heat equations like (28) but also for the Schrödinger-type or Korteweg-de Vries type equations. (see [15]) In [1] Bukhgeim considered initial-Dirichlet or Neumann boundary value problems with point observations at the boundary.*

**Remark 5** *Another approach showing identifiability of coefficients relies on the inverse Sturm-Liouville problems (see [7], [14]).*



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