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K_2 OF A FERMAT QUOTIENT AND THE VALUE OF ITS L FUNCTION

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Introduction. In 1984, Beilinson[Be] formulated a beautiful conjecture which relates the values at each integer of Hasse-Weil L functions of a proper smooth variety X over a number field to the covolume of the image of the regulator map

$$\mathrm{erg}: H^i_{\mathcal{A}}(X, \mathbb{Q}(j)) \to H^i_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(j))$$

where $H^i_{\mathcal{A}}(X,\mathbb{Q}(j)) = K^{(j)}_{2j-i}(X)_{\mathbb{Q}}$ is called *absolute cohomology group* which is a certain eager space of the K-group of X under the action of Adams operator, and

$$H^i_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(j)) = \mathbb{H}^i(0 \to \mathbb{R}(j) \to O_{X_{\mathbb{C}}} \to \Omega^1_{X_{\mathbb{C}}} \to \dots \to \Omega^{j-1}_{X_{\mathbb{C}}} \to 0[-1]).$$

is the *Deligne cohomology group*. The regulator map is defined by Chern class map of K-theory. There are several affirmative examples for this conjecture. (cf.[Ram], [Scha]).

In this article we treat a motive which is a factor of a certain Fermat curve. It is a curve of genus 2, and so $dim_{\mathbb{R}} H^2_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(2))^+ = 2$. Beilinson conjecture tells that there should be two linearly independent elements in the absolute cohomology which corresponds to such a motive, and the main point of this article is to describe such two elements in an explicit manner. Let C be the Fermat curve of exponent 5: $x^5 + y^5 = 1$. Ross [Ross] found in K_2C an element which has nontrivial image under the regulator map. Define the action of $\mathbb{Z}/5\mathbb{Z}$ on C in such a way that $k \in \mathbb{Z}/5\mathbb{Z}$ acts on C by $(x, y) \to (\zeta^k x, \zeta^{-k} y)$ where ζ is a fixed nontrivial fifth root of unity. Let X be the quotient of C by this action.The equation of this curve as an affine curve is:

$$w^5 = u(1-u).$$

Here $w = xy$ and $u = x^5$. Ross' element comes from K_2 of this quotient by pulling back. In this case $L(H^1(X), s)$ equals L function of a Jacobi sum Hecke character, so we have good understanding of the value of it at $s = 0$. Moreover, the recipient of the regulator map is $H^1(X(\mathbb{C}), \mathbb{R}(1))$ so the image of regulator map can be described as a 1-form on X. Since $\mathrm{ord}_{s=0} L(H^1(X), s) = 2$, Beilinson conjecture asserts the existence of another element of K_2X which has an independent image of Ross' element under the regulator map. We first exhibit an element of K_2C which by the projection provides an element of K_2X. Our main result is that this projected element of K_2X is the very element which is asserted to exist by the conjecture. 1Once the elements are exhibited, the linear independence of the regulator images of those can be proved by showing nonvanishing of the determinant of the matrix which is given by integrals of those images along two homologically independent 1-cycles on C. We compute the determinant numerically.

We also compute numerically the value of L function which by the conjecture is prospected to differ from the determinant of the image of the regulator only by a rational number. The ratio of these two computed values nearly equals a simple rational number. The author believes that this is the first example for a motif of rank 2 associated to a curve of genus 2.

§1 Regulator map and the Beilinson's conjecture in the case of a curve/$\mathbb{Q}$. Let X be a projective smooth curve/$\mathbb{Q}$.

The localization sequence of K theory provides with the exact sequence

$$\coprod_{p\in X(\overline{\mathbb{Q}})} K_2\mathbb{Q}(p) \longrightarrow K_2X \longrightarrow K_2\mathbb{Q}(X) \xrightarrow{\tau} \coprod_{p\in X(\overline{\mathbb{Q}})} \mathbb{Q}(p)^*$$

Here $\mathbb{Q}(p)$ denotes the residue field of a point p, and $\mathbb{Q}$(X) denotes the function field of X with coefficients in $\mathbb{Q}$. $\tau = \coprod_{p\in X(\overline{\mathbb{Q}})} \tau_p$ is tame symbol given by

$$\tau_p\{f,g\} = (-1)^{(\mathrm{ord}_p f)(\mathrm{ord}_p g)} \frac{f^{\mathrm{ord}_p g}}{g^{\mathrm{ord}_p f}}(p).$$

Since K_2 of a number field is torsion, K_2X agrees with Kerτ up to torsion.

Let $\mathfrak{X}$ be a proper flat model of X over $\mathbb{Z}$. The natural map induces the pullback $K_2\mathfrak{X} \to K_2$ X. Conjecturally the image of this is independent of the choice of $\mathfrak{X}$. Bloch[Bl] defined the regulator for K_2 X

$$\mathrm{reg}_X : K_2X \to H^1(X(\mathbb{C}), 2\pi i\mathbb{R})^+.$$

the superscript + denotes the invariant subspace under the action of complex conjugate on both $X(\mathbb{C})$ and the coefficient $2\pi i\mathbb{R}$. $\dim_{\mathbb{R}} H^1(X(\mathbb{C}), 2\pi i\mathbb{R})^+$ equals the genus of X. Let g be the genus of X. The Beilinson conjecture in this case is:

Conjecture. 1. $\mathrm{reg}_X(K_2\mathfrak{X})$ is a lattice in $H^1(X(\mathbb{C}), 2\pi i\mathbb{R})^+$.

2. Define $c \in \mathbb{R}^*/\mathbb{Q}^*$ by

$$\Lambda^g[(\mathrm{reg}_X K_2\mathfrak{X}) \otimes \mathbb{Q}] = c.\Lambda^g H^1(X(\mathbb{C}), 2\pi i\mathbb{Q})^+.$$

then $c \equiv L^{(g)}$(X,0) mod$\mathbb{Q}^*$.

An explicit description of the regulator is given as follows. Since $\mathbb{Q}$(X) is a field, $K_2\mathbb{Q}(X)$ is generated by symbols. And so is the elements of K_2(X). Let $\{f,g\} \in K_2$(X) where $f, g \in \mathbb{Q}$(X). $\mathrm{reg}_X(\{f,g\})$ as a 1-form mod. exact forms on X is represented by

$$\mathrm{reg}_X(\{f,g\}) = Im2(\log|f|\partial\log|g| - \log|g|\partial\log|f|)$$

Let γ be a cycle on X($\mathbb{C}$) and suppose that both f and g are holomorphic and nonzero on γ . Then as is stated in [Ram],

$$\int_\gamma \mathrm{reg}(\{f,g\}) = Im\Big(\int_\gamma \log f d\log g - \log|g(p_0)|\int_\gamma d\log f\Big) \tag{1.1}$$

where we take a fixed branch of $\log f$ beginning in $p_0 \in \gamma$.

Let C be the Fermat curve of exponent 5: $X^5 + Y^5 = 1$. Let $\zeta = \exp(\frac{2}{5}\pi i)$ in the following.Define the action of $\mathbb{Z}/5\mathbb{Z}$ on C in such a way that $k \in \mathbb{Z}/5\mathbb{Z}$ acts on C by $(x,y) \to (\zeta^k x, \zeta^{-k} y)$. Let X be the quotient of C by this action.The equation of this curve as an affine curve is:

$$w^5 = u(1-u)$$

where $w = xy$ and $u = x^5$. The genus of this curve is 2.

From now on, We consider on certain two elements of K_2X.

§2 Integration of the regulator images of certain elements of K_2X.

Ross[Ross] found that $\alpha := \{1-w, u\} \in K_2X$ has a nontrivial image under the regulator map.*

According to the Beilinson conjecture, there should be another element of K_2X which is independent of Ross' element. One of our main results is that the projection of a certain element of K_2C on K_2X is such an element. We prove this by numerical integral of the regulator image of these elements along two homologically independent cycles on C.

Let $\beta \in K_2X$ be given by

$$\beta := \pi_*\{x+y, \frac{1-x}{y}\}$$

where $\pi_* : K_2C \to K_2X$ denotes the projection of K theory.

Our main result is that α and β have linearly independent images under the regulator map. We prove this by showing non vanishing of the determinant of the matrix given by integrals of $\pi^*\mathrm{reg}\alpha$ and $\pi^*\mathrm{reg}\beta$ along two homologically independent 1 cycles on C.

$$\pi^*\beta = \sum_{k=0}^{4}\{\zeta^k x + \zeta^{-k} y, \frac{1-\zeta^k x}{\zeta^{-k} y}\}$$

as a symbol.

*In fact,Ross exhibited for each Fermat curve an element of the K_2 with nonzero image under the regulator map.

Let $A_{m,n}$ denote the automorphism of C given by $(x,y) \to (\zeta^m x, \zeta^n y)$ and let $\gamma : [0,1] \to C$ denote the path from (1,0) to (0,1) given by $t \mapsto ((1-t)^{1/5}, t^{1/5})$. For $m,n \in \mathbb{Z}$ let $\gamma_{m,n}$ denote the cycle on C given by

$$\gamma_{m,n} = \gamma - A_{m,0}\gamma + A_{m,n}\gamma - A_{0,n}\gamma.$$

We want to integrate $\mathrm{reg}_C(\pi^*\alpha)$ and $\mathrm{reg}_C(\pi^*\beta)$ along $\gamma_{1,1}$ and along $\gamma_{2,1}$. Since functions which appear in $\pi^*\alpha$ and $\pi^*\beta$ have poles and zeros on these cycles, a slight modification is necessary. The following paths are needed. Let $0 < \epsilon \ll 1$.

$$\begin{aligned}
\gamma^2 :& [1-(1-\epsilon)^5, 1-\epsilon^5] \to C \\
& t \mapsto ((1-t)^{1/5}, t^{1/5}) \\
\gamma^1 :& [1-(1-\epsilon)^5, 1] \to C \\
& t \mapsto ((1-t)^{1/5}, t^{1/5}) \\
C_0^n :& [0,1] \to C \\
& \theta \mapsto (\epsilon \exp(\tfrac{2n}{5}\pi i\theta), (1-\epsilon^5 \exp(2n\pi i\theta))^{1/5}) \\
C_{\zeta^n} :& [0,1] \to C \\
& \theta \mapsto (\zeta^n(1-\epsilon\exp(2\pi i\theta)), (1-(1-\epsilon\exp(2\pi i\theta))^5)^{1/5}) \\
=& (\zeta^n(1-\epsilon\exp(2\pi i\theta)), (5\epsilon)^{1/5}\exp(\tfrac{2}{5}\pi i\theta)(1+F(\epsilon,\theta))^{1/5})
\end{aligned}$$

$$\left(\begin{aligned}
&(1-(1-\epsilon\exp(2\pi i\theta))^5) \\
&= \epsilon\exp(2\pi i\theta)(5 - 10\epsilon\exp(2\pi i\theta) \\
&+10\epsilon^2\exp(4\pi i\theta) - 5\epsilon^3\exp(6\pi i\theta) + \epsilon^4\exp(8\pi i\theta) \\
&\text{and we let} \\
&F(\epsilon,\theta) \\
&= -2\epsilon\exp(2\pi i\theta) + 2\epsilon^2\exp(4\pi i\theta) - \epsilon^3\exp(6\pi i\theta) + \tfrac{1}{5}\epsilon^4\exp(8\pi i\theta).
\end{aligned}\right)$$

We define the cycles $\gamma_{n,1}^2 (n = 1,2)$ to be

$$\gamma_{n,1}^2 := \gamma^2 + C_0^n - A_{1,0}\gamma^2 + C_{\zeta^n} + A_{n,1}\gamma^2 - C_0^n - A_{0,1}\gamma^2 - C_1$$

Here is one of our main results.

Theorem 1. *$\mathrm{reg}_X(\pi^*\alpha)$ and $\mathrm{reg}_X(\pi^*\beta)$ are independent in $H^1(X(\mathbb{C}), 2\pi i\mathbb{R})$. i.e.*

$$\begin{vmatrix} \int_{\gamma_{1,1}^2} \mathrm{reg}_C(\pi^*\alpha) & \int_{\gamma_{2,1}^2} \mathrm{reg}_C(\pi^*\alpha) \\ \int_{\gamma_{1,1}^2} \mathrm{reg}_C(\pi^*\beta) & \int_{\gamma_{2,1}^2} \mathrm{reg}_C(\pi^*\beta) \end{vmatrix} \neq 0$$

remark. Let C_n be the Fermat curve of exponent $5n : X^{5n} + Y^{5n} = 1$, and X_n be a quotient of $C_n : w^{5n} = u(1-u)$. Let $\alpha_n := (\{1-w, u\}) \in K_2 X_n$ and

$\gamma_n := \{x+y, \frac{1-x}{y}\} \in K_2C_n$ and $\beta_n := \pi_*\gamma_n \in K_2X_n$ and for $m \mid n$ let $p_{n,m} : (x,y) \mapsto (x^{\frac{n}{m}}, y^{\frac{n}{m}})$ be the canonical projection. Then it is seen by a straightforward calculation that

$$(p^*_{n,m}p_{n,m*}(\pi^*\alpha_n)) = (p^*_{n,m}(\pi^*\alpha_m)) \text{ and } p^*_{n,m}p_{n,m*}(\gamma_n) = p^*_{n,m}(\gamma_m).$$

We see from this that

$$(p_{n,m*}\pi^*\alpha_n) = (\pi^*\alpha_m) \text{ and } p_{n,m*}\gamma_n = \gamma_m \text{ in } K_2(C_m)_{\mathbb{Q}}$$

because $p_{n,m*}p^*_{n,m}$=multiplication by deg $p_{n,m}$ and since $\pi \circ p_{n,m} = p_{n,m} \circ \pi$, it follows that

$$\begin{aligned} (p_{n,m*}\pi_*\pi^*\alpha_n) &= (\pi_*\pi^*\alpha_m) \\ p_{n,m*}\alpha_n &= \alpha_m \\ \text{and } p_{n,m*}\beta_n &= \beta_m. \end{aligned}$$

Consequently, α_n and β_n are norm compatible system in K_2X_n. So we get the following

Corollary. *$reg(\alpha_n)$ and $reg(\beta_n)$ are also linearly independent in $H^1(X_n, \mathbb{R}(1))$.*

We now explain how to perform the numerical integral.

Integration of $\mathrm{reg}_C(\pi^*\alpha)$ (due to Ross). As is stated before,

$$\begin{aligned} &\int_{\gamma^2_{n,1}} \mathrm{reg}_C(\pi^*\alpha) \\ &= Im\Big(\int_{\gamma^2_{n,1}} \log(1-xy)d\log x - \log|x(p_0)| \int_{\gamma^2_{n,1}} d\log(1-xy)\Big). \end{aligned}$$

Since $|xy| < 1$ both on $\gamma^2_{1,1}$ and $\gamma^2_{2,1}$, $\int_{\gamma^2_{n,1}} d\log(1-xy) = 0$ for n=1,2. Let p_0 be $(1-\epsilon, (1-(1-\epsilon)^5)^{1/5}) \in \gamma$.

Calculation of $\int_{C^k_0} \log(1-xy)d\log x$ $(k=1,2)$. By the definition of C^k_0,

$$\int_{C^k_0} \log(1-xy)d\log x = \int_0^1 \log(1-\epsilon\exp(\frac{2k}{5}\pi i\theta)(1-\epsilon^5\exp(2k\pi i\theta)))2k\pi i d\theta.$$

We see from this that

$$\lim_{\epsilon\to 0}\int_{C^k_0} \log(1-xy)\mathrm{d}\log x = 0.$$

Putting $\epsilon' \to 0$, we get

$$\begin{aligned} &\int_{C^k_0 \text{ for } \epsilon} \log(1-xy)d\log x \\ &= \int_{(\epsilon,(1-\epsilon^5)^{1/5})}^{(0,1)} \log(1-xy)d\log x \quad \text{(integration on } \gamma) \\ &+ \int_{(0,1)}^{(\epsilon,(1-\epsilon^5)^{1/5})} \log(1-xy)d\log x \quad \text{(integration on } A_{k,0}\gamma) \end{aligned} \tag{2.2}$$

Calculation of $\int_{C_{\zeta^k}} \log(1-xy)d\log x (n=0,1,2)$. Since $\log(1-xy), \log x$ are holomorphic and nonzero in the neighborhoods of $(\zeta^k, 0)$,

$$
\begin{aligned}
&\int_{C_{\zeta^k}} \log(1-xy)d\log x \\
&= \int_{(\zeta^k(1-\epsilon),(1-(1-\epsilon)^5)^{1/5})}^{(\zeta^k,0)} \log(1-xy)d\log x \\
&\text{integration on } A_{k,0}\gamma \\
&+ \int_{(\zeta^k,0)}^{(\zeta(1-\epsilon),\zeta(1-(1-\epsilon)^5)^{1/5})} \log(1-xy)d\log x \\
&\text{integration on } A_{k,1}\gamma
\end{aligned}
\tag{2.3}
$$

From (2.2) and (2.3) we see that

$$\int_{\gamma^2_{n,1}} \log(1-xy)\mathrm{d}\log x = \int_{\gamma_{n,1}} \log(1-xy)\mathrm{d}\log x \quad (n=1,2). \tag{2.4}$$

$\int_{\gamma_{n,1}} \log(1-xy)d\log x$ is easily calculated, and we have

$$
\begin{aligned}
&\int_{\gamma^2_{1,1}} \mathrm{reg}_C(\pi^*\alpha) \\
&= Im \int_{\gamma_{1,1}} \log(1-xy)d\log x \\
&= \frac{1}{10}\sum_{k=1}^{\infty}\left(\sin\frac{4}{5}\pi k - 2\sin\frac{2}{5}\pi k\right)\frac{1}{k}B(\frac{k}{5},\frac{k}{5}) \\
&\text{and} \\
&\int_{\gamma^2_{2,1}} \mathrm{reg}_C(\pi^*\alpha) \\
&= Im \int_{\gamma_{2,1}} \log(1-xy)d\log x \\
&= \frac{1}{10}\sum_{k=1}^{\infty}\left(\sin\frac{6}{5}\pi k - \sin\frac{2}{5}\pi k - \sin\frac{4}{5}\pi k\right)\frac{1}{k}B(\frac{k}{5},\frac{k}{5}).
\end{aligned}
\tag{2.5}
$$

Numerical computation of these values is easily done.

Integration of $\mathbf{reg}_C(\pi^*\beta)$. We have

(2.6)

$$\int_{\gamma^2_{n,1}} \mathrm{reg}_C(\pi^*\beta)$$
$$=\int_{\gamma^2_{n,1}} \sum_{k=0}^{4} \mathrm{reg}_C\left(\left\{\zeta^k x+\zeta^{-k}y, \frac{1-\zeta^k x}{\zeta^{-k}y}\right\}\right)$$
$$=Im\left(\sum_{k=0}^{4}\int_{\gamma^2_{n,1}} \log(\zeta^k x+\zeta^{-k}y)(\mathrm{d}\log(1-\zeta^k x)-\mathrm{d}\log(\zeta^{-k}y))\right.$$
$$\left.-\sum_{k=0}^{4}(\log|1-\zeta^k x(p_0)|-\log|\zeta^{-k}y(p_0)|)\int_{\gamma^2_{n,1}} d\log(\zeta^k x+\zeta^{-k}y)\right).$$

We start in $p_0=(1-\epsilon,(1-(1-\epsilon)^5)^{\frac{1}{5}})$ and continue $\log(\zeta^k x+\zeta^{-k}y)$ along $\gamma^2_{n,1}$, n=1,2.

Calculation of $\int_{C^n_0}\log(\zeta^k x+\zeta^{-k}y)\begin{cases} d\log(1-\zeta^k x) \\ d\log(\zeta^{-k}y)\end{cases}(n=1,2).$

Since $\zeta^k x+\zeta^{-k}y, y, 1-\zeta^k x$ are holomorphic and nonzero in the neighborhoods of $(0,1)$, if we let $\gamma^1_{n,1}$ $(n=1,2)$ be

$$\gamma^1_{n,1}:=\gamma^1-A_{n,0}\gamma^1+C_{\zeta^n}+A_{n,1}\gamma^1-A_{0,1}\gamma^1-C_1(n=1,2)$$

we see, by the same argument as in (2.2), that

(2.7)
$$\int_{\gamma^2_{n,1}} \mathrm{reg}_C(\pi^*\beta) =\int_{\gamma^1_{n,1}} \mathrm{reg}_C(\pi^*\beta)$$

Let

$$A(n)=\begin{cases} 1 & n=1 \\ 2 & n=2 \\ -2 & n=3 \\ -1 & n=4.\end{cases}$$

We first take the branches of $\log(\zeta^k x+\zeta^{-k}y)$ so that $\log(\zeta^k 1+\zeta^{-k}0)=\frac{2}{5}\pi A(k)i$ at $(1,0)\in\gamma$, and continue this on $\gamma^1_{n,1}$. Let $\log\zeta^{k+n}$ be the continued value of $\log(\zeta^k x+\zeta^{-k}y)$ at the junction of $A_{n,0}\gamma$ and $A_{n,1}\gamma(=(\zeta^n,0))$. We exhibit tables of the values $\log\zeta^{k+n}$.

The value in the third column (with (1,0) on the top) means the value of $\log(\zeta^k x+\zeta^{-k}y)$ when it is continued along $\gamma_{n,1}$ and comes back to $(1,0)$.

For $\gamma^1_{1,1}$

k	(1,0)	(ζ,0)	(1,0)	$\int_{\gamma^1_{1,1}} d\log(\zeta^k x + \zeta^{-k} y$
0	0	$\frac{2}{5}\pi i$	0	0
1	$\frac{2}{5}\pi i$	$-\frac{6}{5}\pi i$	$-\frac{8}{5}\pi i$	$-2\pi i$
2	$\frac{4}{5}\pi i$	$\frac{6}{5}\pi i$	$\frac{4}{5}\pi i$	0
3	$-\frac{4}{5}\pi i$	$-\frac{2}{5}\pi i$	$-\frac{4}{5}\pi i$	0
4	$-\frac{2}{5}\pi i$	0	$\frac{8}{5}\pi i$	$2\pi i$

For $\gamma^1_{2,1}$

k	(1,0)	(ζ^2,0)	(1,0)	$\int_{\gamma^1_{2,1}} d\log(\zeta^k x + \zeta^{-k} y)$
0	0	$\frac{4}{5}\pi i$	0	0
1	$\frac{2}{5}\pi i$	$-\frac{4}{5}\pi i$	$\frac{2}{5}\pi i$	0
2	$\frac{4}{5}\pi i$	$\frac{8}{5}\pi i$	$\frac{4}{5}\pi i$	0
3	$-\frac{4}{5}\pi i$	$-2\pi i$	$-\frac{14}{5}\pi i$	$-2\pi i$
4	$-\frac{2}{5}\pi i$	$\frac{2}{5}\pi i$	$\frac{8}{5}\pi i$	$2\pi i$

We call the values in the first column(with (1,0) on the top) $\log(\zeta^k)(1)$, the values in the second column (with $(\zeta^n, 0)$ on the top) $\log(\zeta^{k+n})$ and the values in the third column(with (1,0) on the top) $\log(\zeta^k)(2)$. Then we have following proposition.

Proposition. *Let*

$$\begin{aligned}
&\int_{\gamma^1_{n,1}} \left\{\log(\zeta^k x + \zeta^{-k} y) - \log(\zeta^{k+n})\right\} \begin{cases} d\log(1-\zeta^k x) \\ d\log(\zeta^{-k} y) \end{cases} \\
&= \int_{\gamma^1} \{\log(\zeta^k x + \zeta^{-k} y) - \log \zeta^k(1)\} \begin{cases} d\log(1-\zeta^k x) \\ d\log(\zeta^{-k} y) \end{cases} \\
&- \int_{A_{n,0}\gamma^1} \{\log(\zeta^k x + \zeta^{-k} y) - \log(\zeta^{k+n})\} \begin{cases} d\log(1-\zeta^k x) \\ d\log(\zeta^{-k} y) \end{cases} \\
&+ \int_{C_{\zeta^n}} \{\log(\zeta^k x + \zeta^{-k} y) - \log(\zeta^{k+n})\} \begin{cases} d\log(1-\zeta^k x) \\ d\log(\zeta^{-k} y) \end{cases} \\
&+ \int_{A_{n,1}\gamma^1} \{\log(\zeta^k x + \zeta^{-k} y) - \log(\zeta^{k+n})\} \begin{cases} d\log(1-\zeta^k x) \\ d\log(\zeta^{-k} y) \end{cases} \\
&- \int_{A_{0,1}\gamma^1} \{\log(\zeta^k x + \zeta^{-k} y) - \log \zeta^k(2)\} \begin{cases} d\log(1-\zeta^k x) \\ d\log(\zeta^{-k} y) \end{cases} \\
&- \int_{C_1} \{\log(\zeta^k x + \zeta^{-k} y) - \log \zeta^k(2)\} \begin{cases} d\log(1-\zeta^k x) \\ d\log(\zeta^{-k} y). \end{cases}
\end{aligned}$$

Then

$$
\begin{aligned}
&Im \int_{\gamma_{n,1}^1} \sum_{k=0}^{4} \mathrm{reg}_C\left(\left\{\zeta^k x + \zeta^{-k} y, \frac{1-\zeta^k x}{\zeta^{-k} y}\right\}\right) \\
&= Im \int_{\gamma_{n,1}^1} \sum_{k=0}^{4} \{\log(\zeta^k x + \zeta^{-k} y) - \log(\zeta^{k+n})\} \begin{cases} d\log(1-\zeta^k x) \\ d\log(\zeta^{-k} y) \end{cases} \\
&= Im \int_{\gamma_{n,1}} \sum_{k=0}^{4} \{\log(\zeta^k x + \zeta^{-k} y) - \log(\zeta^{k+n})\} \begin{cases} d\log(1-\zeta^k x) \\ d\log(\zeta^{-k} y). \end{cases} \qquad (n=1,2)
\end{aligned}
$$

We actually compute the last member of this equality.

§3 Numerical computation of the value of the L function. We will explain how to compute numerically the value of $L(H^1(X),2)$ which the determinant of the regulator should represent. A general reference for this paragraph is [G-R].

By Weil [W2], $L(H^1(X),s)$ for $Res > 3/2$ can be written as an Euler product

$$
L(H^1(X),s) = \prod_{l \text{ prime} l \neq 5} P_l(l^{-s})^{-1}
$$

$$
\text{where} \quad P_l(T) = \prod_{\mathfrak{l}|l} (1 - \tau(\mathfrak{l})T^f)
$$

and $\mathfrak{l}$ denotes a prime ideal of $\mathbb{Q}(\zeta)$ and f is the order of $l (\mathrm{mod}\ 5)$. We denote $\mathbb{Q}(\zeta)$ by K in the sequel.

$$
\tau(\mathfrak{l}) = - \sum_{a \in O_K/\mathfrak{l} a \neq 0,1} \chi_{\mathfrak{l}}(a)\chi_{\mathfrak{l}}(1-a)
$$

is the Jacobi sum. $\chi_{\mathfrak{l}} : (O_k/\mathfrak{l})^* \longrightarrow \mu_5$ is a character defined by

$$
\chi_{\mathfrak{l}}(a) = \zeta^k \Leftrightarrow a^{\frac{N\mathfrak{l}-1}{5}} \equiv \zeta^k (\mathrm{mod}\mathfrak{l}).
$$

We denote the character of the ideal group of K which is induced by τ also by τ. If we let σ_h be the automorphism of K over $\mathbb{Q}$ given by

$$
\sigma_h(\zeta) = \zeta^h,
$$

the Stickelberger relation gives

$$
\tau(\mathfrak{l}) = \mathfrak{l}\mathfrak{l}^{\sigma_3} \quad (\text{as ideals of} K)
$$

in our case, and we have the congruence

$$
\tau(\mathfrak{l}) \equiv 1 (\mathrm{mod} (1-\zeta)^2).
$$

$\tau(\mathfrak{l})$ can be determined from these conditions. Let $\chi_\infty : K^* \to \mathbb{C}^*$ and $\varphi : K^* \to \mu_5$ be the characters given by

$$\chi_\infty(\alpha) = \alpha\alpha^{\sigma_3}$$

and

$$\varphi(\alpha) = \frac{\tau(\alpha)}{\chi_\infty(\alpha)}.$$

We use the following theta series for computation.

$$\Theta(\varphi, \chi_\infty, y) = \sum_{\alpha \in O_K} \tau((\alpha)) \exp(-2\pi(\alpha\overline{\alpha}y_1 + \alpha^{\sigma_3}\overline{\alpha^{\sigma_3}}y_2)).$$

We can get the value of $L(H^1(X), s)$ by the Mellin transform of this.

Let us denote the fundamental unit of $\mathbb{Q}(\sqrt{5})(=\frac{1+\sqrt{5}}{2})$ by u. Since the norm from K to $\mathbb{Q}(\sqrt{5})$ of the group of units of K is generated by u^2, we have

$$\int_{\mathbb{R}_+\times\mathbb{R}_+/\langle u^2\rangle} \Theta(\varphi, \chi_\infty, y)(Ny)^s d^\times y$$
$$= \sharp\{\text{root of unity in } K\}((2\pi)^{-s}\Gamma(s))^2 L(H^1(C), s)$$

We denote by ω the character $I_K/K^* \to \mathbb{C}^*$ induced by τ. $\omega = \prod_{v \text{ place of } K} \omega_v$ is as follows. For finite $v \nmid 5$, if $\pi_v \in K$ is a generator of v, $\omega_v(\pi_v) = \tau((\pi_v))$.

For $v = (1-\zeta), \omega_v(\alpha) = \varphi(\alpha)$ for $\alpha \in K^*$.

For v infinite, $\omega(x) = x^{-1}$.

Now we consider the standard function $\Phi = \prod \Phi_v$ on $\mathbb{A}_K$ attached to τ in the sense of Weil[W1], and its Fourier transform.

For finite v,we denote the maximal compact subring of K_v by r_v. The local factors Φ_v for each v is:

For finite $v \nmid 5$, Φ_v is the characteristic function of r_v.

For $v = (1-\zeta)$, $\Phi_v = \varphi$ on r_v^* and zero outside r_v^*.

For v infinite, $\Phi_v(x) = x\exp(-2\pi x\overline{x})$.

Let Φ' be the Fourier transform of Φ for some basic character χ of $\mathbb{A}_K$ and by the self dual Haar measure associated to it. Then by Weil[W1]

$$\Phi'(y) = \kappa|b|_{\mathbb{A}}^{1/2}\Phi_{\overline{\omega}}(by)$$

where κ is a complex number with $|\kappa| = 1$, and $b = (b_v) \in I_K$ is such that $\mathrm{ord}_v(b_v) = \mathrm{ord}_v a_v + f_v$

Here (a_v) is a differental idele of χ and f is the conductor of ω. We choose χ so that $(b_v) = 1$ at all finite places (we can do this since the class number of $K = 1$).

Poisson's summation formula gives

$$\Phi(0) + \sum_{\xi \in K^*} \Phi(z\xi) = |z|_{\mathbb{A}_K}^{-1}(\Phi'(0) + \sum_{\xi \in K^*} \Phi'(\xi z^{-1})). \tag{3.1}$$

for $z \in I_K$. For i=1 and 3, let v_i mean the infinite places of K which is given by imbedding K to $\mathbb{C}$ by σ_i. Let $y = (y_v) \in I_K$ be such that
for v finite, $y_v = 1$, and $y_{v_i} = \sqrt{y}_i$.

Let $b_1 \in K^*$ be such that $N_{k/\mathbb{Q}}(b_1) = |D_K|^{-1}N_{K/\mathbb{Q}}(f)^{-1}$ and $b_3 = b_1^{\sigma_3}$. By [G-R] Theorem3.1, $|D_K|^{-1}N_{K/\mathbb{Q}}(f)^{-1} = 5^{-5}$ now.

When (3.1) is applied to $z = y$, we have the following equality.

$$\sum_{\xi\in K^*} \varphi(\xi) \prod_{v \text{ finite}} ch_{\mathfrak{r}_v}(\xi)\xi\xi^{\sigma_3}\sqrt{y_1y_3}\exp(-2\pi(|\xi|^2y_1 + |\xi^{\sigma_3}|^2y_3))$$
$$=\kappa|b|_{\mathbb{A}_K}^{1/2} \sum_{\xi\in K^*} \varphi(\xi) \prod_{v \text{ finite}} ch_{\mathfrak{r}_v}(\xi)\overline{b_1b_3\xi\xi^{\sigma_3}}(\sqrt{y_1y_3})^{-1}$$
$$\times \exp(-2\pi(|\xi b_1|^2/y_1 + |\xi^{\sigma_3}b_3|^2/y_3))(y_1y_3)^{-1}.$$

Dividing both sides by $\sqrt{y_1y_3}$, we get

(3.2)

$$\sum_{\xi\in K^*} \varphi(\xi) \prod_{v \text{ finite}} ch_{\mathfrak{r}_v}(\xi)\xi\xi^{\sigma_3}\exp(-2\pi(|\xi|^2y_1 + |\xi^{\sigma_3}|^2y_3))$$
$$=\kappa|b|_{\mathbb{A}_K}^{1/2}\overline{b_1b_3}(y_1y_3)^{-2} \sum_{\xi\in K^*} \varphi(\xi) \prod_{v \text{ finite}} ch_{\mathfrak{r}_v}(\xi)\overline{\xi\xi^{\sigma_3}}\exp(-2\pi(|\xi b_1|^2/y_1 + |\xi^{\sigma_3}b_3|^2/y_3))$$
$$=\kappa\kappa'|b|_{\mathbb{A}_K}(y_1y_3)^{-2} \sum_{\xi\in K^*} \varphi(\xi) \prod_{v \text{ finite}} ch_{\mathfrak{r}_v}(\xi)\overline{\xi\xi^{\sigma_3}}\exp(-2\pi(|\xi b_1|^2/y_1 + |\xi^{\sigma_3}b_3|^2/y_3)).$$

Here $\kappa' = \overline{b_1b_3}/|b|_{\mathbb{A}_K}^{1/2}$ and $\kappa\kappa'$ equals to the root number associated to $L(H^1(X), s)$. By [G-R](loc. cit),

$$\kappa\kappa' = 1$$

So (3.2) means

$$\Theta(\varphi, \chi_\infty, y_1, y_3) = 5^{-5}(y_1y_3)^{-2}\Theta(\varphi, \chi_\infty, \frac{|b_1|^2}{y_1}, \frac{|b_3|^2}{y_3}).$$

By this formula the problem of convergence of the theta series for y near zero is settled, and the value of $L(H^1(X), 2)$ can be computed quite accurately.

Ratio of the determinant of the regulator and the value of L function. $H^1(X, 2\pi i\mathbb{R})$ has a natural $\mathbb{Q}$ structure given by $H^1(X, 2\pi i\mathbb{Q})$, and from what we have proved so far,it has another $\mathbb{Q}$ structure given by the image of the regulator. The Beilinson conjecture predicts that determinant of the matrix of the base change of these two $\mathbb{Q}$ structures coincides with $L^{(2)}(H^1(X), 0)$ up to a rational multiple. The author computed numerically

$$\frac{\begin{vmatrix} \int_{\gamma_{1,1}^2} \mathrm{reg}_C(\pi^*\alpha) & \int_{\gamma_{2,1}^2} \mathrm{reg}_C(\pi^*\alpha) \\ \int_{\gamma_{1,1}^2} \mathrm{reg}_C(\pi^*\beta) & \int_{\gamma_{2,1}^2} \mathrm{reg}_C(\pi^*\beta) \end{vmatrix}}{(2\pi)^2L^{(2)}(H^1(X), 0)}$$

and verified that it coincides with $\frac{5}{12}$ up to 12 decimal place. We put here the value of entries of the matrix and of L function.

$$\int_{\gamma^2_{1,1}} \mathrm{reg}_C(\pi^*\alpha) = -1.49583966208347069$$

$$\int_{\gamma^2_{2,1}} \mathrm{reg}_C(\pi^*\alpha) = -1.784705124594349710$$

$$\int_{\gamma^2_{1,1}} \mathrm{reg}_C(\pi^*\beta) = -0.6645758775848033518$$

$$\int_{\gamma^2_{2,1}} \mathrm{reg}_C(\pi^*\beta) = -22.2610909571881774599$$

$$L(H^1(X),2) = 0.0006247146595905$$

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