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FLOER'S INFINITE DIMENSIONAL MORSE THEORY AND HOMOTOPY THEORY

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§1 INTRODUCTION

This paper is a progress report[1] on our efforts to understand the homotopy theory underlying Floer homology. Its objectives are as follows:

(A) to describe some of our ideas concerning what, exactly, the Floer homology groups compute;
(B) to explain what kind of an object we think the 'Floer homotopy type' of an infinite dimensional manifold should be;
(C) to work out, in detail, the Floer homotopy type in some examples.

We have not solved the problems posed by the underlying questions, but we do have a programme which we hope will lead to solutions. Thus it seems worthwhile to describe our ideas now, especially in a volume of papers dedicated to the memory of Andreas Floer. We plan to write a complete account of this approach to Floer homotopy theory in a future paper.

Floer homology arises in two different contexts, the study of curves and surfaces in symplectic manifolds, and gauge theory on three- and four-dimensional manifolds. In each of these contexts there are two different perspectives, which one can think of as 'Hamiltonian' and 'Lagrangian'.

The theory began with Floer's proof of the Arnold conjecture. On a compact symplectic manifold M a Hamiltonian flow is generated by a Hamiltonian function $h : M \to \mathbb{R}$, and the stationary points of the flow are the critical points of h. Classical Morse theory tells us that there are at least as many such points as the dimension of the homology $H_*(M;\mathbb{R})$. Arnold conjectured that the same is true of the number of fixed points of a diffeomorphism $\varphi_1 : M \to M$ which arises from a *time dependent* Hamiltonian flow $\{\varphi_t\}_{0\leq t\leq 1}$. The trajectories of such a flow are critical points of the 'action functional' S_h on the space of paths $\gamma : [0,1] \to M$, where

$$S_h(\gamma) = \int_\gamma (pdq - hdt),$$

[1]The following quotation from Blaise Pascal, *Lettres Provinciales* XVI (1657), is quite appropriate: Je n'ai fait celle-ci plus longue que parce que je n'ai pas eu le loisir de la faire plus courte.

and $h : M \times \mathbb{R} \to \mathbb{R}$ is the varying Hamiltonian. The fixed points of φ_1 are therefore the critical points of S_h on the space $\mathcal{L}M$ of loops of length 1 in M. Thus Arnold's conjecture would follow from a version of Morse theory applicable to the function $S_h : \mathcal{L}M \to \mathbb{R}$, but relating its critical points to the homology of M rather than $\mathcal{L}M$. This was the theory that Floer developed.

In conventional Morse theory it is geometrically clear why the homotopy type of a manifold X is reflected in the disposition of the critical points of a function $f : X \to \mathbb{R}$; for example, if there are no critical points apart from the minimum, the gradient flow of f provides a contraction of X. At present there is no comparable homotopy-theoretic underpinning of Floer theory. Ordinary homology can be defined in a great variety of quite different ways, but one does not know how to define Floer's groups without using a Floer-Morse function. One of our purposes in this paper is to speculate about what exactly the Floer groups are describing, or what additional structure an infinite dimensional manifold such as $\mathcal{L}M$ must have for the groups to be defined. We should say at the outset, however, that we have not solved this problem.

One ingredient in the answer is plain. An important feature of a Floer-Morse function, such as S_h, is that at critical points its Hessian has infinitely many negative as well as positive eigenvalues. In fact at every point γ of $\mathcal{L}M$ the Hessian of S_h decomposes the tangent space $T_\gamma(\mathcal{L}M)$ into two parts $T_\gamma^\pm$ corresponding to the positive and negative eigenvalues, with a finite dimensional ambiguity coming from the zero eigenspace. Such an approximate splitting of the tangent bundle of an infinite dimensional manifold X we shall call a *polarization* of X. A formal definition will be given in §2. The significance of the polarization becomes clearer if we turn to the second – or Lagrangian – perspective on Floer theory.

If we choose a Riemannian metric on M making it an almost complex manifold then we can consider *pseudo-holomorphic* maps $\varphi : \Sigma \to M$, where Σ is a Riemann surface. In Arnold's problem the gradient flow lines of the function $S_h : \mathcal{L}M \to \mathbb{R}$ when $h = 0$ are precisely the pseudo-holomorphic maps $S^1 \times \mathbb{R} \to M$. If a closed surface $\Sigma = \Sigma_1 \cup \Sigma_2$ is the union of two pieces intersecting in a common boundary circle then, because a pseudo-holomorphic map $\Sigma_i \to M$ is determined by its boundary values, the finite dimensional space $Z_\Sigma = \mathrm{Hol}(\Sigma; M)$ can be regarded as the intersection of two infinite dimensional submanifolds $Z_{\Sigma_i} = \mathrm{Hol}(\Sigma_i; M)$ of the loop space $\mathcal{L}M$. Here the notation Hol means pseudo-holomorphic maps. The tangent spaces to Z_{Σ_1} and Z_{Σ_2} are close – in a sense explained in §2 – to the positive and negative parts $T_\gamma^\pm$ of the polarization of $T_\gamma(\mathcal{L}M)$. Furthermore Z_{Σ_1} and Z_{Σ_2} define a cycle and a cocycle respectively in the Floer theory of $\mathcal{L}M$, and the pairing between them is, in good cases, the number of isolated pseudo-holomorphic maps $\Sigma \to M$.

This suggests, compare [4], that Floer theory is the homology theory of *semi-infinite dimensional cycles* in a polarized manifold. There is a natural concept of semi-infinite dimensional differential forms on such a manifold, and one might hope to use them to give a de Rham definition of Floer homology; see [11] for interesting work along these lines. To define the Floer groups for an infinite dimensional manifold it seems clear that more structure is needed than just the polarization of X. A crucial point seems to be that the critical manifold of Floer-Morse function is *compact*, and it seems conceivable that some preferred class of compact subspaces of X should be an ingredient in the structure.

Though we cannot answer the question 'what does the Floer homotopy type of a manifold depend on', we can do better with another question, 'what sort of object is the Floer homotopy type of a manifold'. Unfortunately one cannot hope that the Floer groups of X are the ordinary homology groups of a space associated to X, or even of a 'stable space' or a 'spectrum'. We shall show that under reasonable hypotheses one can associate to the flow category, see [9], of a Floer-Morse function an object called a *pro-spectrum*. This is a technical homotopy-theoretic concept which has proved to be central in one of the deepest recent results of homotopy theory. As the pro-spectra involved arise in Floer theory on the projective spaces of polarized vector spaces and also the loop space $\mathcal{L}\mathbb{CP}^n$, it seems worthwhile to explain this result briefly.

For any positive integers $n > m$ let P^n_m be the space obtained from real projective space $\mathbb{RP}^n$ by collapsing the standard linear subspace $\mathbb{RP}^{m-1}$ to a point. Now fix two large positive integers p and q and consider the homotopy groups $\pi_{i+N}(P^{N+p}_{N-q})$ as $N \to \infty$. Notice that P^{N+p}_{N-q} is a space made of cells whose dimensions range from $N+p$ to $N-q$, so that for small $|i|$ we are looking at a kind of 'middle dimensional' homotopy group. The following deep theorem of Lin [20], conjectured by Mahowald [21] and Adams [2], was a crucial step in determining the stable homotopy type of the classifying spaces of finite groups and proving the Segal conjecture; see [3] and [8] for surveys and further references.

Theorem. *If N is a mutiple of 2^{p+q}, then there is a map $S^{N-1} \to P^{N+p}_{N-q}$ which induces isomorphisms*

$$\pi_{i+N}(P^{N+p}_{N-q}) \cong \begin{cases} \mathbb{Z}/2^{a(N)} & \text{if } i = -1 \\ \pi_{i+N}(S^{N-1})_{(2)} & \text{if } i \neq -1 \text{ but } -p \ll i \ll q \end{cases}$$

where $\pi_{i+N}(S^{N-1})_{(2)}$ is the 2-primary component of the homotopy group $\pi_{i+N}(S^{N-1})$, and $a(N) \to \infty$ as $N \to \infty$.

In particular note that, when i is small compared to N, $\pi_{i+N}(P^{N+p}_{N-q})$ is independent of N, except that when $i = -1$ it tends to the 2-adic completion of $\pi_{N-1}(S^{N-1}) = \mathbb{Z}$.

There are natural inclusions $P^{N+p}_{N-q} \to P^{N+p+1}_{N-q}$ and collapsing maps $P^{N+p}_{N-q} \to P^{N+p}_{N-q+1}$. In addition, if N and M are both multiples of 2^{p+q} with

$N \geq M$, there is a homotopy equivalence between P_{N-q}^{N+p} and the $(N-M)$-fold suspension $S^{N-M}P_{M-p}^{M+q}$. It follows that the system of spaces P_{N-q}^{N+p} form a pro-spectrum, and this is the prime example of a pro-spectrum.

Let us now outline the contents of this paper. In §2 we describe some of the homotopy-theoretic properties of polarized manifolds. In §3 we describe the flow categories of Morse functions. In §4 we analyse the flow category of $\mathcal{L}\mathbb{CP}^n$, and explain how to compactify this category. In §5 we describe a method of recovering the stable homotopy type of a finite dimensional manifold from the flow category of a Morse-Bott function. This method is not the same as that used in [9]; in spirit, it is related to the work of Franks [14]. We go on to show how a pro-spectrum can be associated to the idealized flow category of a Floer function. In §6 we describe how the ideas of §5 can be applied to the projective space of a polarized vector space, and to the area function on $\mathcal{L}\mathbb{CP}^n$, and we identify the corresponding pro-spectra. The most surprising point is that the pro-spectrum associated to the compactified flow category of the area function on $\mathcal{L}\mathbb{CP}^n$ is the complex analogue of the one occuring in Lin's theorem.

In an Appendix we give a very brief account, for non-experts, of some of the ideas which lead to the introduction of the stable category of spaces, the category of spectra, and the notion of a pro-spectrum.

§2 Polarized manifolds

A polarization of a real topological vector space E is a *class* of decompositons of $E = E_+ \oplus E_-$ which do not differ too much among themselves. The main example arises when one has an unbounded self adjoint Fredholm operator $D : E \to E$. This splits E according to the positive and negative parts of the spectrum of D: we want to allow the ambiguity of assigning the 0-eigenspace arbitrarily to E_+ or E_-. The most convenient definition is as follows.

Definition 2.1. *A polarisation of E is a class $\mathcal{J}$ of linear operators $J : E \to E$, all congruent modulo the ideal of compact operators, and such that $J^2 = 1$ modulo compact operators. Further, $\mathcal{J}$ must not contain $+1$ or -1.*

If E is polarized we can define the restricted general linear group $GL_{\text{res}}(E)$ which consists of all $g \in GL(E)$ which preserve the polarization. We can also define the restricted Grassmannian $Gr_{\text{res}}(E)$, consisting of all the (-1)-eigenspaces of all $J \in \mathcal{J}$ such that $J^2 = 1$.

If E is a Banach space then $GL_{\text{res}}(E)$ can be regarded as a closed subgroup of $GL(E)$ with the norm topology. But, in general, it is better to give it the topology for which $\{g_\alpha\}$ converges if both $\{g_\alpha\}$ and $\{g_\alpha^{-1}\}$ converge in the compact-open topology, and $[J, g_\alpha]$ converges in the uniform topology for some (and hence all) $J \in \mathcal{J}$.

A polarized manifold X is one whose tangent spaces T_xX are polarized. More precisely, if X is modelled on E the structural group of the tangent bundle of X is reduced to $GL_{\mathrm{res}}(E)$. In all the examples we know the polarizations are *integrable*, that is X has an atlas $\{\varphi_\alpha : U_\alpha \to X\}$ such that $D(\varphi_\beta \varphi_\alpha^{-1})(y) \in GL_{\mathrm{res}}(E)$ for all $y \in \varphi_\alpha(U_\alpha)$; but we shall not need this. The two basic examples which arise in Floer theory are the following.

(i) An almost complex structure on a Riemannian manifold M defines a polarization of the loop space $\mathcal{L}M$. The tangent space T_γ at $\gamma \in \mathcal{L}M$ is the space of tangent vector fields to M along γ, and we have the self-adjoint operator $jD/D\theta : T_\gamma \to T_\gamma$, where j is the almost-complex structure of M and $D/D\theta$ is covariant differentiation. The spectral decomposition of $jD/D\theta$ polarizes $\mathcal{L}M$.

(ii) The space $X = \mathcal{A}^*/\mathcal{G}$, where $\mathcal{A}^*$ is the space of irreducible connections on a complex vector bundle E with compact structural group G on a 3-manifold M and $\mathcal{G} = \mathrm{Aut}(E)$ is the gauge group of E, also carries a natural polarization. The tangent space to $\mathcal{A}^*$ at any point is $\Omega^1(M; \mathrm{End}(E))$, and that of $\mathcal{A}/\mathcal{G}$ at a connection A is the cokernel of

$$d_A : \Omega^0(M; \mathrm{End}(E)) \to \Omega^1(M; \mathrm{End}(E)).$$

If A is a flat connection then $d_A^2 = 0$, and the operator $*d_A$ induces a self-adjoint Fredholm operator, and hence a polarization, on the tangent space $T_A = \Omega^1/d_A\Omega^0$. With more work one can define the polarization at all points of X.

In both these cases the polarization is the same as the polarization induced by a Floer-Morse function. In the first case it is the area functional – the action functional S_h, described in the introduction, with $h = 0$. In the second case it is the Chern-Simons functional

For the usual topological vector spaces of analysis the group $GL(E)$ is contractible, and so the tangent bundle of a manifold X modelled on E carries no homotopy-theoretic information. The position is different when E is polarized. For the group $GL_{\mathrm{res}}(E)$ has – for the usual choices of E – the homotopy type of $\mathbb{Z} \times BO$, the classifying space for stable finite dimensional vector bundles; see [22]. The tangent bundle of a polarized manifold X is therefore described by a map $X \to BGL_{\mathrm{res}}(E)$, determined up to homotopy, which we call the *structural map* of X. By Bott periodicity the homotopy type of $BGL_{\mathrm{res}}(E)$ is U/O, where $U = \bigcup U(n)$ and $O = \bigcup O(n)$ are the infinite unitary group and orthogonal group respectively. The space U/O represents the functor KO^1. Its fundamental group is $\mathbb{Z}$ and its rational cohomology is an exterior algebra on generators of dimensions $4k+1$. Therefore, a polarized manifold X has characteristic classes in $H^{4k+1}(X; \mathbb{Q})$.

The most obvious information provided by the structural map $X \to U/O$ concerns the *grading* of Floer homology. At each point $x \in X$ we have the Grassmannian $Gr_x = Gr_{\mathrm{res}}(T_x X)$, whose connected components correspond to the integers $\mathbb{Z}$, though with no preferred choice of zero: two points of Gr_x have a well-defined *relative dimension*, but no absolute dimension [22]. The sets $\tilde{X}_x = \pi_0(Gr_x)$, as x varies, form a covering space $\tilde{X}$ of X, and when one goes around a path λ in X starting at x the holonomy $\tilde{X}_x \to \tilde{X}_x$ shifts $\tilde{X}_x$ by the image of λ in $\pi_1(U/O) = \mathbb{Z}$. This means that for a particular polarized manifold X the 'dimension' of a semi-infinite subspace of T_xX, and hence of a semi-infinite cycle, or Floer homology class, can be taken to be well-defined modulo the image of $\pi_1(X)$ in $\pi_1(U/O) = \mathbb{Z}$. On the covering space $\tilde{X}$ the dimension – or 'virtual dimension'– is a well-defined element of $\mathbb{Z}$.

If M is an almost-complex manifold of real dimension $2m$, then the tangent bundle of M is classified by a map $\theta : M \to BU(m)$. The structural map of $X = \mathcal{L}M$ is easily seen to be the composite

$$\mathcal{L}M \xrightarrow{\mathcal{L}\theta} \mathcal{L}BU(m) \xrightarrow{\beta} U \to U/O,$$

where β is the transgression. It is important that the structural map factorizes through U, i.e. the structural group of $T\mathcal{L}M$ is reduced to the *complex* restricted general linear group. This means, in particular, that the grading of Floer homology is always well defined modulo 2, for the map $\pi_1(U) \to \pi_1(U/O)$ is multiplication by 2.

The same is true in the gauge theory case, when X has the homotopy type of $\mathrm{Map}(M; BG)$. Then the structural map is the composite

$$\mathrm{Map}(M; BG) \to \mathrm{Map}(M; BU(k)) \to U \to U/O,$$

where the first map is induced by the representation $G \to U(k)$ which defines the bundle E, and the second map is the direct-image map in complex K-theory. (This map represents the element of $K^{-3}(\mathrm{Map}(M; BU(k))$ obtained by pulling back the tautological element of $K(BU(k))$ to $K(M \times \mathrm{Map}(M; BU(k)))$ by the evaluation map, and then 'integrating' over the 3-dimensional manifold M, i.e. evaluating on the K-theory fundamental class in $K_3(M)$.)

From these descriptions of the structural map it is easy to compute its effect on π_1. If M is a simply connected almost complex manifold, we have $\pi_1(\mathcal{L}M) = \pi_2(M)$. The homomorphism

$$\pi_2(M) = \pi_1(\mathcal{L}M) \to \pi_1(U/O) = \mathbb{Z}$$

is the homomorphism defined by $2c_1(M)$, and the grading of Floer homology is well defined modulo its image. In the gauge theory case with structure group $G = SU(2)$, we have $\pi_1(\mathcal{A}^*/\mathcal{G}) = \mathbb{Z}$. The corresponding homomorphism is multiplication by 8, and the grading of Floer homology is well-defined modulo 8.

§3 The flow category

Let us begin by considering a Morse-Bott function $f : X \to \mathbb{R}$ on a finite dimensional compact Riemannian manifold X. Morse-Bott means that the critical set F of f is a smooth manifold, and that the Hessian $D^2 f$ is non-degenerate on the fibres of the normal bundle to F in X.

In this situation we can define a category $\mathcal{C}_f$ whose objects are the critical points of f and whose morphisms from x to y are the piecewise gradient trajectories (or flow lines) γ of f from x to y. This means one permits γ to stop at intermediate critical points en route. More precisely, γ is a sequence $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_p)$ where $\gamma_i : \mathbb{R} \to X$ is a descending trajectory of grad(f) such that

$$\gamma_i(t) \to x_i^{\pm}, \qquad \text{as } t \to \pm\infty$$
$$x = x_0^-, \qquad x_i^+ = x_{i+1}^-, \qquad x_p^+ = y.$$

We identify two such sequences if they differ only by translating the parameters of the γ_i. The category $\mathcal{C}_f$ is a *topological* category [23], in that the sets Ob$(\mathcal{C}_f)$ and Mor$(\mathcal{C}_f)$, of objects and morphisms, have natural topologies, and the structure maps of $\mathcal{C}_f$ are continuous.

In the case where the gradient flow of the function f also satisfes an appropriate version of the Smale transversality condition, see [26], these spaces have a great deal of extra structure which we now describe in detail. Let the critical values of f be $t_n > t_{n-1} \cdots > t_0$, and let the critical manifold with critical value t_i be F_i; then

$$\text{Ob}(\mathcal{C}_f) = \coprod F_i.$$

If $j > i$, let the space of morphisms from points on F_j to points on F_i be $F(j,i)$. It is known, see [26] and [6], that $F(j,i)$ is a compact manifold with corners. By a manifold with corners we mean a manifold modelled on the space $\mathbb{R}_+^d$, where $\mathbb{R}_+$ is the set of real numbers x with $x \geq 0$, and by the boundary of a manifold with corners we mean the set of points which in coordinate charts do not lie in the interior of $\mathbb{R}_+^d$.

If $I = (i_{k+1}, i_k, \ldots, i_0)$ is a sequence with

$$j = i_{k+1} > i_k > \cdots > i_0 = i,$$

let $F(I)$ be the part of $F(j,i)$ consisting of piecewise trajectories which stop at all of the F_{i_r} for $1 \leq r \leq k$. Then $F(j,i)$ is stratified by the $F(I)$, and $F(I)$ is a compact submanifold of codimension k in $F(j,i)$. Furthermore, $\partial F(I)$ is the union of the $F(J)$ with $J \supset I$. In a neighbourhood of a point of $F(I)$ the space $F(j,i)$ is modelled on $\mathbb{R}_+^k \times \mathbb{R}^{m-k}$ where $m = \dim F(j,i)$. Composition in the category maps $F(j,r) \times F(r,i)$ diffeomorphically to $F(j,r,i) \subset \partial F(j,i)$. Finally the beginning and end point maps

$$F_j \xleftarrow{\pi_j} F(j,i) \xrightarrow{\pi_i} F_i$$

are fibrations.

We shall call a category of the type just described a *compact smooth* category. In the finite dimensional case it has one further basic property: it is *framed*, in the following sense.

Let E_i be the downward part of the tangent bundle to X along F_i. That is, E_i is the sub-vector-bundle of the normal bundle of F_i spanned by the eigenvectors of the Hessian $D^2 f$ corresponding to negative eigenvalues. The geometry of the flows gives us canonical isomorphisms of vector bundles on $F(j,i)$

$$\pi_j^* E_j \cong T_{ji} \oplus \mathbb{R} \oplus \pi_i^* E_i, \tag{3.1}$$

where T_{ji} is the tangent bundle along the fibres of the projection $\pi_j : F(j,i) \to F_j$. These isomorphisms are compatible with the compositions in the category C_f. In the case when the function f has isolated critical points the isomorphism (3.1) is a stable framing of the flow manifold in the usual sense.

To see that (3.1) holds, observe that $F(j,i)$ embeds in the sphere bundle $S(E_j)$ of E_j. (Actually the natural map $F(j,i) \to S(E_j)$ is not injective on $\partial F(j,i)$, but that is irrelevant to the present argument, and in any case the map can be made injective by a canonical small deformation.) The normal bundle to $F(j,i)$ in $S(E_j)$ can be identified canonically with $\pi_i^* E_i$, as its fibre at $\gamma \in F(j,i)$ consists of the piecewise trajectories emanating from $\pi_j(\gamma)$ which just miss F_i.

In the infinite dimensional situation which Floer considered the function f always has a compact critical manifold F, and there is a flow category C_f in which each connected component of the space of morphisms is finite dimensional. Three new features, however, need to be considered.

(i) The function f is not usually single-valued. Usually it takes values in $\mathbb{R}/\mathbb{Z}$, but in principle it might be the indefinite integral of any closed 1-form representing a class α in $H^1(X;\mathbb{R})$. Floer theory seems to work well only in the *monotone* case where α is a multiple of the basic element of $H^1(X;\mathbb{Z})$ defined by the structural map $X \to U/O$ of the polarized manifold X. We shall confine ourselves to this case. Then f can be lifted to a map $\tilde{f} : \tilde{X} \to \mathbb{R}$, where $\tilde{X}$ is the infinite cyclic cover of X defined by $X \to U/O$. The critical set of $\tilde{f}$ is then an infinite disjoint union $\coprod_{i\in\mathbb{Z}} F_i$, where each F_i is compact, and is periodic in i with some finite period. Henceforth, when we speak of the flow category of a Floer function we shall mean the flow category of $\tilde{f}$.

(ii) The flow category is no longer framed. We still have the isomorphisms (3.1), but now the bundles E_i are infinite dimensional, and so give no information about the tangent bundle T_{ji}, except to give it a complex structure when the structural map $X \to U/O$ of the polarized manifold X lifts to U. This feature was pointed out long ago by Floer himself. We shall see that whether the flow category is framed is essentially the same question as whether the structural map $\tilde{X} \to U/O$ is homotopic to a constant map.

(iii) Because of the phenomenon of 'bubbling', the flow spaces $F(j,i)$ are no longer compact. This is the most important difference from the finite dimensional case, and the hardest to handle. In the cases that we have studied in detail there is a natural way to compactify the $F(j,i)$ so that one has a compact smooth category, but the precise relation between the categories before and after compactification is still not well-understood.

A topological category $\mathcal{C}$ has a realization $|\mathcal{C}|$ as a topological space; see [23]. For a Morse-Bott function on a finite dimensional manifold it is not difficult to prove that the realization $|\mathcal{C}_f|$ is homotopy equivalent to X. Indeed, in the case of a Morse-Bott-Smale function, that is a Morse-Bott function whose gradient flow satisfies an approriate transversality condition, $|\mathcal{C}_f|$ is even homeomorphic to X. These results are proved in [9].

It is striking that for the flow categories of the usual Floer functions it still seems to be true that $|\mathcal{C}_f|$ is homotopy equivalent to X, if one does *not* compactify the category. Thus if $X = \mathcal{L}M$ is the loop space of a Kähler manifold M with $\pi_2(M) = \mathbb{Z}$ for which one knows that the inclusion

$$\mathrm{Hol}_k(S^2; M) \to \mathrm{Map}_k(S^2; M) \tag{3.2}$$

of holomorphic maps of degree k into smooth maps of degree k tends to a homotopy equivalences as $k \to \infty$, then, as we shall show in a future paper, $|\mathcal{C}_f| \simeq X$. The hypothesis is known to hold when X is $\mathbb{CP}^n$ [24], or more generally a Grassmannian [17]. Furthermore the appropriate version of (3.2) (taking account of the fact that π_2 is free abelian on more than one generator) is also true for a flag manifold [16].

A version of (3.2) is also true for the flow category arising in the context of gauge theory for a compact group G, in virtue of the corresponding homotopy approximation property for the inclusion

$$\mathrm{Hol}_k(S^2; \Omega G) \to \mathrm{Map}_k(S^2; \Omega G).$$

In fact to show that $|\mathcal{C}_f| \simeq X$ one needs only the weak version of this result – called the Atiyah-Jones conjecture [5] – which was proved by Taubes [27], and Gravesen [15], rather than the stronger version proved in [7] and [18].

We should also point out that for any compact smooth category $\mathcal{C}$ with

$$\mathrm{Ob}(\mathcal{C}) = \coprod F_i, \qquad \mathrm{Mor}(\mathcal{C}) = \coprod F(j,i),$$

the tangent bundles along the fibres of $F(j,i) \to F_j$ really define a *functor* from the topological category $\mathcal{C}$ to a topological category $\mathcal{V}$. This category $\mathcal{V}$ has one object; its morphisms $\mathrm{Mor}(\mathcal{V})$ are finite dimensional vector spaces; and the composition law is direct sum. The functor assigns to a morphism $\gamma \in F(j,i)$ the vector space $T_{ji,\gamma} \oplus \mathbb{R}$.

To be more precise, $\mathcal{V}$ is the topological semi-group

$$\mathcal{V} = \coprod_{p \geq 0} BGL_p(\mathbb{R}),$$

and the functor is a coherent collection of maps $F(j,i) \to \mathcal{V}$ which are classifying maps for the bundles $T_{ji} \oplus \mathbb{R}$. The realization $|\mathcal{V}| = B\mathcal{V}$ is the space U/O of §2, and so the functor $\mathcal{C} \to \mathcal{V}$ induces a map $|\mathcal{C}| \to U/O$ which, when $|\mathcal{C}| \simeq X$, is the structural map of the polarization of X.

§4 The area function on $\mathcal{L}\mathbb{CP}^n$

We now analyse the flow category of the area functional on $\mathcal{L}\mathbb{CP}^n$. Our main objective is to show that even though this flow category is not compact it does have a natural compactification, which turns out to be the flow category of a function on an infinite dimensional complex projective space. For simplicity we describe the details for $\mathbb{CP}^1 = S^2 = \mathbb{C} \cup \infty$.

As we saw in §2, we must really consider the area functional on the universal cover $\tilde{\mathcal{L}}S^2$ of $\mathcal{L}S^2$. This is the space of smooth maps $S^1 \to S^2$ together with an extension to a smooth map $D^2 \to S^2$ which is well-defined up to homotopy relative to the boundary. If $f : D^2 \to S^2$ is a smooth map its area is given by

$$\int_{D^2} f^*\omega,$$

where ω is the standard symplectic 2-form on S^2. This gives smooth functions

$$\tilde{\mathcal{A}} : \tilde{\mathcal{L}}S^2 \to \mathbb{R}, \qquad \mathcal{A} : \mathcal{L}S^2 \to \mathbb{R}/4\pi\mathbb{Z}.$$

The critical points of $\mathcal{A}$ are the constant loops, and those of $\tilde{\mathcal{A}}$ are pairs (γ, n), where γ is a constant loop and n is the degree of the extension. Thus the critical manifold of $\mathcal{A}$ is S^2, and that of $\tilde{\mathcal{A}}$ is $S^2 \times \mathbb{Z}$.

It is easy to see, compare [12], that the gradient vector field of $\mathcal{A}$ at a loop γ is the vector field along γ given by $j\dot{\gamma}$, where j is the complex structure on TS^2. This means that flow lines of $\mathcal{A}$ are given by holomorphic maps $h : S^2 \to S^2$ in the following way. Consider the path in $\mathcal{L}S^2$ given by $t \mapsto h_t$ where

$$h_t(s) = h(e^{-t+is}).$$

(We have parametrized loops by the closed interval $[0, 2\pi]$.). Then h_t converges to the constant loop at $h(\infty) = a$ as $t \to -\infty$, and as $t \to \infty$ it converges to the constant loop at $h(0) = b$. This path h_t is a flow line of $\mathcal{A}$, and every flow line from a to b is of this form.

The holomorphic map h gives a natural exension of the loop h_t to the lower hemisphere of S^2 and this defines a path $\tilde{h}_t$ in $\tilde{\mathcal{L}}S^2$. This path $\tilde{h}_t$ is a flow line of $\tilde{\mathcal{A}}$ from (a,k) to $(b,0)$, where k is the degree of h, and every such flow line arises in this way. The flow lines from $(a,n+k)$ to (b,n) are given by applying the appropriate covering translation to flow lines from (a,k) to $(b,0)$.

Let $W(n,m) \subset \tilde{\mathcal{L}}S^2$ be the space of points which lie on flow lines from a critical point of the form (a,n) to one of the form (b,m). Thus a point is in $W(n,m)$ if and only if it is on a flow line of $\mathcal{A}$ which starts at level n and ends at level m. The above identification of the flow lines of $\mathcal{A}$ shows that

$$W(n,m) = \mathrm{Rat}_{n-m}$$

where Rat_{n-m} is the space of holomorphic maps, or rational functions, $h: S^2 \to S^2$ of degree $n-m$.

The space Rat_k is not compact and it is very important to understand this non-compactness. A rational function $h: S^2 \to S^2$ of degree k is given by $h = p/q$ where p and q are polynomials of degree $\leq k$ with no roots in common. Throughout we allow roots at infinity: thus if p has degree r with $r < k$ then we say p has $k-r$ roots at infinity. This is a convenient device which, for example, allows us to say that the zeroes of the rational function p/q are the roots of p, and its poles are the roots of q. Suppose we now take a sequence of rational functions $h_n = p_n/q_n$ where a root α_n of p_n converges to a root β_n of q_n. Then this sequence does not converge in Rat_k. This is the bubbling phenomenon for rational functions and we shall say that a bubble occurs at the point a which is the common limit of α_n and β_n.

The reason for this terminology is as follows. Suppose that $|\alpha_n - \beta_n|$ is extremely small, and that $\varepsilon > 0$ is also very small but much greater than $|\alpha_n - \beta_n|$. Let D be the disc of radius ε around β_n. Then p_n/q_n is almost constant on ∂D. Outside D the function p_n/q_n is almost equal to a rational function of degree $k-1$, namely $(x-\beta_n)p_n/(x-\alpha_n)q_n$. The interior of D, however, is mapped by a map which is almost surjective with degree 1.

If we have a sequence of rational functions h_n in which a bubble occurs at either ∞ or 0 then the corresponding sequence of paths in $\tilde{\mathcal{L}}S^2$ converges to a piecewise flow line. However, if the bubble occurs at any other point the corresponding sequence of paths, no matter how it is parametrized, does not even converge to a path in $\tilde{\mathcal{L}}S^2$. Thus the flow category $\mathcal{C}_{\mathcal{A}}$ is not compact.

We now construct a compactification of $\mathcal{C}_{\mathcal{A}}$. Let $W = \mathbb{C}[z, z^{-1}]$ be the vector space of Laurent polynomials topologized as a space of maps $\mathbb{C}^\times \to \mathbb{C}$, where $\mathbb{C}^\times = \mathbb{C}\setminus 0$. Now the linear flow $z^n \mapsto e^{nt}z^n$ defines a flow Φ on $\mathbb{P}(\mathbb{C}^2 \otimes W)$. It is straightforward to check that the space of stationary points of Φ is $S^2 \times \mathbb{Z}$, where $S^2 \times n$ is the subspace $\mathbb{P}(\mathbb{C}^2 \otimes z^n) \subset \mathbb{P}(\mathbb{C}^2 \otimes W)$. Let W_m^n be the subspace of W spanned by z^i with $n \leq i \leq m$. The space of points which lie on piecewise flow lines of Φ which go from level n to level m is $\mathbb{P}(\mathbb{C}^2 \otimes W_m^n) = \mathbb{CP}^{2(n-m)+1}$, and since this space is compact the flow category $\mathcal{C}_\Phi$ is compact.

A pair (f_0, f_1) of elements of W with no roots in common except, possibly, at 0 and ∞ defines a map

$$f : \mathbb{C}^\times \to \mathbb{CP}^1 = S^2.$$

Since this map is algebraic it extends to a holomorphic map $f : \mathbb{C} \cup \infty = S^2 \to S^2$. Let $\mathcal{U}$ be the open subset of the projective space $\mathbb{P}(\mathbb{C}^2 \otimes W)$ defined by the pairs (f_0, f_1) with no roots in common in $\mathbb{C}^\times$. Then there is a map

$$i : \mathcal{U} \to \tilde{\mathcal{L}}S^2$$

defined by restricting the holomorphic map $f : S^2 \to S^2$ given by the pair (f_0, f_1) to the unit circle and using the extension of this loop to the lower hemisphere to get an element of $\tilde{\mathcal{L}}S^2$. It is clear that $\mathcal{U} \subset \mathbb{P}(\mathbb{C}^2 \otimes W)$ is invariant under the flow Φ and that $i : \mathcal{U} \to \tilde{\mathcal{L}}S^2$ is equivariant with respect to the flow Φ on $\mathcal{U}$ and the gradient flow of $\mathcal{A}$ on $\tilde{\mathcal{L}}S^2$. Furthermore it is straightforward to check that $i : \mathcal{U} \to \tilde{\mathcal{L}}S^2$ defines an isomorphism of flow categories. In fact, this map i is a homotopy equivalence, as we will show in a future paper, but we do not need this for our present purpose.

The diagram

$$\tilde{\mathcal{L}}S^2 \xleftarrow{i} \mathcal{U} \xrightarrow{j} \mathbb{P}(\mathbb{C}^2 \otimes W)$$

(where j is the inclusion), together with the fact that i induces an isomorphism of flow categories, gives us an embedding of flow categories

$$\mathcal{C}_\mathcal{A} \to \mathcal{C}_\Phi.$$

The flow category $\mathcal{C}_\Phi$ is compact and $\mathcal{C}_\mathcal{A}$ is embedded as an open dense subcategory; therefore $\mathcal{C}_\Phi$ is a compactification of $\mathcal{C}_\mathcal{A}$. Moreover, it is natural to view the flow Φ on $\mathbb{P}(\mathbb{C}^2 \otimes W)$ as a 'compactification' of the gradient flow of $\tilde{\mathcal{A}}$ on $\tilde{\mathcal{L}}S^2$.

The above compactification of $\mathcal{C}_\mathcal{A}$ gives a compactification of the space of rational functions $\mathrm{Rat}_k = W_\mathcal{A}(n+k, n)$ as $\mathbb{CP}^{2k+1}$. This is the precise analogue for rational functions of the Donaldson-Uhlenbeck compactification of the moduli space of instantons on a 4-manifold, compare [10, §4.4]. To see the analogy regard $\mathbb{CP}^{2k+1}$ as the projective space of the vector space of pairs of polynomials (p, q) where $\deg p, \deg q \leq k$. Then

$$\mathbb{CP}^{2k+1} = \bigcup_l \mathrm{Rat}_{k-l} \times \mathrm{SP}^l(S^2)$$

where $\mathrm{SP}^l(S^2)$ is the l-th symmetric product of S^2, that is the space of unordered sets of l, not necessarily distinct, points in S^2. To a pair of polynomials (p, q) we associate the rational function $f = p/q$ which has degree $k - l$ where p and q have l roots in common, and the point of $\mathrm{SP}^l(S^2)$ given by the l common roots allowing, as above, roots at infinity.

This construction of a compactification works equally well for the area functional on $\mathcal{L}\mathbb{CP}^n$ and it gives the flow category of the flow $\Phi^{(n)}$ on $\mathbb{P}(\mathbb{C}^{n+1} \otimes W)$ defined by the linear flow $v \otimes z^n \mapsto v \otimes e^{nt} z^n$ on $\mathbb{C}^{n+1} \otimes W$. It is striking that the compactification of the flow category of the area functional on the loop space of $\mathbb{CP}^n = \mathbb{P}(\mathbb{C}^{n+1})$ is given by the flow category of a function on $\mathbb{P}(\mathbb{C}^{n+1} \otimes \mathbb{C}[z, z^{-1}])$.

§5 Morse theory and homotopy theory

The most important result of finite dimensional Morse theory is the relation between the topology of a compact manifold X and that of the manifold F of critical points of a smooth function $f : X \to \mathbb{R}$. It asserts that after changing the grading of the chain groups $C_*(F)$ appropriately there is a differential $\tilde{d}$ on $C_*(F)$ whose homology is $H_*(X)$.

In fact more is true. Let F_i be the part of the critical set F where the downward part of the tangent bundle of X, defined in §3, has dimension i. (Note that, compared to §3, we have made a slight change in notation.) This means that F_i may be empty for some values of i, but that for $j > i$ the manifold with corners $F(j, i)$ has dimension $j - i - 1$ whenever it is non-empty. We get a filtration of X

$$X_0 \subset X_1 \subset \cdots \subset X_n = X$$

by closed subspaces, where X_i consists of the points on downward piecewise trajectories emanating from F_i. (Here n is the dimension of X.) The successive quotient spaces $Y_k = X_k / X_{k-1}$ are the Thom spaces $Y_k = F_k^{E_k}$ of the downward bundles E_k on F_k. Recall that the Thom space X^E of a bundle E over a compact space X is the one-point compactification E^+ of the total space of E.

This leads to the homological assertion above because of the following general principle. Although the homotopy type of a filtered space X is not determined by the quotients $Y_k = X_k / X_{k-1}$, nevertheless the *stable* homotopy type – more precisely, the homotopy type of the n-fold suspension $S^n X$ – *is* determined by the Y_k together with certain maps between them.

Thus if $n = 1$, the Puppe construction for the inclusion $X_0 \to X_1$ tells us that the suspension SX_1 is obtained by attaching a cone $C(Y_1)$ on Y_1 to SX_0 by a map $\partial_1 : Y_1 = X_1 / X_0 \to SX_0$:

$$SX_1 \simeq SX_0 \cup C(Y_1).$$

When $n = 2$, one finds that

$$S^2 X_2 \simeq S^2 X_0 \cup C(SY_1) \cup C^2(Y_2).$$

To reconstruct S^2X_0 in this way we need the maps $\partial_2 : Y_2 \to SY_1$ and $\partial_1 : Y_1 \to SX_0$, obtained from the Puppe construction, together with a null-homotopy of the composite $S\partial_1 \circ \partial_2$. Explicitly, this null-homotopy provides a map $SY_1 \cup_{\partial_2} C(Y_2) \to S^2(X_0)$ whose restriction to SY_1 is equal to $S\partial_1$, and the mapping cone of this map is homotopy equivalent to S^2X_2. In general, one finds that

$$S^nX_n \simeq S^nX_0 \cup C(S^{n-1}Y_1) \cup \cdots \cup C^n(Y_n).$$

To describe the maps and homotopies needed to reconstruct S^nX_n in this way requires some technology.

Let $\mathcal{J}$ be the topological category whose objects are the integers $\mathbb{Z}$, and whose non-identity morphisms $j \to i$, when $j > i$, form a space $J(j,i)$ which is the one-point compactification of the space of sequences of real numbers $\{\lambda_k\}_{k\in\mathbb{Z}}$ such that

$$\begin{aligned} \lambda_k &\geq 0, &\quad &\text{for all } k, \text{ and} \\ \lambda_k &= 0, & &\text{unless } i < k < j. \end{aligned}$$

There are no non-identity morphisms unless $j > i$. Composition of morphisms is the map $J(k,j) \times J(j,i) \to J(k,i)$ induced by addition of sequences. Thus $J(j,i)$ is a compact space of dimension $j - i - 1$, with ∞ as a distinguished base-point. If $j - i = 1$, then $J(j,i)$ has just two points 0 and ∞; if $j - i = 2$, then $J(j,i)$ is a closed interval $[0,\infty]$. Indeed, if $j - i \geq 2$, then $J(j,i)$ is homeomorphic to a disc of dimension $j - i - 1$. We shall also consider the full subcategory $\mathcal{J}_a^b$ of $\mathcal{J}$ spanned by the objects $a, a+1, \ldots, b$.

There is a close relation between sequences of compact spaces

$$\underline{X} = \{X_a \to X_{a+1} \to \cdots \to X_b\}$$

and base-point-preserving covariant functors $Z : \mathcal{J}_a^b \to \mathcal{T}_*$. Here $\mathcal{T}_*$ is the category of compact spaces with base-point, and a base-point-preserving functor is one that maps ∞ in $J(j,i)$ to the zero map (i.e. the constant map with value the base-point) $Z(j) \to Z(i)$.

Let us assume for simplicity that the maps in $\underline{X}$ are inclusions. Then, the sequence $\underline{X}$ gives rise to a functor $Z : \mathcal{J}_a^b \to \mathcal{T}_*$ with

$$Z(i) = S^{b-i}(X_i/X_{i-1}),$$

for $i > a$, and

$$Z(a) = S^{b-a}(X_a^+) = (\mathbb{R}^{b-a} \times X_a)^+.$$

Here, if X is compact the notation X^+ means X with a disjoint base-point, denoted by ∞, adjoined, and if X is not compact it means the one-point compactification of X.

We will give the construction of the functor Z later, but for the moment, let us note two of its properties.

(i) The map $Z(i+1) \to Z(i)$ induced by the nontrivial morphism $i+1 \to i$ in $\mathcal{J}$ is the $(b-i-1)$-fold suspension of the map

$$X_{i+1}/X_i \to S(X_i/X_{i-1})$$

obtained by applying the Puppe construction to the inclusion $X_i/X_{i-1} \to X_{i+1}/X_{i-1}$.

(ii) The functor Z gives a map

$$J(i+2,i) \times S^{b-i-2}(X_{i+2}/X_{i+1}) \to S^{b-i}(X_i/X_{i-1}).$$

The space of morphisms $J(i+2,i)$ is the closed interval $[0,\infty]$, and this map is given by the $(b-i-2)$-fold suspension of a null-homotopy of the composite

$$X_{i+2}/X_{i+1} \to SX_{i+1}/X_i \to S^2X_i/X_{i-1}.$$

In the other direction, a functor $Z : \mathcal{J}_a^b \to \mathcal{T}_*$ has a realization $|Z|$ as a compact space. This is constructed from the disjoint union

$$\coprod_{a \le i \le b} Z(i) \wedge J(i, a-1)$$

by identifying the image of $Z(j) \times J(j,i) \times J(i,a-1)$ in $Z(j) \wedge J(j,a-1)$ with its image in $Z(i) \wedge J(i,a-1)$ whenever $a \le i \le j \le b$. Notice that

$$Z(i) \wedge J(i,a-1) = C^{i-a}(Z(i))$$
$$|Z| = Z(a) \cup C(Z(a+1)) \cup \cdots \cup C^{b-a}(Z(b)).$$

If Z is the functor defined by a sequence of spaces $X_a \to \cdots \to X_b$, then the comparison between this decomposition of $|Z|$ and the decomposition of $S^{b-a}X_b$ described above leads, very naturally, to the following result.

Proposition 5.1. *(i) If Z is the functor associated to a sequence of compact spaces $\underline{X}$, then there is a canonical homotopy equivalence*

$$|Z| \simeq S^{b-a}(X_b^+).$$

(ii) For any functor $Z : \mathcal{J}_a^b \to \mathcal{T}_*$, the homology $H_*(|Z|)$ can be calculated from the double complex

$$C_*(Z) = \bigoplus_{a \le i \le b} C_*(Z(i)).$$

The proof of Proposition 5.1 is straightforward, given the construction of the functor $Z : \mathcal{J}_a^b \to \mathcal{T}_*$ from a sequence of compact spaces

$$\underline{X} = \{X_a \to X_{a+1} \to \cdots \to X_b\}.$$

To construct Z we shall, as above, assume that the maps in $\underline{X}$ are inclusions. If $a \le i \le b$, let Z_i be the open subspace of $X_b \times \mathbb{R}_+^{b-a}$ consisting of all points $(x; \lambda_a, \ldots, \lambda_{b-1})$ such that:

(i) $\lambda_r > 0$ if $r \ge i$, and
(ii) if $\lambda_r > 0$ then $x \in X_r$.

We now show that the one-point compactification Z_i^+ is homotopy equivalent to $S^{b-i}(X_i/X_{i-1})$. First note that Z_i is the product of a subset of $X_i \times \mathbb{R}_+^{i-a}$ with the extra factor $(0,\infty)^{b-i}$. The factor $(0,\infty)^{b-i}$ accounts for the suspension, and it is enough to consider Z_b^+. This space is obtained from X_a by attaching a cone $C(X_{a+1})$ on X_{a+1} and then a cone on $C(X_{a+2}) \subset C(X_{a+1})$, and so on. Contracting these cones in the standard way shows that Z_b^+ is homotopy equivalent to X_b/X_{b-1}.

If $j > i$ then Z_i is an open subset of Z_j, so there is a natural map $Z_j^+ \to Z_i^+$. Let W_{ij} be the open subspace of Z_i consisting of all points with $\lambda_j > 0$. Then $Z_i \subset W_{ij} \subset Z_j$, and so the map $Z_j^+ \to Z_i^+$ factors through W_{ij}^+. Furthermore there is a proper map $\mathbb{R}_+^{j-i-1} \times W_{ij} \to W_{ij}$ which simply adds the coordinates labelled $i+1, i+2, \ldots, j-1$. This induces $J(j,i) \wedge W_{ij} \to W_{ij}$, and hence $J(j,i) \wedge Z_j^+ \to Z_i^+$ giving us the desired functor.

Our next task is to see that the compact smooth framed category which arises in §3 from a Morse-Bott-Smale function gives rise to a functor $Z : \mathcal{J}_0^n \to \mathcal{T}_*$ such that

$$Z(i) = S^{n-i}(F_i^{E_i}).$$

This is just a version of the Pontryagin-Thom construction. We have seen that the manifold with corners $F(j,i)$ embeds in the sphere bundle $S(E_j)$, with normal bundle $\pi_i^* E_i$. Let us map it further into E_j with normal bundle $\pi_i^* E_i \oplus \mathbb{R}$. We can choose a map $F(j,i) \to \mathbb{R}_+^k$ inducing the stratification of $F(j,i)$, where $k = j - i - 1 = \dim F(j,i)$. This gives us an embedding of $F(j,i)$ in $E_j \times \mathbb{R}_+^k$, compatible with the boundary structure, with normal bundle

$$\nu_{ji} = \pi_i^* E_i \oplus \mathbb{R}^{j-i}.$$

In other words, we have maps

$$\mathbb{R}_+^k \times E_j \leftarrow \nu_{ji} \to E_i \times \mathbb{R}^{j-i},$$

where the first is an open inclusion, and the second is proper. Passing to the one-point compactifications this gives

$$J(j,i) \wedge F_j^{E_j} \to \nu_{ji}^+ \to S^{j-i}(F_i^{E_i}),$$

and, after applying S^{n-j}, this is exactly what we need to define a base-point-preserving functor $Z : \mathcal{J}_0^n \to \mathcal{T}_*$ with $Z(i) = S^{n-i}(F_i^{E_i})$.

The method, described above, of reconstructing the manifold X from the data provided by the function f is quite different from that used in [9]. It uses the framings of the spaces $F(j,i)$, and their compatibility under the composition law in $\mathcal{C}_f$ to recover the stable homotopy type of X. For example, in the case where f is a Morse-Smale function (that is one with isolated non-degenerate critical points whose gradient flow satisfies Smale's transversality condition) the method amounts to the following construction. Take a cell of dimension j for each critical point of index j; now one constructs a CW complex inductively, using the framings of the spaces of flow lines to give the maps needed to attach appropriate suspensions of these cells to the lower skeleta. In this way we recover the stable homotopy type of M. Thus, the construction is similar in spirit to the work of Franks [14]. Note that the Morse-Smale chain complex simply uses the framings of the zero dimensional spaces $F(i+1,i)$ to define the boundary map.

Now let us consider what happens when we apply this method to the infinite dimensional situations studied by Floer. Here we confine ourselves to summarising the basic points; we will give a complete account in a future paper.

Clearly, the first step is to consider functors Z defined on the whole category $\mathcal{J}$. If we have a functor $Z : \mathcal{J} \to \mathcal{T}_*$, then we get a functor $Z_a^b : \mathcal{J}_a^b \to \mathcal{T}_*$, and a compact space $|Z|_a^b = |Z_a^b|$ for each $a < b$. It is important to observe that, from the construction of the realizations, there are maps

$$S^{b'-b}|Z|_a^b \to |Z|_a^{b'}, \qquad |Z|_{a'}^b \to S^{a-a'}|Z|_a^b$$

when $a' \leq a < b \leq b'$. Such a system of spaces and maps defines a *pro-spectrum*; see the Appendix for a brief discussion of pro-spectra, and further references. Thus, in the case of a functor $Z : \mathcal{J} \to \mathcal{T}_*$, the output is a pro-spectrum, rather than a stable homotopy type.

If we have a compact smooth framed category with objects $\{F_i\}_{i\in\mathbb{Z}}$ we do not quite get a functor $\mathcal{J} \to \mathcal{T}_*$. The framing only provides us with 'stable' or 'virtual' bundles E_i instead of genuine finite dimensional vector bundles. The space $F_i^{E_i}$ is then an object in the 'stable category' $\mathcal{S}$, which is described in the Appendix. Thus we get a functor $Z : \mathcal{J} \to \mathcal{S}$, with $Z(i) = S^{-i}(F_i^{E_i})$ and such a functor still defines a pro-spectrum.

What happens when we have a compact smooth category which is not framed? The essential point is to understand how to extract some kind of *stable map* $S^{2m}A \to B$ from a diagram of compact manifolds

$$A \xleftarrow{\pi_1} C \xrightarrow{\pi_2} B, \tag{5.2}$$

where π_1 is a fibration whose fibres are closed almost complex manifolds of (real) dimension $2m$. Evidently, we can lift π_1 to an embedding $C \to$

$A \times \mathbb{C}^{m+p}$ for some p. Let the normal bundle be ν. Then we have

$$S^{2m+2p}(A^+) \to C^\nu, \qquad C \to B.$$

To proceed we must pass to a category in which C and C^ν are equivalent. If ν is trivialized, then $C^\nu = S^{2p}(C^+)$, and the usual stable category will serve. In general, we must do something more brutal, which we will digress to explain.

The stable category of compact spaces is described in the Appendix. In this category, two compact spaces X and Y become homotopy equivalent if their suspensions S^pX and S^pY are homotopy equivalent for large p. The notion of a *spectrum*, a sequence of spaces $\mathbf{K} = \{K_p\}$ with maps $S^qK_p \to K_{p+q}$, is also described in the Appendix. We are concerned here with *ring spectra*, where there are associative pairings $K_p \wedge K_q \to K_{p+q}$. The spheres themselves form a natural example. For a ring spectrum $\mathbf{K}$ we can define the $\mathbf{K}$-homotopy category: its objects are compact spaces and its morphisms from X to Y are

$$\mathrm{Mor}_{\mathbf{K}}(X,Y) = \lim_{p\to\infty}[S^pX, K_p \wedge Y].$$

Thus if $K_p = S^p$ this is the stable homotopy category.

There is an optimal spectrum $\mathbf{M}$ with the property that for any p-dimensional complex vector bundle on a compact space C the Thom space C^ν is *canonically* $\mathbf{M}$-homotopy equivalent to the suspension $S^{2p}(C^+)$. For this choice of $\mathbf{M}$, the diagram (5.2) induces a map $S^{2m}(A^+) \to B^+$ in the $\mathbf{M}$-homotopy category.

We can now carry out the Pontryagin-Thom construction for an arbitrary compact smooth category, and we shall obtain an object $|Z|_a^b$ of the $\mathbf{M}$-homotopy category for each pair of integers $a < b$. If $a' \leq a < b \leq b'$ there will be a natural $\mathbf{M}$-maps

$$S^{b'-b}|Z|_a^b \to |Z|_a^{b'}, \qquad |Z|_{a'}^b \to S^{a-a'}|Z|_a^b.$$

This system of spaces and maps again defines a pro-spectrum and it is our desired output.

For fixed a, the spaces $|Z|_a^b$ and $\mathbf{M}$-maps $S^{b'-b}|Z|_a^b \to |Z|_a^{b'}$ define an object $|Z|_a$ of the $\mathbf{M}$-homotopy category of spectra. Furthermore, for the flow categories of Floer functions $S^d|Z|_a^b$ and $|Z|_{a+d}^{b+d}$ are $\mathbf{M}$-homotopy equivalent, where d is the periodicity of the virtual dimension, compare §2. This gives a periodicity map $|Z|_a \to S^d|Z|_a$. Therefore the pro-spectrum, which is the output of the construction, is of a particularly simple kind; it is given by the inverse system of spectra

$$Z \leftarrow S^{-d}Z \leftarrow S^{-2d}Z \leftarrow \cdots$$

where $Z = |Z|_0$ and the map $S^{-d}Z \to Z$ is the periodicity map.

The spectrum M is traditionally called the MU-spectrum. The space M_{2p} is the Thom space of the universal $\mathbb{C}^p$ bundle on $BGL_p(\mathbb{C})$, and $M_{2p+1} = SM_{2p}$. If ν is a $\mathbb{C}^p$-bundle on C the classifying map $C \to BGL_p(\mathbb{C})$ extends to $C^\nu \to M_{2p}$. Putting this together with the projection $\nu \to C$ gives

$$C^\nu \to M_{2p} \wedge (C^+)$$

which is an M-equivalence $C^\nu \to S^{2p}(C^+)$. This M-equivalence does not depend on the choice of a classifying map for ν though it would take us too far afield to explain that here: the point is the functoriality of the transformation $\mathcal{C} \to \mathcal{V}$ mentioned at the end of §3.

Spectra are the same thing as generalized cohomology theories. If we define $h^p(X;\mathrm{K}) = \mathrm{Mor}_{\mathrm{K}}(X;S^p)$ then $h^*(-;\mathrm{K})$ is a cohomology theory, and the correspondence $\mathrm{K} \leftrightarrow h^*(-;\mathrm{K})$ is one-to-one. The theory $h^*(-;\mathrm{M})$, called complex cobordism, is universal among so-called complex oriented theories [25], which include ordinary cohomology and K-theory and it determines them algebraically. So our construction gives us definitions of Floer cobordism and Floer K-theory, as well as Floer cohomology.

§6 Floer theory for the projective space of a polarized vector space and $\mathcal{L}\mathbb{CP}^n$

We now explain what the method described in the previous section gives in some infinite dimensional examples. Motivated by the fact that the compactification of the flow category of the area functional on $\mathcal{L}\mathbb{CP}^n$, constructed in §4, is the flow category of a function on an infinite dimensional projective space, we begin by considering projective spaces.

Example 6.1 – Real projective space. Let V be the real vector space of sequences $x = \{x_n\}_{n\in\mathbb{Z}}$ with only a finite number of non-zero terms, topologized as the direct limit of its finite dimensional subspaces. We use the usual Hilbert norm $\|-\|$ on V; of course V is not complete in this norm. Let $S(V)$ be the sphere in V and consider the function $f : S(V) \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=-\infty}^{\infty} n x_n^2.$$

This descends to a function

$$f : \mathbb{P}(V) \to \mathbb{R}$$

with critical points $c_i = [\delta_i]$, $i \in \mathbb{Z}$, where δ_i is the i-th element in the standard basis for V. The gradient flow of f, with respect to the Hilbert norm, is the flow on $\mathbb{P}(V)$ defined by the linear flow ψ on V where $\psi_t(\delta_n) = e^{nt}\delta_n$. We could replace V by a space of sequences of suitably rapid decay, but this does not make any real difference.

The unstable manifold $W^u(c_i)$ and the stable manifold $W^s(c_i)$ are given by

$$W^u(c_i) = \{[x] \in \mathbb{P}(V) : x_i \neq 0, x_j = 0 \quad \text{if } j < i\}$$
$$W^s(c_i) = \{[x] \in \mathbb{P}(V) : x_i \neq 0, x_j = 0 \quad \text{if } j > i\}.$$

Neither $W^u(c_i)$ nor $W^s(c_i)$ is finite dimensional but the intersection $W^u(c_i) \cap W^s(c_j)$ is transverse, finite dimensional, and

$$\dim W^s(c_i) \cap W^u(c_j) = j - i \qquad \text{if } j \geq i.$$

From this it is not difficult to identify the flow category $\mathcal{C} = \mathcal{C}_f$ explicitly.

Now consider the question of whether this category is framed in the sense of §3. Fix a pair of integers $a < b$ and let $\mathcal{C}_a^b$ be the full subcategory generated by the critical points c_i with $a \leq i \leq b$. This category is the flow category of the function

$$f(x) = \sum_{n=a}^{b} n x_n^2$$

on $\mathbb{RP}^{b-a} = \mathbb{P}(V_a^b)$ where V_a^b is the finite dimensional subspace of V with basis δ_i, $a \leq i \leq b$, and we are using the natural homogeneous coordinates $x = [x_a, \ldots, x_b]$ with $\sum x_n^2 = 1$ on $\mathbb{P}(V_a^b)$. Therefore, $\mathcal{C}_a^b$ is a framed category and each of the spaces of morphisms $F(j,i)$ with $a \leq i < j \leq b$ inherits a framing φ_a^b from the flow category $\mathcal{C}_a^b$. The framing φ_a^b comes from embedding $F(j,i)$ in the unstable sphere of the critical point c_j in $\mathbb{RP}^{b-a}$, which, since c_j has index $j-a$ in $\mathbb{RP}^{b-a}$, is a sphere of dimension $j-a-1$. In particular, this shows that the framings φ_a^b and $\varphi_a^{b'}$ are identical. However, the framings φ_a^b and $\varphi_{a'}^b$ are not the same, as we will show.

Let us work in the stable category $\mathcal{S}$, described in the Appendix, and define

$$|(\mathcal{C}_a^b, \varphi)| = S^{a-b}|Z|$$

where $Z : \mathcal{J}_a^b \to \mathcal{T}_*$ is the functor defined by the category $\mathcal{C}_a^b$ equipped with the framing φ. This has the effect of removing the suspensions which occur in the statement of Proposition (5.1) and it simplifies notation; it is a straightforward matter to keep track of the suspensions, if necessary.

The manifold $F(i+1,i)$ has dimension zero and, using φ_a^b, is framed in S^{i-a}, the unstable sphere of the critical point c_{i+1} in $\mathbb{RP}^{b-a}$. Thus it gives a map of spheres $S^{i-a} \to S^{i-a}$. This map is the relative attaching map between the $(i-a+1)$-cell and the $(i-a)$-cell in $|(\mathcal{C}_a^b, \varphi_a^b)| = \mathbb{RP}^{b-a}$. Therefore it has degree $1-(-1)^{i-a}$. Using the framing φ_{a-1}^b of $F(i+1,i)$ in the unstable S^{i-a+1} of the critical point c_{i+1} in $\mathbb{RP}^{b-a+1}$ we get a map $S^{i-a+1} \to S^{i-a+1}$ which is the relative attaching map between the $(i-a+2)$-cell and the $i+1-a$-cell of $|(\mathcal{C}_{a-1}^b, \varphi_{a-1}^b)| = \mathbb{RP}^{b-a+1}$; therefore it has degree $1-(-1)^{i-a+1}$. So the framings φ_a^b and φ_{a-1}^b produce different maps.

For $a' < a$ the framings $\varphi_{a'}^b$ and φ_a^b differ because the normal bundle to $\mathbb{RP}^{b-a}$ in $\mathbb{RP}^{b-a'}$, that is $(a-a')\eta$ where η is the real Hopf line bundle, is non-trivial. Furthermore, it is straightforward to check that

$$|(C_a^b, \varphi_{a'}^b)| = (\mathbb{RP}^{b-a})^{(a-a')\eta}.$$

Since the framings $\varphi_{a'}^{b'}$ and $\varphi_{a'}^b$ agree it follows that

$$|(C_a^b, \varphi_{a'}^{b'})| = (\mathbb{RP}^{b-a})^{(a-a')\eta}$$

whenever $a' \leq a < b \leq b'$.

From the construction of the realization there are maps

$$|(C_a^b, \varphi_a^b)| \to |(C_a^{b'}, \varphi_a^{b'})|, \qquad |(C_{a'}^b, \varphi_{a'}^b)| \to |(C_a^b, \varphi_{a'}^b)|$$

for $a' \leq a < b \leq b'$ which we now identify. The first is the inclusion

$$\mathbb{RP}^{b-a} \to \mathbb{RP}^{b'-a}$$

and the second is the map

$$\mathbb{RP}^{b-a'} \to (\mathbb{RP}^{b-a})^{(a-a')\eta}$$

obtained from the Pontryagin-Thom construction applied to the embedding $\mathbb{RP}^{b-a} \to \mathbb{RP}^{b-a'}$; we describe this construction briefly.

Suppose we have an embedding $P \to M$ of compact manifolds. Let ν be the normal bundle of the embedding and let N_P be an open tubular neighbourhood of P in M. Then the inclusion $N_P \to M$ is an open embedding and so it gives a map $N_P^+ \leftarrow M$ where N_P^+ is the one-point compactification of N_P. Now N_P^+ is just the Thom complex of ν and so the embedding $N \to M$ gives a map $M \to P^\nu$. Applied to the embedding $\mathbb{RP}^{b-a} \to \mathbb{RP}^{b-a'}$, with normal bundle $(a-a')\eta$, this gives the required map.

We now explain how to assemble the spaces $|(C_a^b, \varphi_{a'}^{b'})|$ into a single, a pro-spectrum, which correctly reflects the relation between the different spaces. To do this, we use the maps we have just described. The bundle $(a-a')\eta$ extends over $\mathbb{RP}^{b-a}$ and so, using the theory of Thom spaces of virtual bundles described in the Appendix, we can convert the map $\mathbb{RP}^{b-a'} \to (\mathbb{RP}^{b-a})^{(a-a')\eta}$ into a stable map

$$(\mathbb{RP}^{b-a'})^{-(a-a')\eta} \to \mathbb{RP}^{b-a}.$$

Therefore we can construct the sequence

$$\mathbb{RP}^{b-a} \leftarrow (\mathbb{RP}^{b-a+1})^{-\eta} \leftarrow (\mathbb{RP}^{b-a+2})^{-2\eta} \leftarrow \dots$$

in the stable category of compact spaces. As explained in the Appendix, this sequence defines a pro-spectrum.

Using the inclusions $\mathbb{RP}^{b-a} \to \mathbb{RP}^{b'-a}$ we can take the limit over b and then, using Thom spaces of virtual vector bundles over CW complexes of finite type (see the Appendix), we get the pro-spectrum defined by the sequence of Thom spectra

$$\mathbb{RP}^\infty \leftarrow (\mathbb{RP}^\infty)^{-\eta} \leftarrow (\mathbb{RP}^\infty)^{-2\eta} \leftarrow \cdots.$$

This pro-spectrum is the final output of the construction; it is the *Floer homotopy type* associated to the function $f : \mathbb{P}(V) \to \mathbb{R}$. In fact, this pro-spectrum, which is usually denoted by $\mathbb{RP}^\infty_{-\infty}$, is exactly the pro-spectrum which occurs in the theorem of Lin mentioned in the introduction.

Floer's method of associating a chain complex to the function f : $\mathbb{P}(V) \to \mathbb{R}$ gives the chain complex C_* with $C_p = \mathbb{Z}$ for all $p \in \mathbb{Z}$, and the boundary operator $\partial_p : C_p \to C_{p-1}$ is multiplication by $1 + (-1)^p$. It is easy to check that the homology of this chain complex is the same as the homology of the pro-spectrum $\mathbb{RP}^\infty_{-\infty}$ for any coefficient group. So the Floer groups do compute the homology of $\mathbb{RP}^\infty_{-\infty}$.

Example 6.2 – Complex projective space. Now consider the complex analogue of the previous example. Let W be the complex vector space of sequences $z = \{z_n\}_{n\in\mathbb{Z}}$ with only a finite number of non-zero terms, equipped with the direct limit topology, and Hilbert norm $||-||$. This time the function $S(W) \to \mathbb{R}$ defined by

$$z \mapsto \sum_{n=-\infty}^{\infty} n|z_n|^2$$

descends to a function $f : \mathbb{P}(W) \to \mathbb{R}$, and the flow of this function is exactly the flow $\Phi^{(0)}$ on the projective space $\mathbb{P}(W)$ which arose in §4. The construction used in the case of the real projective space $\mathbb{P}(V)$ shows that the Floer homotopy type associated to this function is the pro-spectrum $\mathbb{CP}^\infty_{-\infty}$ defined by the sequence of Thom spectra

$$\mathbb{CP}^\infty \leftarrow (\mathbb{CP}^\infty)^{-\zeta} \leftarrow (\mathbb{CP}^\infty)^{-2\zeta} \leftarrow \cdots$$

where ζ is the complex Hopf line bundle. Once more we find by direct computation that the Floer chain complex of f does indeed compute the cohomology of this pro-spectrum.

Example 6.3 – The area function on $\mathcal{L}\mathbb{CP}^n$. To associate a Floer homotopy type to the area functional $\mathcal{A}$ on $\mathcal{L}\mathbb{CP}^n$ we first compactify the flow category $C_{\mathcal{A}}$ to give the flow category of the flow $\Phi^{(n)}$ on the projective space $\mathbb{P}(\mathbb{C}^{n+1} \otimes \mathbb{C}[z, z^{-1}])$, as in §4. Now the method of (6.1) and (6.2) gives the pro-spectrum defined by the sequence of Thom spectra

$$\mathbb{CP}^\infty \leftarrow (\mathbb{CP}^\infty)^{-(n+1)\zeta} \leftarrow (\mathbb{CP}^\infty)^{-2(n+1)\zeta} \leftarrow \cdots$$

where ζ is the complex Hopf line bundle. This pro-spectrum is $\mathbb{CP}^{\infty}_{-\infty}$.

The reason why $(n+1)\zeta$ appears in this construction is that the category C_a^b which occurs in this example is the flow category of a Morse-Bott-Smale function on $\mathbb{CP}^{(n+1)(b-a)}$, and the normal bundle to the embedding of $\mathbb{CP}^{(n+1)(b-a)}$ in $\mathbb{CP}^{(n+1)(b-(a-1))}$ is $(n+1)\zeta$.

As explained in the Appendix, the cohomology of $\mathbb{CP}^{\infty}_{-\infty}$ with integral coefficients is $\mathbb{Z}[u, u^{-1}]$, the ring of Laurent polynomials in u where u has degree 2. Thus, as a group, this is one copy of $\mathbb{Z}$ in every even dimension. Computing the first Chern class of $\mathbb{CP}^n$ shows that the Floer homology $\mathrm{HF}_*(\mathcal{L}\mathbb{CP}^n)$ is $\mathbb{Z}/(2n+2)$ graded and Floer [12] shows that these groups are one copy of $\mathbb{Z}$ in even degrees and 0 in odd degrees.

The relation between these groups is as follows. Let $e((n+1)\zeta)$ be the Euler class of the bundle $(n+1)\zeta$, which is the bundle which naturally occurs in the above sequence of Thom spectra; of course $e((n+1)\zeta) = u^{n+1}$. If we now set $e((n+1)\zeta)$ to be 1, we get Floer's groups with their $\mathbb{Z}/(2n+2)$ grading:

$$\mathrm{HF}_*(\mathcal{L}\mathbb{CP}^n) = \frac{\mathbb{Z}[u, u^{-1}]}{(u^{n+1}-1)} = \frac{H^*(\mathbb{CP}^{\infty}_{-\infty})}{(e((n+1)\zeta)-1)}.$$

The Floer cohomology of $\mathcal{L}\mathbb{CP}^n$ has a ring structure using the 'pair of pants' product and what is more the above isomorphisms are isomorphisms of rings.

We close this section with some final comments.

(i) It seems very likely that the same method will work for the area functional on $\mathcal{L}Gr_k(\mathbb{C}^n)$ where $Gr_k(\mathbb{C}^n)$ is the Grassmannian of complex k planes in $\mathbb{C}^n$. In this case the ring strucuture of Floer cohomology ring has been computed, by Witten; it is the 'deformed cohomology ring' of the Grassmannian. Once more, we expect the Floer homotopy type to come from an inverse system of Thom spectra, and to find the Floer cohomology ring is given by a formula similar to the one which arises in the case of $\mathbb{CP}^n$.

(ii) In the projective space examples the pro-spectra $\mathbb{RP}^{\infty}_{-\infty}$ and $\mathbb{CP}^{\infty}_{-\infty}$ are indeed the natural candidates for the semi-infinite homotopy type of the polarized manifolds $\mathbb{P}(V)$ and $\mathbb{P}(W)$. The polarization of these projective spaces is defined by the natural polarization of V and W. For example the construction of $\mathbb{RP}^{\infty}_{-\infty}$ can be phrased so that it only depends on the polarization $V = V^- \oplus V^0 \oplus V^+$ where V^- has basis $\{\delta_i\}_{i<0}$, V^+ has basis $\{\delta_i\}_{i>0}$ and V^0, the 'finite dimensional ambiguity', is the one dimensional space spanned by δ_0.

Let W be a finite dimensional subspace of V such that $W = W^- \oplus W^0 \oplus W^+$ where $W^{\pm} = W \cap V^{\pm}$ and $W^0 = W \cap V^0$. We refer to $W^0 \oplus W^+$ as the positive part of W and W^- as the negative part. To W we associate the Thom space $\mathbb{P}(W)^{-\xi}$ of the virtual bundle $-\xi$ where ξ is the vector bundle over $\mathbb{P}(W)$ defined by W^-. If we choose bases then we get an isomorphism $\xi \cong (\dim W^-)\eta$ where η is the Hopf line bundle. In the stable category

$\mathbb{P}(W)^{-\xi}$ has one i-cell for each i with $-\dim W_- \leq i \leq \dim W - \dim W^-$.

Now suppose we have an inclusion $W_1 \to W_2$ which preserves the decompositions. If it is an isomorphism on the negative part, i.e. increases the positive part, then we get an obvious map

$$\mathbb{P}(W_1)^{-\xi_1} \to \mathbb{P}(W_2)^{-\xi_2}.$$

On the other hand, if it is an isomorphism on the positive part, i.e. increases the negative part, then we get a map

$$\mathbb{P}(W_1)^{-\xi_1} \leftarrow \mathbb{P}(W_2)^{-\xi_2}.$$

This map is constructed as follows. The bundle ξ_1 is a sub-bundle of ξ_2 restricted to $\mathbb{P}(W_1)$ and therefore the virtual bundle $-\xi_2$ restricted to $\mathbb{P}(W_1)$ is a sub-virtual-bundle of $-\xi_1$. This gives a map (in the stable category) of Thom spaces $\mathbb{P}(W_1)^{-\xi_1} \leftarrow \mathbb{P}(W_3)^{-\xi_2}$.

Thus we get a system of Thom spectra indexed by subspaces W of V. An inclusion which is an isomorphism on the negative part (i.e increases the positive part) gives a map in the same direction whereas an inclusion which is an isomorphism on the positive part (i.e. increases the negative part) gives a map in the other direction. Using the basis for V we can use the subspaces V_a^b to reduce this system to one indexed by pairs of integers integers and this gives the pro-spectrum $\mathbb{RP}^{\infty}_{-\infty}$. This construction fits in rather well with the coordinate free theory of spectra described in [19], where the spaces defining the spectrum are indexed by finite dimensional subspaces of an infinite dimensional real vector space, rather than the integers. Here we have spaces indexed by the finite dimensional subspaces of a polarized vector space.

Therefore, we are able to associate a 'semi-infinite homotopy type to $\mathbb{P}(V)$ which depends only on the polarization of V, and the Floer function $f : \mathbb{P}(V) \to \mathbb{R}$ does indeed compute the 'semi-infinite' cohomology of $\mathbb{P}(V)$. However the construction depends very heavily on special features of $\mathbb{P}(V)$.

Appendix – Spectra and pro-spectra

In stable homotopy theory it is convenient, and it greatly simplifies many arguments, to be able to work in a suitable stable category of spaces. The *stable category* $\mathcal{S}$ of finite CW complexes is defined, essentially, by inverting the suspension functor on the category of finite CW complexes. The objects of $\mathcal{S}$ are defined to be S^nX where X is a finite CW complex, and $n \in \mathbb{Z}$. If n is positive then S^nX is just the n-th suspension of X, but by allowing n to be negative we have introduced formal desuspensions of X. More precisely, the objects of $\mathcal{S}$ are pairs (X, n), where X is a finite CW complex and $n \in \mathbb{Z}$, modulo the equivalence relation generated by identifying (X, n) with $(S^nX, 0)$ if n is positive, and S^nX is the equivalence class of (X, n). A map from $S^nX \to S^mY$ is defined by giving a map

$S^{n+k}X \to S^{m+k}Y$ where k is chosen large enough so that both $S^{n+k}X$ and $S^{m+k}Y$ are genuine CW complexes. Two maps are identified if they are equal after a suitably large number of suspensions. Thus,

$$\text{Map}_{S}(S^nX, S^mY) = \lim_{k\to\infty} \text{Map}(S^{n+k}X, S^{m+k}Y)$$

where the maps in the direct system are given by suspension.

The main difficulty with this stable category is that it is not closed under direct and inverse limits. The category of *spectra* $\mathcal{SP}$ is defined, essentially, to be the category one obtains by formally adjoining direct limits of sequences in the stable category. Thus a spectrum $\mathbf{K}$ is given by a sequence of CW complexes $\{K_p\}$ and maps $SK_p \to K_{p+1}$. One thinks of the spectrum $\mathbf{K}$ as the direct limit of the sequence

$$K_0 \to S^{-1}K_1 \to \cdots \to S^{-p}K_p \to S^{-p+1}K_{p+1} \to \cdots$$

in $\mathcal{S}$. Note that it is only necessary to define the spaces K_{p_n} and the maps $S^{p_{n+1}-p_n}K_{p_n} \to K_{p_{n+1}}$ for a strictly increasing sequence of integers p_n; for if $p_n < m < p_{n+1}$ we can define $K_m = S^{m-p_n}K_{p_n}$ and use the identity map $SK_m \to K_{m+1}$ if $m+1 \neq p_{n+1}$.

It takes some work to define maps between spectra and set up a good category of spectra. The essential difficulty is the usual one when dealing with maps from a direct limit $\mathbf{K}$ to a direct limit $\mathbf{L}$; it is certainly the case that a map of direct systems defines such a map but there are many different direct systems with limits $\mathbf{K}$ and $\mathbf{L}$. However the technicalities involved in the definition of a suitable category of spectra are well-understood; see [1] and [19]. In particular [19] gives the definition from the 'coordinate-free' point of view which leads to a category of spectra with all the good properties one could expect.

Spectra define generalised cohomology theories: if X is a finite CW complex

$$h^p(X;\mathbf{K}) = \lim_{k\to\infty} [S^{k-p}X; K_k].$$

The maps in the direct system are defined by

$$[S^{k-p}X, K_k] \to [S^{k-p+1}X, SK_k] \to [S^{k-p+1}X, K_{k+1}],$$

where the first map is given by suspension, and the second by the structure maps of the spectrum $\mathbf{K}$. In the literature this group $h^p(X;\mathbf{K})$ is often denoted by $\mathbf{K}^p(X)$. Furthermore every generalised cohomology theory arises in this way. This is one of the main justifications for introducing spectra.

For some purposes the category of spectra is still not big enough. In the study of the Segal conjecture it becomes clear that one also needs pro-spectra, inverse systems of spectra. The most convincing argument for the necessity of pro-spectra is given in [2, pages 5-6]. By definition, a *pro-spectrum* is a doubly indexed family of finite CW complexes $\{X_{p,q}\}$ equipped with maps

$$SX_{p,q} \to X_{p,q+1}, \qquad X_{p,q} \to X_{p,q-1}.$$

Thus if we fix p the sequence of spaces $X_{p,q}$ with structure maps $SX_{p,q} \to X_{p,q+1}$ form a spectrum $\mathbf{X}_p$ and the maps $X_{p,q} \to X_{p-1,q}$ give a map of spectra $\mathbf{X}_p \to \mathbf{X}_{p-1}$; so we have an inverse system of spectra

$$\cdots \leftarrow \mathbf{X}_{p-1} \leftarrow \mathbf{X}_p \leftarrow \cdots .$$

It is only necessary to define these spaces and maps for a strictly increasing sequences of integers p_n and q_n. Furthermore, to define a pro-spectrum it is sufficient to give the structure maps in the stable category $\mathcal{S}$.

The first example of a pro-spectrum which arises in the main text occurs in §5, where for any pair of integers a and b with $a < b$, we have spaces $|Z|_a^b$, and for $a' \le a < b \le b'$ we have maps

$$S^{b'-b}|Z|_a^b \to |Z|_a^{b'}, \qquad |Z|_{a'}^b \to S^{a-a'}|Z|_a^b.$$

The corresponding pro-spectrum is defined by $X_{a,b} = S^a|Z|_a^b$ and the natural structure maps. Note that since a can be negative, these spaces and maps really do lie in the stable category; as noted above, they nonetheless define a pro-spectrum.

A good example which illustrates these ideas and is of considerable importance in our approach to Floer homotopy type arises from the theory of Thom spaces. Recall that the Thom space X^E of a vector bundle E over a compact space X is defined to be the one-point compactification E^+ of the total space of E. Now consider the problem of defining the Thom space of a virtual vector bundle ξ over X. A virtual vector bundle is an element $\xi \in KO(X)$ and its dimension, which is an integer, is defined by the homomorphism $KO(X) \to KO(\mathrm{pt}) = \mathbb{Z}$ given by the inclusion of a point in X. We are assuming, of course, that X is connected. By standard properties of K-theory we can find a genuine vector bundle E over X such that

$$\xi = E - k \in KO(X)$$

where k is a trivial bundle of dimension k. Now X^ξ is defined to be the object $S^{-k}X^E$ in the stable category $\mathcal{S}$. It is not difficult to check that in $\mathcal{S}$ the homotopy type of X^ξ does not depend on the choice of E and k.

Now suppose that X is CW complex of finite type (this means that the n-skeleton $X^{(n)}$ of X is a finite CW complex for each n) and ξ is a virtual vector bundle on X. In this case we cannot necessarily choose a vector bundle E such that $\xi = E - k$, but we can choose vector bundles $E^{(n)}$ over $X^{(n)}$ such that

$$\xi^{(n)} = E^{(n)} - k_n \in KO(X^{(n)}),$$

where $\xi^{(n)}$ is the restriction of ξ to $X^{(n)}$. Furthermore these bundles can be chosen so that there is a bundle map $E^{(n)} + k_m - k_n \to E^{(m)}$, covering the inclusion of $X^{(n)} \to X^{(m)}$, which is an isomorphism on fibres. Thus we get maps

$$S^{k_{n+1}-k_n}(X^{(n)})^{E^{(n)}} \to (X^{(n+1)})^{E^{(n+1)}}.$$

and the spaces $\{(X^{(n)})^{E^{(n)}}\}$ with these maps define a spectrum. This is the Thom spectrum of ξ, denoted by X^ξ. Once more the homotopy type of X^ξ does not depend on the choices made in its definition. For example the MU-spectrum used in §5 is the Thom spectrum of the universal bundle over BU.

On a compact space X, if E is a sub-vector-bundle of F we get a map of Thom complexes $X^E \to X^F$. Similarily if X has finite type and ξ is a sub-virtual-vector-bundle of η, which means that there is a genuine vector bundle E such that $\xi + E = \eta$, then we get a map of Thom spectra

$$X^\xi \to X^\eta.$$

In particular, suppose that X has finite type, and E is a vector bundle over X. Then $-E$ is a virtual vector bundle over X, and if $k \geq 0$ then $-kE$ is a sub-virtual-vector-bundle of $(-k+1)E$. So we get maps $X^{-kE} \to X^{(-k+1)E}$ and we can form the inverse system of Thom spectra

$$X \leftarrow X^{-E} \leftarrow X^{-2E} \leftarrow \cdots$$

with k-th term X^{-kE} where $k \geq 0$. This inverse system of Thom spectra is a pro-spectrum which we will denote by $X^{-\infty E}$. The examples which occur in the main text are the cases where $X = \mathbb{RP}^\infty$ and E is the real Hopf line bundle; $X = \mathbb{CP}^\infty$ and E is the complex Hopf line bundle. These pro-spectra are denoted by $\mathbb{RP}^\infty_{-\infty}$ and $\mathbb{CP}^\infty_{-\infty}$ respectively.

The cohomology of a pro-spectrum $\mathbf{X}$ is defined as follows. The pro-spectrum is an inverse system of spectra

$$\cdots \leftarrow \mathbf{X}_{p-1} \leftarrow \mathbf{X}_p \leftarrow \cdots$$

and then

$$H^*(\mathbf{X}) = \lim_{p\to\infty} H^*(\mathbf{X}_p).$$

In good cases, that is where there is no $\lim^1$-term, we get

$$H^*(\mathbf{X}) = \lim_{p\to\infty} \lim_{\infty\leftarrow q} H^*(X_{p,q})$$

where the spaces $\{X_{p,q}\}$, with the appropriate structure maps, define the pro-spectrum $\mathbf{X}$.

In the case of the pro-spectrum $X^{-\infty E}$ defined by a vector bundle over a CW complex of finite type, it is straightforward to compute cohomology. If E is orientable it has an Euler class $e(E) \in H^{\dim(E)}(X;\mathbb{Z})$, and

$$H^*(X^{-\infty E};\mathbb{Z}) = H^*(X;\mathbb{Z})[e(E)^{-1}].$$

If E is not orientable then it has an Euler class in mod 2 cohomology, and the mod 2 cohomology of $X^{-\infty E}$ is given by the same formula. For $\mathbb{RP}^{\infty}_{-\infty}$ and $\mathbb{CP}^{\infty}_{-\infty}$ this shows that

$$H^*(\mathbb{RP}^{\infty}_{-\infty};\mathbb{Z}/2) = \mathbb{Z}/2[x,x^{-1}]$$
$$H^*(\mathbb{CP}^{\infty}_{-\infty};\mathbb{Z}) = \mathbb{Z}[u,u^{-1}].$$

Here x has degree 1 and corresponds to the Euler class of the real Hopf line bundle in $H^1(\mathbb{RP}^\infty;\mathbb{Z}/2)$; u has degree 2 and corresponds to the Euler class of the complex Hopf line bundle in $H^2(\mathbb{CP}^\infty;\mathbb{Z})$.

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