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Nijenhuis tensors in pseudoholomorphic curves neighborhoods

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Abstract

In this paper* we consider the normal forms of almost complex structures in a neighborhood of pseudoholomorphic curve. We define normal bundles of such curves and study the properties of linear bundle almost complex structures. We describe 1-jet of the almost complex structure along a curve in terms of its Nijenhuis tensor. For pseudoholomorphic tori we investigate the problem of pseudoholomorphic foliation of the neighborhood. We get some results on nonexistence of the tori deformation.

Introduction

Let M^{2n} be an almost complex manifold, i.e. it is equipped with the tensor field $J \in T^*M \otimes TM$ such that $J^2 = -\mathbf{1}$. A submanifold $N \subset M$ is called *pseudoholomorphic* (PH-submanifold) if its tangent bundle $TN \subset TM$ is invariant under the operator J. In this paper we study pseudoholomorphic curves, which are 2-dimensional submanifolds. Generically there are no other PH-submanifolds.

The investigation of pseudoholomorphic curves was initiated by Gromov in [G] (but first they appeared in [NW]). The structure of the moduli space of such curves plays an important role in symplectic geometry ([MS]). Generically nonexceptional PH-spheres occur not discretely but in families. The same situation is also with PH-disks ([K3]). However it is known that even in the complex situation generically exist discrete holomorphic tori ([A1]).

In [Mo] Moser posed a question what type of KAM-theory can be constructed for a PH-foliation of an almost complex torus T^{2n}. If this is a foliation by PH-tori T^2 then such a theory can be expected only under very special conditions. Namely in general position PH-tori are discrete and their number was investigated by Kuksin in [Ku1]. Moser considered a foliation by entire PH-lines $\mathbb{C} \to T^{2n}$ with the slope of general position. The main result states that under

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a small almost complex perturbations of the standard complex structure J_0 on T^{2n} many leaves persist. If the perturbation is big but tame-restricted then only some of the leaves persist. This was proved by Bangert in [B]. Slight generalization of his result with another approach see in [KO].

In section 1 we study the structure of the neighborhood of a PH-torus in a 4-dimensional manifold. The dimension four is quite specific for the Nijenhuis tensor and generically it leads to some canonical $\{e\}$*-structure* (theorem 1). This gives sufficient number of moduli for classification of almost complex structures in a germ of a point as well as of a PH-curve.

The moduli are absent in the complex situation and many results of the complex theory fail in the almost complex case. In particular this is so with Arnold theory of elliptic curves neighborhoods (which he called Floquet-type theory [A1]). In this paper we recover some traces of normal bundles to holomorphic curves theory in almost complex situation.

The Nijenhuis tensor is determined by 1-jet of the almost complex structure. 1-jet of a complex structure on a holomorphic curve determines its holomorphic normal bundle. So to define pseudoholomorphic normal bundle we should involve the Nijenhuis tensor of the structure J.

In section 2 we define abstract linear bundle almost complex structures and prove that the normal bundle $N_{\mathcal{C}}M = TM/T\mathcal{C}$ to a PH-curve $\mathcal{C}$ has a canonical structure of this kind. Then we show that the Nijenhuis tensor of such structure is quite special in a sense that its characteristic 2-dimensional distribution is integrable and the leaves coincide with the fibers. This allows to find normal forms of linear bundle almost complex structures and thus normal forms of 1-jet of an almost complex structure along a PH-curve. Of course the formula involves the Nijenhuis tensor only because it is a complete invariant of the almost complex structure ([K1]).

In section 3 we investigate the germ of neighborhoods of a PH-torus $T^2 \subset M^4$ by means of pseudoholomorphic curves. To get some analogy with the complex situation we note that for a neighborhood of an elliptic curve there is a special foliation by cylinders. Part of these cylinders persists in the almost complex category. We construct families of cylinders and investigate some natural questions of transports and monodromy of the transversal PH-disks.

The last part of section 3 is devoted to the problem of pseudoholomorphic tori deformation. It mainly concerns the linear almost complex bundles since these model normal bundles of pseudoholomorphic curves. We deduce the equation on the deformed tori, study it and describe some cases when there exists a unique solution. This means that the torus is isolated and persistent. The first example is a sort of geometric version of the Moser's non-deformation example [Mo].

Since the most of the paper is about normal bundles in almost complex category we sketch at the appendix what happens in other geometries, namely Riemannian.

The paper was started by a question of Arnold ([A2]; 1993-25) about almost complex version of his theory. As we show the answers to most of his questions are negative (in a sense: "linearizable" along a curve complex structures form a typical set, while in the almost complex case, codimension of such "linearizable" structures is ∞. However we describe the obstructions for "linearizations" – equivalence to the normal bundle structure – in §2.2). But this does not bring the issue to a close. For example there are still questions about pseudoholomorphic *foliations* of a PH-torus neighborhood by cylinders (see section 3) to be investigated. I expect this can be done by a method similar to Moser's [Mo].

1 Moduli of germs of pseudoholomorphic curves neighborhoods

1.1 Nijenhuis tensor characteristic distribution

Here we recall basic facts about Nijenhuis tensors in dimension 4. They generalize theorem 7 [K1].

Nijenhuis tensor of almost complex manifold (M, J) is the $(2, 1)$-tensor $N_J \in \Lambda^2 T^*M \otimes TM$ given by the formula:

$$N_J(\xi, \eta) = [J\xi, J\eta] - J[J\xi, \eta] - J[\xi, J\eta] - [\xi, \eta]. \qquad (1)$$

This tensor satisfies J-linearity $N_J(J\xi, \eta) = N_J(\xi, J\eta) = -JN_J(\xi, \eta)$. So it can be considered as an antilinear map $N_J : \Lambda^2\mathbb{C}^2 \to \mathbb{C}^2$, $\mathbb{C}^2 = (T_xM^4, J)$. Thus the image is invariant under J and hence is a complex line $\mathbb{C} \subset \mathbb{C}^2$ (if $N_J \neq 0$) or a point 0 (if $N_J = 0$).

Suppose $N_J \neq 0$ in a neighborhood of $x \in M$. We consider the above complex line as J-invariant 2-dimensional real subspace of T_xM and denote it by Π_x. Thus we obtain a distribution of 2-dimensional planes in a neighborhood of x.

Definition 1. *Let us call* $\Pi^2 = \operatorname{Im} N_J$ *the* Nijenhuis tensor characteristic distribution *on 4-dimensional almost complex manifold* (M^4, J).

Remark 1. *At the points, where* $N_J = 0$, *we have* $\operatorname{Im} N_J = 0$. *Therefore we should call* Π *differential systems. However singular points of* Π, *where the rank is 0, are generically isolated and so are not relevant to our future discussion. Thus we keep the term distribution.*

This distribution Π^2 is in general situation nonintegrable. Therefore it has nontrivial derivative $\Pi^3 = \partial\Pi^2$, that is the differential system with the $C^\infty(M)$-module of sections $\mathcal{P}_3 = C^\infty(\Pi^3)$ generated by the module of sections $\mathcal{P}_2 = C^\infty(\Pi^2)$ and its self-commutator: $\mathcal{P}_3 = [\mathcal{P}_2, \mathcal{P}_2]$. Π^3 is not distribution everywhere, it can have singularities. These singularities form a (stratified) manifold Σ_2' of codim $= 2$ with singularities.

Consider a point outside Σ_2'. Generically the distribution Π^3 is not integrable and so $\mathcal{P}_4 = [\mathcal{P}_3, \mathcal{P}_3] = C^\infty(TM) = \mathcal{D}(M)$ is the module of all vector fields, so that the next distribution is the whole tangent space. Moreover we can assume $[\mathcal{P}_2, \mathcal{P}_3] = \mathcal{D}(M)$. This again fails to be so outside some other (stratified) manifold with singularities Σ_2'' of codim $= 2$.

Let $x \notin \Sigma_2'$. Then $\Pi_x^2 \subset \Pi_x^3$ has a transversal measure. Actually there exist vectors $\xi_1, \xi_2 \in \Pi_x^2$, $\xi_3 \in \Pi_x^3 \setminus \Pi_x^2$ such that $N_J(\xi_1, \xi_3) = \xi_1$, $N_J(\xi_2, \xi_3) = -\xi_2$ (because a J-antilinear isomorphism $N_J(\cdot, \xi_3) : \Pi_x^2 \to \Pi_x^2$ is orientation reversing). The fields ξ_1, ξ_2 are defined up to multiplication by a constant, while $\xi_3(\text{mod } \Pi_x^2)$ is defined up to multiplication by ± 1. Therefore Π^3/Π^2 is normed.

By a similar reason T_xM/Π_x^3 is normed outside $\Sigma_2' \cup \Sigma_2''$. The transversal measure for $\Pi_x^3 \in T_xM$ is given by the vector $\xi_4 = J\xi_3$.

Note that Π_x^3/Π_x^2 is oriented. Actually $[\xi_1, \xi_2] \bmod \Pi_x^2$ depends only on the values of ξ_1, ξ_2 at the point x. It is a vector $f\xi_3 \bmod \Pi_x^2$ for some f. So if we require $\xi_2 = J\xi_1$ then ξ_3 can be chosen so that $f > 0$. This produces coorientation of $\Pi_x^2 \subset \Pi_x^3$ and then via J coorientation of $\Pi_x^3 \subset T_xM$.

Moreover the requirement $f = 1$ determines canonically vector field ξ_1 (still however up to ± 1) and hence $\xi_2 = J\xi_1$. Then we set $\xi_3 = [\xi_1, \xi_2]$ and $\xi_4 = J\xi_3$. So the pair (ξ_1, ξ_2) is defined canonically up to a sign and the pair (ξ_3, ξ_4) is absolutely canonical.

Theorem 1. *Let almost complex structure J be of general position. Then at generic points $x \in M^4$ a canonical frame $(\xi_1, \xi_2, \xi_3, \xi_4)$ is defined. It restores uniquely the almost complex operator J and the tensor N_J by the tables:*

X	JX
ξ_1	ξ_2
ξ_2	$-\xi_1$
ξ_3	ξ_4
ξ_4	$-\xi_3$

$N_J(\uparrow, \leftarrow)$	ξ_1	ξ_2	ξ_3	ξ_4
ξ_1	0	0	ξ_1	$-\xi_2$
ξ_2	0	0	$-\xi_2$	$-\xi_1$
ξ_3	$-\xi_1$	ξ_2	0	0
ξ_4	ξ_2	ξ_1	0	0

Note that reducing a structure to the frame (or $\{e\}$-structure) solves completely the equivalence problem. We recall briefly the main idea. Consider the moduli of the problem, i.e. functions c_{jk}^i given by the formula $[\xi_j, \xi_k] = \sum c_{jk}^i \xi_i$. Denote by $\mathbb{A} = \{c_{jk}^i\}$ the space of all invariants and by $\Phi : M \to \mathbb{A}$ the "momentum map" $x \mapsto \{c_{jk}^i(x)\}$. Then two equivalent structures have the same images and this gives an equivalence of the structures in most cases. For more details see [S].

Remark 2. *If both derivatives of Π^2 are nontrivial, i.e. $x \notin \Sigma_2' \cup \Sigma_2''$ then the distribution Π^2 has a canonical normal form. It is called* Engel distribution *([BCG]). It is shown in [K2] that this and other cases are realized as Nijenhuis tensors characteristic distributions.*

1.2 Invariants of a PH-curve neighborhood

Let $\mathcal{C}^2$ be a PH-curve. At every point $x \in \mathcal{C}$ we have two J-invariant planes $T_x\mathcal{C}^2$ and Π_x^2. Obviously in general position they intersects by zero. When these distributions are considered along the torus they are generically transversal everywhere except a finite number of points. Denote this set by

$\tilde{\Sigma}_0$. In general (transversal) case there are also finite sets $\Sigma_0' = \Sigma_2' \cap \mathcal{C}$ and $\Sigma_0'' = \Sigma_2'' \cap \mathcal{C}$. The arrangement of these points

$$\Sigma_0 = \tilde{\Sigma}_0 \cup \Sigma_0' \cup \Sigma_0'' \subset \mathcal{C}$$

gives a (finite-dimensional) invariant.

For points $x \in \mathcal{C} \setminus \Sigma_0'$ we define field of directions $L^1 = T\mathcal{C} \cap \Pi^3$. The integral curves of this 1-distribution foliate the set $\mathcal{C} \setminus \Sigma_0'$ and in general $\mathcal{C}$ foliates with only nondegenerate singular points. Denote the number of elliptic points by $e(L^1)$ and the number of hyperbolical by $h(L^1)$. Note that (topologically stable) points of $\tilde{\Sigma}_0$ are usually regular points of L^1.

Lemma 2. *Under C^1-small perturbation of the structure J the foliation L^1 has minimal number of singularities,* $\min\{e(L), h(L)\} = 0$, $\max\{e(L), h(L)\} = \frac{1}{2}|\chi(L)|$. *In particular for torus $\mathcal{C} = T^2$ we get foliation without singularities.*

Proof. Actually this perturbation is C^0-small for Π^2 and large for Π^3 and hence L^1. To perform it one collects elliptic-hyperbolic points at pairs (after a small perturbation one can assume there's nothing more complicated than separatrix self-connection) and kill them one by one. The possibility of such a perturbation is an easy calculational argument similar to theorem 5 [K1]. $\square$

As follows from the previous subsection the foliation L^1 is equipped with orientation and parallel measure outside Σ_0. Thus there exists a canonical vector field v_1 along L^1. So the curve $\mathcal{C}$ has many invariants from dynamical systems point of view, for example winding number of v_1 for every two independent cycles, linearizations at critical points etc.

But since the foliation is equipped with a coorientation and a transversal measure we have more. Let v_2 be positively cooriented, have transversal measure 1 and be directed along JL^1. Then (v_1, v_2) is a canonical frame outside Σ_0. Writing

$$[v_1, v_2] = \gamma_1 v_1 + \gamma_2 v_2.$$

we obtain two invariant (under pseudoholomorphic isomorphisms) functions γ_1, γ_2. We can also construct a function γ_3 defined by $Jv_1 = \gamma_3 v_2$. These together with the functions c^i_{jk} from the previous subsection and others form *moduli* of the $\mathcal{C}$-neighborhoods germ.

Note that all constructed invariants are nontrivial. We illustrate this for the simplest case of arbitrary winding number on the torus.

Example. Consider the manifold $M = T^2(\varphi, \psi) \times \mathbb{R}^2(x, y)$ with almost complex structure given by

$$\begin{aligned} &J\partial_x = \partial_y; \quad && J\partial_\varphi = \tfrac{2+\rho y^2}{2}\partial_\psi + \tfrac{y^2}{2}\partial_\varphi + x\partial_x; \\ &J\partial_y = -\partial_x; \quad && J\partial_\psi = -\tfrac{4+y^4}{4+2\rho y^2}\partial_\varphi - \tfrac{y^2}{2}\partial_\psi - \tfrac{xy^2}{2+\rho y^2}\partial_x - \tfrac{2x}{2+\rho y^2}\partial_y, \end{aligned}$$

where $\rho \in \mathbb{R}$. The torus $\mathcal{C} = \{x = y = 0\}$ is pseudoholomorphic. We calculate $N_J(\partial_x, \partial_\varphi) = \xi$ for $\xi = y(\partial_\varphi + \rho\partial_\psi) - \partial_y$. Thus $\Pi^2 = \langle \xi, J\xi \rangle$ is transversal to $\mathcal{C}$. Since $[\xi, J\xi]_{x=y=0} = -\partial_\psi + \rho\partial_\varphi$ we conclude that the foliation $\langle L^1 \rangle$ is given by the equation $\{\dot\varphi = \rho, \dot\psi = -1\}$, whence the winding number is $-\rho$.

1.3 Holomorphic recollections

Let $\mathcal{C}$ be a holomorphic curve in some complex surface and let it be not self-linked, $\mathcal{C}\cdot\mathcal{C}=0$. Consider its co-normal bundle defined as the first term of the following exact sequence:

$$0\to\mathcal{I}/\mathcal{I}^2\to T^*M|_{\mathcal{C}}\to T^*\mathcal{C}\to 0,$$

where $\mathcal{I}$ is the ideal of holomorphic functions corresponding to the curve $\mathcal{I}=\{f:\mathcal{O}\to\mathbb{C}\,|\,f(\mathcal{C})=0\}$ and $\mathcal{O}$ is the germ of neighborhoods of $\mathcal{C}\subset M$. The dual sequence

$$0\to T\mathcal{C}\to TM|_{\mathcal{C}}\to N_{\mathcal{C}}M\to 0 \tag{2}$$

defines the normal bundle.

Let the curve be elliptic $\mathcal{C}=T^2$. This torus is uniquely characterized by its periods $(2\pi,\nu)$, where $\operatorname{Im}(\nu)\neq 0$. The complex number ν is determined almost uniquely in the sense that the corresponding lattice $\mathbb{Z}^2(2\pi,\nu)$ is unique. The torus is given then as the quotient of $\mathbb{C}$ by the lattice.

All holomorphic topologically trivial 1-dimensional vector bundles over this elliptic curve are described as follows. Consider $\mathbb{C}^2$ with coordinates (z,φ) and for $\lambda\in\mathbb{C}\setminus\{0\}$, $\nu\in\mathbb{C}\setminus\mathbb{R}$ make the identifications

$$(z,\varphi)\simeq(z,\varphi+2\pi)\simeq(\lambda z,\varphi+\nu). \tag{3}$$

The bundle $E\to T^2$ is now given by $(z,\varphi)\mapsto\varphi$.

Similarly one classifies S^2-bundles for which two distinguished holomorphic sections "zero" and "infinity" are chosen.

Arnold in [A1] §27 considers normal bundle $N_{T^2}M^4$ to an elliptic curve and prove that if the pair (λ,ν) is normal nonresonant ($\lambda^n\neq e^{ik\nu}$ plus some diophantine condition for this pair) then a small neighborhood of $T^2\subset M$ is biholomorphically equivalent to a neighborhood of the zero section in $N_{T^2}M$.

Note that the number λ is defined by 1-jet of the complex structure on the torus T^2. But in almost complex situation 1-jet is determined by the field of the Nijenhuis tensors N_J along the torus, so this λ is not defined. We are going to define a family of complex structures $J_{(\lambda)}$ which have the same values as J at the points of the torus T^2.

1.4 Complex structures in a PH-torus neighborhood

Let $\mathcal{C}\subset M^4$ be a pseudoholomorphic curve. To define some local normal coordinates in a neighborhood of $\mathcal{C}$ we prove

Proposition 3. *Small neighborhood $\mathcal{O}$ of a PH-curve $\mathcal{C}\subset M^4$ can be foliated by transversal PH-disks D^2.*

This actually follows from Nijenhuis-Wolf theorem [NW] of existence of small PH-disk in a given direction (which is normal in our case) and smooth dependence on this direction. We give another proof based on the (visibly different) idea we exploit later on.

Proof. Let us change the almost complex structure J outside a small neighborhood $\mathcal{O}$ of $\mathcal{C}$ in such a manner that it be integrable. Moreover it can be done so that some bigger neighborhood is isomorphic to $\mathcal{C} \times D^2$ near the boundary. Thus we can glue the neighborhood and obtain the manifold $M_0 = \mathcal{C} \times S^2$ with the almost complex structure J' being integrable outside the fixed neighborhood of our PH-curve. (cf. prop. 2[K3]).

Now we introduce a symplectic product-structure $\omega = \omega_1 \oplus \omega_2$ on (M_0, J') and note that for sufficiently small neighborhood $\mathcal{O}$ symplectic structure ω tames the structure J'. If $\chi(\mathcal{C}) < 2$ the pseudoholomorphic spheres can lie only in the multiple of the homology class of the second factor in $M_0 = \mathcal{C}^2 \times S^2$. So we seek for them in the primitive class $[S^2]$. The Gromov theory ([G]) implies that in the case of general position almost complex structure J' the manifold (M_0, J') is foliated by PH-spheres. Taking the intersection of this foliation with the neighborhood $\mathcal{O}$ we prove the statement.

If the curve is rational $\mathcal{C} = S^2$ some more delicate arguments are required ([M2]). If the almost complex structure J' is not of general position we apply the compactness theorem from [MS] for a generic sequence $J_i \to J'$. $\square$

Now if $z = x + iy$ is a complex coordinate on some transversal PH-disk we can define coordinates on all close disks by parallel transport (along specified rays) using some connection ∇. If the connection is almost complex $\nabla J = 0$ then on every nearby transversal the coordinate is complex. However we cannot define the coordinates globally in such a way because of the holonomy. We consider the special case of elliptic curve $\mathcal{C} = T^2$ (the case of rational curve is trivial).

Note that the number ν is defined for $\mathcal{C}$. Actually almost complex structure J on any 2-dimensional manifold is integrable and hence the torus T^2 has periods $(2\pi, \nu)$. In particular we have a (double-periodic) complex coordinate $\varphi = \varphi_1 + i\varphi_2$ on the curve.

We extend the coordinate φ to the neighborhood $\mathcal{O}$ by the projection along the disks, i.e. the disks are given by $D_\varphi = \{\varphi = \text{const}\}$. We introduce now complex coordinate $z = x + iy$ on every transversal disk in such a way that they specify the same complex structure on D_φ as the restriction of J and our PH-torus is $\{z = 0\}$. This coordinates are multivalued because of the holonomy but we can choose z so that the gluing rules (monodromies along the cycles) on the torus are given by (3) with any prescribed λ. Now assuming the coordinates (z, φ) are complex we get the complex structure $J_{(\lambda)}$ in $\mathcal{O}$.

The previous discussion and Arnold theorem imply that these structures form an almost complete family in the following sense

Proposition 4. *Let $\tilde{J}$ be a complex structure in a neighborhood of $T^2 \subset M^4$ equal to the almost complex structure J at the points of the torus. Then if the pair (λ, ν), determined by $\tilde{J}$ on T^2, is normal nonresonant the germ of $\tilde{J}$ is equivalent to the germs of $J_{(\lambda)}$.* $\square$

2 Differential geometry of a pseudoholomorphic curve neighborhood

2.1 Normal bundle of a pseudoholomorphic curve

In this section we do not require $\dim M = 4$.

In almost complex case the bundle $TM|_{\mathcal{C}}$ for a PH-curve $\mathcal{C} \subset M$ is no longer holomorphic. So one needs to consider almost complex bundles. A bundle $\pi : (E, J) \to (\mathcal{C}, J_0)$ is called almost complex if

$$\pi_*(J\xi) = J_0(\pi_*\xi).$$

Proposition 5. *Consider the normal bundle to a PH-curve $\mathcal{C} \subset M$ given as vector space by the sequence (2). There is a canonical almost complex structure $\hat{J}$ on $N_{\mathcal{C}}M$ such that $(N_{\mathcal{C}}M, \hat{J})$ is an almost complex bundle.*

Proof. Recall that on almost complex manifold (M, J) there is always an almost complex connection $\nabla J = 0$. Actually if ∇' is any linear connection then $\nabla = \frac{1}{2}\big(\nabla' - J\nabla' J\big)$ is also a linear connection, which preserves the structure J. Moreover we can assume that ∇ is minimal, i.e. the torsion of ∇ equals to its antilinear by each argument part $T_\nabla = T_\nabla^{--} = \frac{1}{4}N_J$ ([L]).

Lemma 6. *There exists a minimal almost complex connection such that the curve $\mathcal{C}$ is a totally geodesic submanifold, i.e. parallel transport of any vector $v \in T\mathcal{C}$ along a path in $\mathcal{C}$ belongs again to $T\mathcal{C}$.*

Proof. To see this let us make a gauge transformation $\tilde{\nabla} = \nabla + A$, where $A \in \Omega^1(M, \mathrm{end}_{\mathbb{C}}\, \tau_M)$ is a 1-form with values in complex endomorphisms of the tangent bundle. If we require that this 1-form is symmetric $A \in S^2T^*M \otimes_{\mathbb{C}} TM$, then the new connection $\tilde{\nabla}$ is also almost complex and minimal.

Let ξ be a vector field on $\mathcal{C}$ with nondegenerate critical points, $\xi \in \mathcal{D}(\mathcal{C})$. Let $\nabla_\xi \xi = \eta \in \mathcal{D}(M)$. We define $A(\xi, \xi) = -\eta$, $A(J\xi, \xi) = A(\xi, J\xi) = -J\eta$, $A(J\xi, J\xi) = \eta$ and for other vectors somehow preserving symmetry and J-linearity (near the critical points there's a more work, we need to choose ξ so that η has a zero of the second order at these critical points).

Then $\tilde{\nabla}_\xi \xi = 0$, $\tilde{\nabla}_\xi J\xi = 0$. Therefore by minimality $\tilde{\nabla}_{J\xi}\xi = \tilde{\nabla}_\xi J\xi + [J\xi, \xi] + \frac{1}{4}N_J(J\xi, \xi) = [J\xi, \xi] \in \mathcal{D}(\mathcal{C})$ and also $\tilde{\nabla}_{J\xi} J\xi \in \mathcal{D}(\mathcal{C})$. So $\tilde{\nabla}$ preserves $T\mathcal{C}$. $\square$

Another way to prove this is to introduce trivial connection in $\mathcal{O}(\mathcal{C}) \simeq \mathcal{C} \times D$ and then to check that procedures of making connection almost complex and then minimal do not destroy the property of $\mathcal{C}$ being totally geodesic. Let us denote this new connection again by the symbol ∇.

We introduce a connection $\hat{\nabla}$ to the bundle $N_{\mathcal{C}}M$ by means of parallel transport as follows. Let $v = [\theta] \in (N_{\mathcal{C}}M)_x$ be the class of $\theta \in T_xM$ and let $\gamma(t) \subset \mathcal{C}$ be a curve, $\gamma(0) = x$. Calculate parallel transport $\theta(t)$ of θ along $\gamma(t)$. We define $v(t) = [\theta(t)]$ to be the parallel transport of v along $\gamma(t)$. Since $\mathcal{C}$ is totally geodesic, the definition is correct ($\hat{\nabla}$-parallel transport of 0 is 0). Moreover the connection $\hat{\nabla}$ is $\mathbb{R}$-linear. So as usual in the theory of generalized connections we conclude that $\hat{\nabla}$ is a linear connection.

Let $a \in N_{\mathcal{C}}M$ be a point in the normal bundle with projection $\pi(a) = x$. Let $T_a(N_{\mathcal{C}}M) = H_a \oplus V_a$ be the decomposition by horizontal and vertical subspaces induced by $\hat{\nabla}$. The first space $H_a \overset{\pi_*}{\simeq} T_x\mathcal{C}$ has a canonical complex structure J_1 induced from J by π_*, while the second $V_a \simeq T_xM/T_x\mathcal{C}$ inherits canonical complex structure J_2 from J. So we introduce the structure on $N_{\mathcal{C}}M$ by the rule $\hat{J} = J_1 \oplus J_2$.

Let us show that the constructed almost complex structure $\hat{J}$ does not depend on the choice of minimal connection ∇ preserving TC. Let us change the connection $\tilde{\nabla} = \nabla + A$, $A \in \Omega^1(M; \mathrm{end}_{\mathbb{C}}\,\tau_M) \cap S^2\tau_M^* \otimes \tau_M$. This affects in the change of decomposition $T_a(N_{\mathcal{C}}M) = \tilde{H}_a \oplus V_a$, where $\tilde{H}_a = \mathrm{graph}\{\lambda_a : H_a \to V_a\}$ and $\lambda_a = A(\cdot)a$. We state that λ_a is a complex linear map. Actually

$$\lambda_a(Jw) = A(Jw)a = A(a)Jw = JA(a)w = JA(w)a = J\lambda_a(w).$$

So the complex structure $\hat{J}$ on $T_a(N_{\mathcal{C}}M)$ is canonically defined. □

Remark 3. *Let $\phi_t : (\mathcal{C}, J_0) \to (M^4, J)$ be a family of J-holomorphic curves of the same holomorhpic type with $\phi_0 = \mathrm{id}$. Then $\frac{d}{dt}\big|_{t=0}\phi_t$ is the pseudoholomorphic curve in $N_{\mathcal{C}}M$. Moreover we have a one-parametric (scaled) family of PH-curves. Since $(N_{\mathcal{C}}M)_x = T_xM/T_x\mathcal{C}$ this family does not depend on the reparametrization $\tilde{\phi}_t = \phi_t \circ g_t$ for $g_t \in \mathrm{Aut}(\mathcal{C}, J_0)$ (note that a curve in the bundle bijectively projected to the base has a natural parametrization). We will use it in §3.4.*

Remark 4. *There is another approach to get an almost complex structure on $N_{\mathcal{C}}M$ ([M2]). Consider a foliation of a neighborhood of $\mathcal{C} \subset M$ to transversal PH-disks (proposition 3). Let A_t be the dilation along the disks. We define $J_t = A_tJA_t^{-1}$ and $\hat{J} = \lim_{t\to\infty} J_t$. Now the bundle $(N_{\mathcal{C}}M, \hat{J})$ is almost complex. However it is difficult to see canonicity and the properties of this almost complex structure.*

2.2 Almost complex bundles

Now we return to the case $\dim M = 4$.

Proposition 7. *Let $\pi : (E^4, J) \to (\mathcal{C}^2, J_0)$ be an almost complex bundle over a curve. Then the characteristic distribution $\Pi^2 = \mathrm{Im}\, N_J$ is integrable and is tangent to the fibers $F_x = \pi^{-1}(x)$.*

Proof. Actually:

$$\begin{aligned}
\pi_*N_J(\xi,\eta) &= \pi_*[J\xi, J\eta] - \pi_*J[\xi, J\eta] - \pi_*J[J\xi, \eta] - \pi_*[\xi, \eta] \\
&= [J_0\pi_*\xi, J_0\pi_*\eta] - J_0[\pi_*\xi, J_0\pi_*\eta] - J_0[J_0\pi_*\xi, \pi_*\eta] - [\pi_*\xi, \pi_*\eta] \\
&= N_{J_0}(\pi_*\xi, \pi_*\eta) = 0.
\end{aligned}$$

□

Corollary 8. *Codimension of the set of almost complex structures, the germs of which on the PH-curve $\mathcal{C} \subset M$ are isomorphic to these of the normal bundle $\mathcal{C} \subset N_{\mathcal{C}}M$, in the set of all almost complex structures is infinity.* $\square$

Proof. Actually if the distribution Π^2 is nonintegrable in a neighborhood of the PH-curve $\mathcal{C}$, then a neighborhood $(\mathcal{O}, J)$ of the curve is not isomorphic to a neighborhood of the zero section in the normal bundle $(N_{\mathcal{C}}M, \hat{J})$. $\square$

This property is contrary to its analog in the complex category, see Arnold theorem [A1] about neighborhoods of elliptic curves (§1.3). So extending the category the discussed property becomes exceptional.

As the following example shows the integrability of Π^2 is necessary but by no means sufficient condition on the structure J to be locally isomorphic to its representative on the normal bundle.

Example. Consider the almost complex structure given on a $T^2(\varphi) \times D^2(z)$ with $\varphi = \varphi_1 + i\varphi_2$, $z = x + iy$ by the formula:

$$J\partial_x = \partial_y,\ J\partial_{\varphi_1} = \partial_{\varphi_2} + A_1\partial_{\varphi_1} + A_2\partial_{\varphi_2} + B_1\partial_x + B_2\partial_y, \tag{4}$$

with $A_i|_{T^2} = B_i|_{T^2} = 0$. The condition $\Pi^2 = T_*\{\varphi = \text{const}\}$ is equivalent to the following PDE system

$$\begin{cases} \dfrac{\partial A_1}{\partial y} = A_1\dfrac{\partial A_1}{\partial x} - \dfrac{1+A_1^2}{1+A_2}\dfrac{\partial A_2}{\partial x} \\ \dfrac{\partial A_2}{\partial y} = (1+A_2)\dfrac{\partial A_1}{\partial x} - A_1\dfrac{\partial A_2}{\partial x} \end{cases} \tag{5}$$

If the functions A_i satisfy this (Cauchy-Kovalevsky) system the projection along the leaves is given by the formula $(z, \varphi) \mapsto \varphi$. Therefore our structure J is projectable iff $A_i \equiv 0$. But there are nonzero solutions of (5). For example:

$$A_1 = -\frac{x}{1+y},\ A_2 = -\frac{y}{1+y}.$$

So the space of projectable structures J are of codim $= \infty$ among the structures with Π^2 integrable, which are of codim $= \infty$ among all almost complex structures with a fixed 0-jet on the torus T^2.

2.3 Nijenhuis tensor of normal bundles

Let $N_{\mathcal{C}}M$ be a normal bundle of pseudoholomorphic curve $\mathcal{C} \subset M^4$. Then at the points $x \in \mathcal{C}$ two different Nijenhuis tensors N_J of ambient structure and $N_{\hat{J}}$ of the structure of the normal bundle are defined. Are there some relations between these two tensors? Or between characteristic distributions of these structures? As the following examples show the answer is negative.

Example. "Parallel distribution Π^2". Let $M = \mathbb{R}^4(x_1, y_1, x_2, y_2)$ be equipped with almost complex structure

$$J\partial_{x_1} = \partial_{y_1},\ J\partial_{y_1} = -\partial_{x_1},\ J\partial_{x_2} = \partial_{y_2} + x_1\partial_{x_1},\ J\partial_{y_2} = -\partial_{x_2} - x_1\partial_{y_1}.$$

Then the curve $\mathcal{C} = \{x_2 = y_2 = 0\}$ is pseudoholomorphic. We calculate that the following are the Nijenhuis tensor of the structure J and a minimal almost complex connection (note that $\Pi^2 = \operatorname{Im} N_J = \langle \partial_{x_1}, \partial_{y_1} \rangle$):

$N_J(\uparrow, \leftarrow)$	∂_{x_1}	∂_{y_1}	∂_{x_2}	∂_{y_2}
∂_{x_1}	0	0	$-\partial_{y_1}$	$-\partial_{x_1}$
∂_{y_1}	0	0	$-\partial_{x_1}$	∂_{y_1}
∂_{x_2}	∂_{y_1}	∂_{x_1}	0	$-x_1\partial_{y_1}$
∂_{y_2}	∂_{x_1}	$-\partial_{y_1}$	$x_1\partial_{y_1}$	0

$\nabla_{\uparrow}\leftarrow$	∂_{x_1}	∂_{y_1}	∂_{x_2}	∂_{y_2}
∂_{x_1}	0	0	$-\frac{1}{4}\partial_{y_1}$	$-\frac{3}{4}\partial_{x_1}$
∂_{y_1}	0	0	$-\frac{1}{4}\partial_{x_1}$	$-\frac{1}{4}\partial_{y_1}$
∂_{x_2}	0	0	0	0
∂_{y_2}	$-\frac{1}{2}\partial_{x_1}$	$-\frac{1}{2}\partial_{y_1}$	$\frac{1}{4}x_1\partial_{y_1}$	$\frac{1}{4}x_1\partial_{x_1}$

So we find that the horizontal planes are $H = \langle \partial_{x_1}, \partial_{y_1} \rangle$, whence the structure on $N_{\mathcal{C}}M$ is

$$\hat{J}\partial_{x_1} = \partial_{y_1},\ \hat{J}\partial_{y_1} = -\partial_{x_1},\ \hat{J}\partial_{x_2} = \partial_{y_2},\ \hat{J}\partial_{y_2} = -\partial_{x_2}$$

and $N_{\hat{J}} = 0$.

Example. "Transversal distribution Π^2". Let the structure be now

$$J\partial_{x_1} = \partial_{y_1} + x_2\partial_{x_2},\ J\partial_{y_1} = -\partial_{x_1} - x_2\partial_{y_2},\ J\partial_{x_2} = \partial_{y_2},\ J\partial_{y_2} = -\partial_{x_2}.$$

Again the curve $\mathcal{C} = \{x_2 = y_2 = 0\}$ is pseudoholomorphic and the Nijenhuis tensor and a minimal almost complex connection are (now $\Pi^2 = \langle \partial_{x_2}, \partial_{y_2} \rangle$):

$N_J(\uparrow, \leftarrow)$	∂_{x_1}	∂_{y_1}	∂_{x_2}	∂_{y_2}
∂_{x_1}	0	$-x_2\partial_{y_2}$	∂_{y_2}	∂_{x_2}
∂_{y_1}	$x_2\partial_{y_2}$	0	∂_{x_2}	$-\partial_{y_2}$
∂_{x_2}	$-\partial_{y_2}$	$-\partial_{x_2}$	0	0
∂_{y_2}	$-\partial_{x_2}$	∂_{y_2}	0	0

$\nabla_{\uparrow}\leftarrow$	∂_{x_1}	∂_{y_1}	∂_{x_2}	∂_{y_2}
∂_{x_1}	0	0	0	0
∂_{y_1}	$\frac{1}{4}x_2\partial_{y_2}$	$\frac{1}{4}x_2\partial_{x_2}$	$-\frac{1}{2}\partial_{x_2}$	$-\frac{1}{2}\partial_{y_2}$
∂_{x_2}	$-\frac{1}{4}\partial_{y_2}$	$-\frac{3}{4}\partial_{x_2}$	0	0
∂_{y_2}	$-\frac{1}{4}\partial_{x_2}$	$-\frac{1}{4}\partial_{y_2}$	0	0

The horizontal planes are $H = \langle \partial_{x_1}, \partial_{y_1} + \frac{1}{2}e^{\frac{1}{2}y_1}(x_2\partial_{x_2} + y_2\partial_{y_2}) \rangle$. So the structure on $N_{\mathcal{C}}M$ is

$$\begin{aligned} \hat{J}\partial_{x_1} &= \partial_{y_1} + \tfrac{1}{2}e^{\frac{1}{2}y_1}(x_2\partial_{x_2} + y_2\partial_{y_2}), \qquad & \hat{J}\partial_{x_2} &= \partial_{y_2}, \\ \hat{J}\partial_{y_1} &= -\partial_{x_1} - \tfrac{1}{2}e^{\frac{1}{2}y_1}(x_2\partial_{y_2} - y_2\partial_{x_2}), & \hat{J}\partial_{y_2} &= -\partial_{x_2}. \end{aligned}$$

Now $N_{\hat{J}}(\partial_{x_1}, \partial_{x_2}) = -e^{\frac{1}{2}y_1}\partial_{y_2}$ and so the characteristic distribution of the normal structure is "the same" as for the structure J: $\hat{\Pi}^2 = \operatorname{Im} N_{\hat{J}} = \langle \partial_{x_2}, \partial_{y_2} \rangle$.

So the answer to the above question is positive if N_J is of special type.

2.4 Linear bundle almost complex structures

Consider an almost complex vector bundle $\pi : (E, J) \xrightarrow{F} (\mathcal{C}, J_0)$ of the rank $\dim F = 2n$ and suppose the restriction $J|_F$ is a linear complex structure on the fiber. So we can also consider $(E, \pi, \mathcal{C})$ just as vector bundle with complex structures in the fibers.

Definition 2. *We call the almost complex structure J on E* linear bundle structure *if there exists a linear minimal almost complex connection $\hat{\nabla}$ on this bundle such that the lift $T_bE \stackrel{\hat{\nabla}}{\longleftarrow} T_a\mathcal{C}$ is a complex mapping, splitting the exact sequence*

$$0 \to F \to T_aE \to T_x\mathcal{C} \to 0, \qquad x = \pi(a).$$

In particular the zero section $\mathcal{C} \subset E$ is a J-pseudoholomorphic curve.

We note now that the almost complex structure $\hat{J}$ on the normal bundle $N_{\mathcal{C}}M$ is a linear bundle structure.

Lemma 9. *The Nijenhuis tensor of linear bundle almost complex structure J is constant along the fibers and determines J-invariant differential system $\Pi = \operatorname{Im} N_J \subset TE$ which is a subsystem of vertical distribution F.*

Proof. There are local coordinates (φ, z) on $\pi^{-1}(U) = U \times F$, $\varphi = \varphi_1 + i\varphi_2$, $z_k = x_k + iy_k$, $1 \le k \le n$, such that $\hat{\nabla}$-lift of ∂_{φ_i} is $\partial_{\varphi_i} + \sum b_{ij}\partial_{x_j} + c_{ij}\partial_{y_j}$, where the coefficients are linear functions of z_k and the vertical coordinates are complex linear coordinates but horizontal coordinates are complex only on the zero section. So the structure J is given by relations

$$J\partial_{\varphi_1} = \partial_{\varphi_2} + \sum (b_{2j} + c_{1j})\partial_{x_j} + (c_{2j} - b_{1j})\partial_{y_j}, \qquad J\partial_{x_k} = \partial_{y_k}$$

and so $N_J(\partial_{\varphi_1}, \partial_{x_j}) = \sum \alpha_j^k \partial_{x_k} + \beta_j^k \partial_{y_k}$ with constant coefficients α_j^k, β_j^k.

The last claim follows because $N_J(F, F) = 0$ and $\Pi = \mathbb{C}N_J(\partial_{\varphi_1}, F)$ is a linear subspace of F (of course invariant under J). Note that $\operatorname{rk}\Pi$ can vary with $\varphi \in \mathcal{C}$. $\square$

Remark 5. *The conditions $N_J(F, F) = 0$ and $\operatorname{Im} N_J \subset F$ allow to lift naturally this tensor from $\mathcal{C}$ to E. Actually let $\xi \in T_xC$ and $\eta, \theta \in T_xF \subset T_xE$. Let us choose some lift $\tilde{\xi} \in T_aE$ (i.e. $(\pi_*)\tilde{\xi} = \xi$). Let $\tilde{\eta}, \tilde{\theta} \in T_aF = F$ be identified with η, θ. Then we define $N_J(\tilde{\xi}, \tilde{\eta}) = \tilde{\theta}$ at $a \in E$ if $N_J(\xi, \eta) = \theta$ at $x = \pi(a)$. Our assumptions imply that this extension does not depend on a lift of ξ. So N_J on E is the canonical tensor.*

Consider now the case $\dim E = 4$. For every point $a \in E$ denote by $r = r_a \in T_aE$ the vertical vector equal to $\vec{xa} \in F \simeq T_aF$ with $x = \pi(a) \in \mathcal{C}$.

Theorem 10. *Let J be a linear bundle structure on a vector bundle $\pi : E \to \mathcal{C}$ with 2-dimensional fibers over a curve $(\mathcal{C}, J')$. Then for some complex structure J_0 on E, making the bundle holomorphic, we have:*

$$J = J_0 + \frac{1}{2}J_0N_J(r, \cdot).$$

Proof. Let us define the structure by the formula

$$J_0 = J - \frac{1}{2}JN_J(r, \cdot). \tag{6}$$

Since $N_J|_F \equiv 0$ this structure $J_0|_F = J|_F$ is linear on fibers. This proves the formula for J of the theorem.

We first show that the structure J_0 is almost complex. Note that by lemma 9 $N_J(r,\xi) \in F$ for any ξ and $N_J(r,\xi) = 0$ for $\xi \in F$. So

$$J_0^2 = J^2 - \frac{1}{2}J^2 N_J(r,\cdot) - \frac{1}{2}JN_J(r,J\cdot) + \frac{1}{4}JN_J(r,JN_J(r,\cdot)) = J^2 = -\mathbf{1}.$$

Now we show that this J_0 is integrable. By Newlander-Nirenberg theorem [NW] this is equivalent to the vanishing of the tensor N_{J_0}. Let us choose local coordinates (z,φ) as in proposition 3. In these coordinates $J\partial_x = J_0\partial_x = \partial_y$ and we deduce from (6):

$$\begin{aligned} N_{J_0}(\partial_x,\partial_{\varphi_1}) &= N_J(\partial_x,\partial_{\varphi_1}) - [\partial_y, \tfrac{1}{2}JN_J(x\partial_x + y\partial_y, \partial_{\varphi_1})] \\ &\qquad + J[\partial_x, \tfrac{1}{2}JN_J(x\partial_x + y\partial_y, \partial_{\varphi_1})] \\ &= N_J(\partial_x,\partial_{\varphi_1}) - \tfrac{1}{2}N_J(\partial_x,\partial_{\varphi_1}) - \tfrac{1}{2}N_J(\partial_x,\partial_{\varphi_1}) = 0. \end{aligned}$$

Since the bivector $\partial_x \wedge \partial_{\varphi_1}$ generates $\Lambda^2_{\mathbb{C}}TE$ the claim follows. □

Note that the tensor N_J on the right-hand side in the formula of the theorem can be defined only at the points of $\mathcal{C} \subset E$ and then lifted to E as in remark 5. Since one easily describes complex structures on vector bundles (for trivial bundles see §1.3) we get the complete description of linear bundle almost complex structures.

2.5 Normal form of 1-jet of an almost complex structure

Consider the ideal of real-valued functions corresponding to the curve $\mathcal{C}$

$$\mu = \{f \in C^\infty(M^4) \,|\, f(\mathcal{C}) = 0\}.$$

Degrees of this ideal give the filtration μ^k on every $C^\infty(M)$-module, in particular we can talk about jets of tensor fields.

Theorem 11. *Let $\mathcal{C} \subset M^4$ be a pseudoholomorphic curve with respect to two almost complex structures J_1 and J_2 on M^4. Assume $H^1(\mathcal{C}; N_{\mathcal{C}}M) = 0$, i.e. the curve is a disk D_R, the entire line $\mathbb{C}$ or the sphere S^2 with trivial normal bundle $S^2 \cdot S^2 = 0$. If $J_1|_a = J_2|_a$ and $N_{J_1}|_a = N_{J_2}|_a$ at every point $a \in \mathcal{C}$ then the structures J_1 and J_2 are 1-equivalent, i.e. there exists a diffeomorphism ψ of a neighborhood $\mathcal{O}(\mathcal{C})$, preserving $\mathcal{C}$, such that*

$$J_2 = \psi^* J_1 \bmod \mu^2.$$

Proof. This statement is an analog of theorem 1 [K1]. Details of the construction can be found there. Here we present a short proof with the indication of differences.

By the hypothesis the desired diffeomorphism ψ has 1-symbol $\Phi^{(1)} = \mathrm{id} \in \tau^* \otimes \tau$ along the curve $\mathcal{C}$, where $\tau = TM|_{\mathcal{C}}$. Its 2-symbol $\Phi^{(2)} \in S^2\tau^* \otimes \tau$ in local coordinates is given by $\Phi^{(2)}(\xi,\eta)^i = \dfrac{\partial^2\psi^i}{\partial x^r \partial x^s}\xi^r\eta^s$. Moreover these symbols are compatible with the condition $\psi|_{\mathcal{C}} = \mathrm{id}$:

$$\xi,\eta \in T\mathcal{C} \Rightarrow \Phi^{(2)}(\xi,\eta) = 0. \tag{7}$$

The symbol satisfies the following condition:

$$dJ_1(\xi,\eta) + J_1 \circ \Phi^{(2)}(\xi,\eta) = \Phi^{(2)}(J_2\xi,\eta) + dJ_2(\xi,\eta),$$

where $d = d^{\nabla}$ is the differential w.r.t. some symmetric (and so not almost complex) curvature-free connection ∇.

Equivalently this means that the tensor

$$P(\xi,\eta) = J_1 \circ \Phi^{(2)}(\xi,\eta) - \Phi^{(2)}(J_2\xi,\eta) \tag{8}$$

satisfies the equation

$$P(\xi,\eta) = dJ_2(\xi,\eta) - dJ_1(\xi,\eta). \tag{9}$$

Now given J_1 and J_2 we have P and would like to find the corresponding Φ from (8). Form equality (9) we deduce

$$J_1 \circ P(\xi,\eta) = -P(J_2\xi,\eta),$$

which implies $P(\xi,\eta) = J_1B(\xi,\eta) - B(J_2\xi,\eta)$ for some $(2,1)$-tensor B. Now the identity

$$P(\xi,\eta) - P(\eta,\xi) = P(J_2\xi, J_2\eta) - P(J_2\eta, J_2\xi) \tag{10}$$

implies the possibility to satisfy equation (8) by the choice

$$\Phi^{(2)}(\xi,\eta) = \frac{1}{2}[B(\xi,\eta)+B(\eta,\xi)] - \frac{J}{4}[B(\xi,J\eta)+B(\eta,J\xi)-B(J\xi,\eta)-B(J\eta,\xi)].$$

Identity (10) means exactly $N_{J_1} = N_{J_2}$. Note that the tensor B can be chosen to satisfy (7) and all the constructions respect this condition ($T\mathcal{C}$ is J_k-invariant). So we get $\Phi^{(2)}$ satisfying (7) and (8).

Now the symbols Φ stand for the differential of the mapping ψ we seek for. Calculations above show the $\tau^* \otimes \tau$-valued 1-form generated by Φ is closed. Then the condition $H^1(\mathcal{C}; N_{\mathcal{C}}M) = 0$ imply that it is exact. Hence the symbols Φ are integrated to the diffeomorphism ψ we sought for. $\square$

Consider PH-sphere $\mathcal{C} = S^2 \subset M^4$, $\mathcal{C}\cdot\mathcal{C} = 0$. On the family of transversal disks D_φ, constructed in proposition 3, we can choose smooth global complex coordinate z so that z is complex w.r.t. the restriction $J|_{D_\varphi}$. So a neighborhood $\mathcal{O}$ of S^2 is represented as a product $D^2 \times S^2$ with coordinates (z,φ), φ taking values in $\bar{\mathbb{C}}$. Let us define a complex structure by the product-formula $J_0 = J_v \times J_h$, with the vertical part J_v induced by the projection along the spheres $S^2 = \{z = \mathrm{const}\}$ from the structure $J|_{D_\varphi}$ and the horizontal part J_h induced by the projection along the disks D_φ from the structure $J|_{\mathcal{C}}$.

Corollary 12. *Almost complex structure J in a neighborhood $\mathcal{O} \subset M^4$ of PH-sphere $\mathcal{C} = S^2$ is 1-equivalent to the almost complex structure $J' = J_0 + \frac{1}{2}J_0 N_J(r, \cdot) + \dots$, where N_J is the field of the Nijenhuis tensors along $\mathcal{C}$. In other words there exists a local diffeomorphism ψ, $\psi|_{\mathcal{C}} = \mathrm{id}$, such that*

$$\psi^* J\xi = J_0\xi + \frac{1}{2}J_0 N_J(r_a, \xi) \operatorname{mod} \mu^2$$

for any vector $\xi \in T_a M$, $a \in \mathcal{O}$.

Proof. The values of the almost complex structure J and the almost complex $\operatorname{mod} \mu^2$ structure $\tilde{J} = J_0 + \frac{1}{2}J_0 N_J(r_a, \cdot) + \dots$ on the right-hand side of the formula are equal at the points of the sphere S^2. Let us show that the same is true for their Nijenhuis tensors. Namely we show that the Nijenhuis tensor $N_{\tilde{J}}$ calculated according to definition (1) is equal to the prescribed Nijenhuis tensor N_J at all points S^2.

The calculations are local and it is sufficient to consider the value of $N_{\tilde{J}}$ on two complex independent vectors, say on ∂_x and ∂_{φ_1}, where $\varphi = \varphi_1 + i\varphi_2$ and $z = x + iy$ are local coordinates defining J_0. Let us denote by $\doteq$ the equality $\operatorname{mod} \mu$. We have:

$$\begin{aligned} N_{\tilde{J}}(\partial_x, \partial_{\varphi_1}) &\doteq [J_0\partial_x, \tilde{J}\partial_{\varphi_1}] - J_0[\partial_x, \tilde{J}\partial_{\varphi_1}] - J_0[J_0\partial_x, \partial_{\varphi_1}] - [\partial_x, \partial_{\varphi_1}] \\ &\doteq [\partial_y, \partial_{\varphi_2} + \tfrac{1}{2}J_0 N_J(x\partial_x + y\partial_y, \partial_{\varphi_1})] \\ &\quad - J_0[\partial_x, \partial_{\varphi_2} + \tfrac{1}{2}J_0 N_J(x\partial_x + y\partial_y, \partial_{\varphi_1})] \\ &\doteq \tfrac{1}{2}J_0 N_J(\partial_y, \partial_{\varphi_1}) + \tfrac{1}{2}N_J(\partial_x, \partial_{\varphi_1}) \doteq N_J(\partial_x, \partial_{\varphi_1}). \end{aligned}$$

Thus $N_{\tilde{J}} = N_J$ along S^2 and the claim is proved. □

Remark 6. *Another way to construct a complex structure J_0' in a neighborhood of PH-sphere $\mathcal{C} = S^2 \subset M^4$ with trivial normal bundle ($\mathcal{C} \cdot \mathcal{C} = 0$) is to foliate this neighborhood by homologous PH-spheres ([M2]) and to induce the vertical part J_v from some particular transversal disks D_{φ_0}.*

Consider now arbitrary PH-curve $\mathcal{C}$ with trivial normal bundle $\mathcal{C} \cdot \mathcal{C} = 0$. Similarly to the proposition 3 we find coordinates (z, φ) in a neighborhood of $\mathcal{C} \subset M$, where φ is multivalued and $\{\varphi = \text{const}\}$ is a family of transversal PH-disks D_φ. D_φ can be equipped with vector space structure and so for every $a \in D_\varphi \subset M^4$ we set $r \in T_a D_\varphi$ be the radius vector $\vec{0z}$ attached at the point a. Here we use the canonical identification $T_a F \simeq F$ for the vector spaces.

Theorem 10 expresses the linear bundle almost complex structure in terms of some complex structure and a Nijenhuis tensor. When almost complex structure is arbitrary then the conclusions of proposition 7 and theorem 1 are wrong (the formula of theorem 1 can give the operator with $J^2 \neq -\mathbf{1}$). However on the level of 1-jet the theorem remains true.

Theorem 13. *Let $\mathcal{C} \subset (M^4, J)$ be a PH-curve and $N_J \in \Lambda^2\tau^* \otimes \tau$ be the field of Nijenhuis tensors of J along $\mathcal{C}$, where $\tau = TM|_{\mathcal{C}}$. Assume the Nijenhuis* *tensor characteristic distribution* Π^2 *is transversal everywhere to* $\mathcal{C}$*. Then for some complex structure* J_0 *in a neighborhood of* $\mathcal{C}$ *we have:*

$$J\xi = J_0\xi + \frac{1}{2}J_0 N_J(r,\xi) \mod \mu^2. \tag{11}$$

Proof. We can assume the disks D_φ have Π^2 as tangent planes at the points of $\mathcal{C}$. We use the symbol $\doteq$ as in the proof of the corollary above. First note that formula (11) implies $J \doteq J_0$ for vertical vectors (belonging to TD_φ). Thus we define $J_0 \doteq J - \frac{1}{2}JN_J(r,\xi)$. Similarly to the proof of theorem 10 it is proved that $J_0^2 \doteq -\mathbf{1}$ and that $N_{J_0} \doteq 0$. Thus J_0 is a complex structure $\mod \mu^2$ [K1] and we can take instead of it any *complex* structure $\tilde{J}_0 = J_0 \mod \mu^2$. $\square$

In this theorem J_0 is some complex structure while in the corollary 12 we had some definite complex structure. In the case $C = T^2$ we can also assume the complex structure J_0 is $J_{(\lambda)}$ from proposition 4.

Actually, if the complex structure J_0 in the torus neighborhood has normal bundle determined by the normal nonresonant pair (λ,ν), then we can find coordinates (z,φ) in a neighborhood with the gluing rule (3) in which J_0 is standard. In these coordinates

$$J_\alpha^\gamma = \delta_\alpha^{\gamma+n} + \frac{1}{2}N_{\alpha\beta}^{\gamma+n}r^\beta + \bar{\bar{o}}(|r|), \tag{12}$$

with $z = r^2 + ir^4$ and indices $1,3$ being used for the real and imaginary part of $\varphi = r^1 + ir^3$. Here $N_{\alpha\beta}^\gamma$ are the components of the Nijenhuis tensor. We assume every object changes its sign when one index is changed by 4, e.g. $N_{\alpha\beta}^{\gamma+4} = -N_{\alpha\beta}^\gamma$. The summation index β is assumed to take only values 2,4.

Remark 7. *Similar statement holds also for 1-jet of almost complex structure at a point. It says that for some coordinate system* (x^i) *near the point the operator* J *has components*

$$J_\alpha^\gamma = \delta_\alpha^{\gamma+n} + \frac{1}{4}N_{\alpha\beta}^{\gamma+n}x^\beta + \bar{\bar{o}}(|x|), \tag{13}$$

where $N_{\alpha\beta}^\gamma$ *are components of the Nijenhuis tensor and we assume* $2n$*-twisted periodicity, i.e.* $x^{i+2n} = -x^i$ *etc.*

This formula is obtained by studying the structural function (Weyl tensor) of the corresponding geometric structure ([K1],[KL]). The difference of the factors (1/2 instead of 1/4) in the formulas is connected with the fact that we consider different coordinate systems: in (12) we consider the radial vector field r *which connects the point to its projection on the torus and in (13) we use the usual radius-vector directing from a fixed origin. Such formulas for* $J \mod \mu^2$ *can be considered along any PH-submanifolds of* (M^{2n},J) *but generically there are only 0- and 2-dimensional PH-submanifolds.*

3 Curves in a neighborhood of a PH-torus

3.1 Locally foliating families of cylinders

In this section we consider only pseudoholomorphic tori $\mathcal{C} = T^2$. Results of section 1 imply there are moduli of almost complex structures on a germ of $\mathcal{C}$ contrary to the complex case in which Arnold found a normal form of a neighborhood of an elliptic curve $T^2 \subset M^4$. To get some analogs of the complex situation we consider foliations of a neighborhood of T^2 by pseudoholomorphic curves. Generically there are no compact curves in a neighborhood and PH-tori appear discretely because index of the corresponding linearized Cauchy-Riemann operator is zero ([Ku2]). So one should consider noncompact curves. We begin with PH-lines. Let $\|\cdot\|$ be some fixed norm.

Proposition 14. *There are two neighborhoods $\mathcal{O}' \subset \mathcal{O}$ of $T^2 \subset M^4$, a change of almost complex structure in $\mathcal{O} \setminus \mathcal{O}'$ and a number $C > 0$ with the following properties. For every $R > 0$ there exists a smooth family of PH-disks $f_\alpha : D_R \to \mathcal{O}$ with uniformly bounded norms $\|(f_\alpha)_*(z)\| \le C$ and $\|(f_\alpha)_*(0)\| = 1$. Moreover this family fills some smaller neighborhood $\mathcal{O}'' \subset \mathcal{O}'$ of T^2, i.e.*

$$\mathcal{O}'' \subset \cup_\alpha f_\alpha(D_R).$$

Proof. Let us take the universal covering $\hat{\mathcal{O}} \simeq \mathbb{C} \times D^2$ of $\mathcal{O}$. The torus is covered by the entire line $\mathbb{C} \to T^2$. Changing the structure J at infinity in $\hat{\mathcal{O}}$ and near the boundary to the integrable one we glue the manifold to the product $S^2 \times S^2$ with the line $\mathbb{C}$ being glued to the first factor S^2_1. Then the introduction of the taming symplectic product-structure $\omega = \omega_1 \oplus \omega_2$ yields a foliation of $S^2_1 \times S^2_2$ by PH-spheres S^2 in the homology class of the first factor if we additionally demand $\omega_1(S^2_1) < \omega_2(S^2_2)$, i.e. the homology class $[S^2_1]$ of the first sphere-factor is symplectically simple. Here we use the fact that the dimension is 4: due to positivity of intersections [M1] we actually have a foliation ([M2]).

This foliation of $S^2 \times S^2$ gives a family of big PH-disks on $\hat{\mathcal{O}}$ parametrized by the radius ρ of disk in $\mathbb{C}$ out of which the almost complex structure is changed. To get estimates we use Brody reparametrization lemma as in [KO]. The filling property is given by the choice of pertrubation of the structure in $\mathcal{O} \setminus \mathcal{O}'$: This is simple if we require the boundary of $\mathcal{O}$ to be J-pseudoconvex ([G] 2.4.D). $\square$

We now consider filling by pseudoholomorphic cylinders $\mathcal{C}_R = [-R; R] \times S^1$. They can be considered also as annuli $\mathcal{C}_R \subset \mathbb{C} \setminus \{0\}$, but the next construction differs from the previous because the cylinders are now non-contractible in $\mathcal{O}$ (Fig.1).

Proposition 15. *In the statement of proposition 14 we can change disks D_R to $\mathcal{C}_R$ and get for every $R > 0$ a family of PH-cylinders $f_\alpha : \mathcal{C}_R \to \mathcal{O}$ with uniformly bounded norms and normalization $\|(f_\alpha)_*(0)\| = 1$. In addition this family is filling:*

$$\mathcal{O}'' \subset \cup_\alpha f_\alpha(\mathcal{C}_R).$$

Figure 1. Filling by PH-cylinders

Proof. Actually take a covering of the neighborhood $\mathcal{O}$ which corresponds to one cycle of the torus. The torus is covered by the entire cylinder $\hat{\mathcal{C}} \to T^2$. We can change the almost complex structure J at infinity so that it makes possible to "pinch" each end of the cylinder. This means we perturb the structure J so that it is standard integrable outside some $\mathcal{C}_{R_2} \subset \hat{\mathcal{C}}$ and the support is also a big cylinder $\mathcal{C}_{R_1}$. Then we glue the ends to the disks. This operation gives us a sphere S^2 instead of the cylinder $\hat{\mathcal{C}} = \mathbb{R} \times S^1$. We can also assume that neighborhoods of two cylinder ends are pinched (Fig.2).

Figure 2. Cutting and Gluing

Thus we have a neighborhood U of the sphere S^2_0. It is foliated by PH-spheres close to S^2_0. Actually, we can change the structure J near the boundary of this neighborhood, glue and get the manifold-product $\hat{M} = S^2 \times S^2$. As before it is foliated by PH-spheres. Thus U is foliated by PH-spheres and in the preimage they give a PH-foliation by cylinders. $\square$

Remark 8. *Consider the infinite cylinder $\hat{\mathcal{C}} = \mathcal{C}_\infty$. One can prove a more general statement: There exists a change of the structure J in $\mathcal{O} \setminus \mathcal{O}'$ such that $\mathcal{O}'$ is filled with images of entire PH-cylinders $f_\alpha : \hat{\mathcal{C}} \to \mathcal{O}$: $\mathcal{O}' \subset \cup_\alpha f_\alpha(\hat{\mathcal{C}})$. The technical difficulty is however that we need reparametrization in the sequence of finite PH-cylinders passing the given point and the limiting curve can shift and not pass through the point (I am indebted to V.Bangert for a discussion about it). To achieve filling one needs to control the sequence. We will consider the problem elsewhere.*

The positivity of intersections theory ([M1]) is applicable only to closed submanifolds and it cannot guarantee that leaves of the considered filling do not intersect. If there are no intersections of the leaves with close values of parameters and there are no self-intersections we call the family *locally foliating*.

Recall that PH-torus T^2 is parametrized by periodic coordinate $z \in \mathbb{C}$, i.e. the map $f_0 : \mathbb{C} \to T^2$ satisfies $f_0(z) = f_0(z + 2\pi) = f_0(z + \nu)$, where $2\pi, \nu$ are periods. We parametrize f_α by the condition $f_\alpha(z) = f_\alpha(z + 2\pi)$.

Fix some transversal PH-disk D_{φ_0} to the torus T^2. Consider the return map $\Psi : D_{\varphi_0} \to D_{\varphi_0}$ for our family of surfaces f_α. This is defined as follows: if a point $z \in D_{\varphi_0}$ belongs to f_α, $z = f_\alpha(\varphi)$ we set $\Psi(z) = z'$ with $z' = f_\alpha(\varphi + \sigma(\varphi))$, $\sigma(\varphi) \approx \nu$, chosen to belong to D_{φ_0} again. This return map Ψ is determined non-uniquely since several cylinders of the family f_α can pass through z.

Proposition 16. *Let the return map be either attracting or repelling, $|\Psi(z)/z| \in (0, \varepsilon)$ or $(\varepsilon^{-1}, \infty)$ for all $z \in D_{\varphi_0}$ close to zero and some fixed $\varepsilon < 1$ (we assume the inequality holds notwithstanding the nonuniqueness of Ψ). Then the constructed family $f_\alpha : \hat{\mathcal{C}} \to \mathcal{O}$ is locally foliating in a neighborhood of T^2.*

Proof. The condition that Ψ is attracting or repelling means that there are no self-intersections of the leaves $f_\alpha(\hat{\mathcal{C}})$. If a leaf $f_\alpha(\hat{\mathcal{C}})$ transversally intersect another $f_\beta(\hat{\mathcal{C}})$ with $\alpha \approx \beta$ then because of the convergence $f_\alpha(\mathcal{C}_R) \to f_\alpha(\hat{\mathcal{C}})$ there is an intersection of the spheres $S^2 \subset \hat{M}$ from which cylinders are constructed. This is certainly impossible because our family of spheres is foliating. If the leaves $f_\alpha(\hat{\mathcal{C}})$ and $f_\beta(\hat{\mathcal{C}})$ are tangent let us consider the first order of their jets which are different. This is the next number after the tangency order. Since the maps of finite cylinders $\mathcal{C}_R$ tends to the maps of $\hat{\mathcal{C}}$ in C^∞-topology the spheres in the manifold $\hat{M}$ do intersect. Every such intersection even nontransversal contributes positively to the intersection number ([M1]) which contradicts $[S^2] \cdot [S^2] = 0$. It remains to note that if two pseudoholomorphic curves in an almost complex manifold have the infinite tangency they must coincide ([MS]). $\square$

For arbitrary almost complex structures it is not clear if there are foliations of the T^2-neighborhood by cylinders. The following can be probably solved using a method similar to the Moser's proof [Mo] of PH-lines foliation persistence.

Question: Let T^2 be normal nonresonant elliptic curve. Let J be a small almost complex perturbation of the complex structure J_0. The curve T^2 will be perturbed into close PH-curve $\mathcal{C}$. Is it true that a neighborhood $\mathcal{O}(\mathcal{C})$ is foliated by PH-cylinders?

3.2 Floquet theory for the PH-foliations

Holomorphic bundle (3) possesses a distinguished foliation $\{z = \text{const}\}$. We call this foliation the *foliation by λ-twisted cylinders*. Arnold's "Floquet-type" result in [A1]§27 implies that a neighborhood of every elliptic curve has a foliation by twisted cylinders which is biholomorphic to the standard one. Here we study PH-foliations.

Let $f_\alpha : \hat{\mathcal{B}} \to \mathcal{O}$ be a foliating family of a neighborhood of a PH-curve $\mathcal{C}$ with trivial self-intersection. Let $D_\varphi = \{\varphi = \text{const}\}$ be a family of normal disks from proposition 3. Then every path $\gamma(t)$ on $\mathcal{C}$ with $\gamma(0) = \varphi_0$, $\gamma(1) = \varphi_1$ gives a mapping $\Phi_\gamma : D_{\varphi_0} \to D_{\varphi_1}$ of shift along the leaves of f_α. For a loop γ we have an automorphism of D_φ. Since f_α is foliation there is no local holonomy: $\Phi_\gamma = \text{id}$ for contractible loops γ. Thus we can consider the map $\pi_1(\mathcal{C}) \to \text{Aut}(D_\varphi)$.

Definition 3. *We call $\Phi_\gamma \in \text{Aut}(D_\varphi)$ the monodromy map along $\gamma \in \pi_1(\mathcal{C})$.*

For example there is no monodromy for the sphere $\mathcal{C} = S^2$ and each choice of local coordinates in a normal disk D_{φ_0} gives coordinates for the others D_φ.

Let now $\mathcal{C} = T^2(2\pi, \nu)$. Since our foliating family f_α consists of cylinders there is no monodromy along one generating cycle, let along the cycle $\varphi \mapsto \varphi + 2\pi$. Denote by Φ_ν the monodromy along the other cycle $\varphi \mapsto \varphi + \nu$.

Let $f_\alpha : \hat{\mathcal{C}} \to \mathcal{O}$ be a foliating family of PH-cylinders in a neighborhood $\mathcal{O}' \subset \mathcal{O}$ of the PH-torus T^2. Assume the monodromy of the family f_α is

$$\Phi_\nu(z) = \lambda z + \bar{\bar{o}}(|z|). \tag{14}$$

Recall [A1] that $\lambda \in \mathbb{C} \setminus \{0\}$ is of (C, σ)-type if $|\lambda^m - 1| \geq C/m^{1+\sigma}$ for all $m \in \mathbb{Z}_{>1}$ (in particular such are all nonzero from Poincaré domain $|\lambda| \neq 1$).

Theorem 17. *Let the monodromy Φ_ν be a germ of biholomorphic mapping with the number λ (14) of (C, σ)-type. Then the foliating family of cylinders f_α is similar to λ-twisted foliation in the following sense: There exist coordinates (z, φ) with the gluing rule (3) such that transversal disks $D_\varphi = \{\varphi = \text{const}\}$ are pseudoholomorphic, φ is a complex coordinate on the curve $\mathcal{C}$, z is a complex coordinate on one D_{φ_0} and the foliation f_α is given by $\{z = \text{const}\}$.*

Proof. A coordinate system z on a transversal disk D_{φ_0} provides coordinates on all others D_φ. These coordinates are multivalued because rotation along the second cycle $\varphi \mapsto \varphi + \nu$ yields the monodromy mapping $z \mapsto \psi(z) = \lambda z + \bar{\bar{o}}(|z|)$.

Now we apply Poncaré-Siegel theorem ([A1]) in complex dimension 1 which states that the monodromy map is conjugate to the map $\psi(z) = \lambda z$. □

3.3 Monodromy and transports

Unlike complex case almost complex monodromy can be non-holomorphic mapping of the fibers. Actually there are simple examples of PH-foliations with any prescribed monodromy.

Moreover even if the monodromy is complex (as required in theorem 17) the transport maps $\Phi_\gamma : (D_{\varphi_0}, J) \to (D_{\varphi_1}, J)$ can be not. In fact this is the occasion of codim $= \infty$.

Proposition 18. *Let $\mathcal{C} \subset M$ be a PH-curve in a 4-dimensional manifold and let $f_\alpha : P \to \mathcal{O}$ be a local PH-foliating family of some neighborhood $\mathcal{O}(\mathcal{C})$. Let also D_φ be a transversal PH-foliation and Φ_γ be the corresponding transport map $D_{\varphi_0} \to D_{\varphi_1}$ along curves $\gamma(t) \subset \mathcal{C}$, $\gamma(0) = \varphi_0$, $\gamma(1) = \varphi_1$. If all Φ_γ are holomorphic then the Nijenhuis tensor characteristic distribution Π^2 is tangent to the leaves of f_α.*

Proof. Actually the foliation provides a local bundle $\pi : \mathcal{O} \to D_\varphi$. The hypothesys that all transports are complex is equivalent to the claim that the bundle π with fibers $f_\alpha(P)$ is almost complex. Therefore the statement follows from proposition 7. □

Remark 9. *If the distribution Π^2 is not integrable the transports are not complex. But even integrability is not sufficient, see §2.2.*

3.4 PH-tori deformation problem

Generically holomorphic 2-torus in a complex manifold cannot be deformed ([A1]). The same situation is also with PH-tori in almost complex manifolds. This follows from vanishing of the index of the linearized Cauchy-Riemann operator ([Ku1]). By the deformation we mean existence of close homologous PH-torus of the same periods $T^2 = T^2(2\pi, \nu)$. In this section we consider some examples where we can make the condition of non-existence explicit.

$\underline{\mathbf{1}}^\circ$) Let us consider linear bundle almost complex structure J on the bundle $E \to T^2$. There exist coordinates (z, φ) with the gluing rule (3) such that

$$\begin{cases} J\partial_x = \partial_y, & J\partial_{\varphi_1} = \partial_{\varphi_2} + x \cdot v - y \cdot Jv, \\ J\partial_y = -\partial_x, & J\partial_{\varphi_2} = -\partial_{\varphi_1} - x \cdot Jv - y \cdot v. \end{cases}$$

This formula follows from theorem 10 and the coordinates are determined by J_0. Vector field $v = \frac{1}{2} J N_J(\partial_x, \partial_{\varphi_1})$ can be decomposed $v = \alpha\partial_x + \beta\partial_y$ with $\alpha = \alpha(\varphi)$, $\beta = \beta(\varphi)$. The complexified vector bundle is decomposed $T_{\mathbb{C}}E = E_+ + E_-$, where $E_\pm = \{\xi \mid J\xi = \pm i\xi\}$; $E_- = \bar{E}_+$. Vectors

$$U_1 = \partial_\varphi - z\bar{b}\,\partial_{\bar{z}}, \quad U_2 = \partial_z,$$

form a basis of E_+. Here $\partial_\varphi = \frac{1}{2}(\partial_{\varphi_1} - i\partial_{\varphi_2})$, $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\bar{b} = \dfrac{\beta + i\alpha}{2}$. Thus the basis of E_+^* in the decomposition $T_{\mathbb{C}}^*E = E_+^* + E_-^*$ $(E_-^* = \bar{E}_+^*)$ is

$$\omega_1 = d\varphi, \quad \omega_2 = dz + \bar{z}b\,d\bar{\varphi}.$$

Now every pseudoholomorphic torus in E homologous to the zero section T^2 is of the form $f(T^2)$ for some section f of the bundle $E \to T^2$. This is to say each PH-torus in E has unique transversal intersection with every fiber. Actually we may compactify the fibers of the bundle to the spheres and the claim follows from the positivity of intersections (or even simpler by studying the degree of the projection of this torus to the torus-base).

Let us deduce the equation for f. The curve $f(T^2)$ is pseudoholomorphic iff

$$\omega_2|_{z=f(\varphi)} = c \cdot \omega_1.$$

Substituting $df = f_\varphi d\varphi + f_{\bar\varphi} d\bar\varphi$ we have:

$$f_{\bar\varphi} + b\bar f = 0. \tag{15}$$

Theorem 19. *Let J be a linear bundle almost complex structure and J_0 be the corresponding complex structure from the decomposition of theorem 10. Suppose the number λ, determined by the complex structure J_0 in the bundle E via* (3)*, is of unit length: $|\lambda| = 1$. Assume also that the function $\Lambda \in C^\infty(T^2)$, determined uniquely by the equation $\frac{1}{2} JN_J(\partial_z, \partial_\varphi) = \Lambda \partial_{\bar z}$, is nonzero holomorphic function: $\partial_{\bar\varphi}\Lambda = 0$, $\Lambda \neq 0$. Then zero section T^2 is the unique PH-torus in E.*

Proof. First note that since

$$\begin{array}{ll} \frac{1}{2} JN_J(\partial_z, \partial_\varphi) = (\alpha - i\beta)\partial_{\bar z}, & \frac{1}{2} JN_J(\partial_{\bar z}, \partial_\varphi) = 0, \\ \frac{1}{2} JN_J(\partial_z, \partial_{\bar\varphi}) = 0, & \frac{1}{2} JN_J(\partial_{\bar z}, \partial_{\bar\varphi}) = (\alpha + i\beta)\partial_z, \end{array} \tag{16}$$

we have $\Lambda = -2i\bar b$. Thus $\Lambda_{\bar\varphi} = 0 \Leftrightarrow \bar\Lambda_\varphi = 0 \Leftrightarrow b_\varphi = 0$.

Let us show the equation (15) has no nonzero solutions. Complex Laplacian of f equals $f_{\bar\varphi\varphi} = -b\bar f_\varphi = -b\overline{f_{\bar\varphi}} = |b|^2 f$. Our torus neighborhood is the trivial cylinder $\mathcal{C}^2 = \{\varphi \in \mathbb{C} \mid \operatorname{Im}\varphi \in [0, \operatorname{Im}\nu)\}/2\pi\mathbb{Z}$ neighborhood glued by the rule $(z, \varphi) \mapsto (\lambda z, \varphi + \nu)$. Thus when $\varphi \mapsto \varphi + \nu$ we have: $f \mapsto \lambda f$. So integrating over the cylinder gives:

$$\begin{aligned} \int_{\mathcal{C}^2} (f_{\bar\varphi\varphi}\bar f + f_{\bar\varphi}\overline{f_{\bar\varphi}}) \frac{i}{2} d\varphi \wedge d\bar\varphi \ &= \int_{\mathcal{C}^2} \frac{\partial}{\partial\varphi}(f_{\bar\varphi}\bar f) \frac{i}{2} d\varphi \wedge d\bar\varphi = \int_{\mathcal{C}^2} \frac{i}{2} d(f_{\bar\varphi}\bar f d\bar\varphi) = \\ &= (\lambda\bar\lambda - 1) \oint_{S^1} \frac{i}{2} f_{\bar\varphi}\bar f d\bar\varphi = 0. \end{aligned}$$

So using the calculation with the Laplacian we have:

$$\int_{\mathcal{C}^2} (|b|^2|f|^2 + |f_{\bar\varphi}|^2) d\varphi_1 \wedge d\varphi_2 = 0. \tag{17}$$

Therefore since $|b| \neq 0$ we get $f = 0$. Thus there are no homologous to the zero section PH-tori $\tilde T^2$ with $f \neq 0$. If the homology class of $\tilde T^2$ is a multiple of the zero section $[\tilde T^2] = k[T^2]$ a k-finite covering finishes the proof. $\square$

Remark 10. *If $|b| = 0$, i.e. almost complex structure J is integrable $J = J_0$, equality (17) implies that f is holomorphic section. Thus if $\lambda^n \neq 1$ we get again $f = 0$ comparing the Fourier coefficients of f.*

2°) Consider a general almost complex structure J with Nijenhuis tensor characteristic distribution Π^2 transversal to some PH-torus T^2. The linearized equation for close PH-tori can be written in the form

$$f_{\bar{\varphi}} + af + b\bar{f} = 0. \tag{18}$$

Actually, the linearization does not depend on a change of the structure J by second order quantities. Thus we can perturb J to make the distribution Π^2 integrable in $\mathcal{O} \supset T^2$. This new almost complex structure is given by the formula (4).

Let's write the equation for close PH-tori. The basis of E_+ is

$$U_1 = \partial_\varphi + \frac{\bar{A}}{A+2i}\partial_{\bar{\varphi}} + \frac{\bar{B}}{A+2i}\partial_{\bar{z}}, \quad U_2 = \partial_z,$$

where $A = A_1 + iA_2$, $B = B_1 + iB_2$. The corresponding basis of E_+^* is

$$\omega_1 = d\varphi - \frac{A}{\bar{A}-2i}d\bar{\varphi}, \quad \omega_2 = dz + \frac{1}{4}\frac{\bar{A}B}{A_2+1}d\varphi - \frac{1}{4}\frac{(A+2i)B}{A_2+1}d\bar{\varphi}.$$

So the equation $\left.(\omega_2 - c\cdot\omega_1)\right|_{z=f(\varphi)} = 0$ implies the required equation on f:

$$f_{\bar{\varphi}} + \frac{A}{\bar{A}-2i}f_\varphi - \frac{B}{\bar{A}-2i} = 0. \tag{19}$$

Denote by A^0 and B^0 linearizations by fiber coordinate of the functions A and B respectively. Note that linearization of equation (5) implies that A^0 is holomorphic w.r.t. z, that is we can bring our equations to the constant A^0.

Since $\left.A\right|_{z=0} = 0$, $\left.B\right|_{z=0} = 0$, linearization of equation (19) is

$$f_{\bar{\varphi}} - \frac{i}{2}B^0(f) = 0,$$

which has the form (18) if we set $-\frac{i}{2}B^0 = az + b\bar{z}$.

Since we have equations (5) the Nijenhuis tensor of (4) is

$$\begin{aligned} N_J(\partial_x, \partial_{\varphi_1}) = &\left(\frac{\partial B_1}{\partial y} - \frac{\partial A_1}{\partial x}B_1 + \frac{\partial A_2}{\partial x}\frac{A_1B_1 - B_2}{A_2+1} + \frac{\partial B_2}{\partial x}\right)\partial_x \\ &+ \left(\frac{\partial B_2}{\partial y} - \frac{\partial A_1}{\partial x}B_2 + \frac{\partial A_2}{\partial x}\frac{A_1B_2 + B_1}{A_2+1} - \frac{\partial B_1}{\partial x}\right)\partial_y. \end{aligned}$$

Therefore linearizing A and B we conclude that its values on T^2 are given by the formula (see also (16))

$$\frac{1}{2}JN_J(\partial_z, \partial_\varphi) = -2i\bar{b}\,\partial_{\bar{z}}.$$

Remark 11. *Since the only invariant of 1-jet of J on a PH-curve is the Nijenhuis tensor, which we expressed by the function $b(\varphi)$, we can bring the function $a(\varphi)$ in (18) to the simplest form. Namely we can introduce coordinates (φ, z) using the complex structure J_0 of theorem 13. This gives $a = 0$ for normal coordinate z on the torus with gluing rule (3). Alternatively we can have global well-defined coordinate z but then $a = \mathrm{const}$. This proves a suggestion on p. 430 [Mo] that* "the linearized equation can be brought into the form (18) with $a = \mathrm{const}$".

Theorem 20. *Let almost complex structure J in a neighborhood of PH-curve T^2 be described by formula (11) with complex structure J_0 having $|\lambda| = 1$. If the characteristic distribution Π^2 is transversal to T^2 and for linearized structure $b(\varphi)$ is anti-holomorphic, then the curve T^2 is isolated and persistent.*

Proof. Actually as Moser [Mo] noticed if the linearized equation $f_{\bar{\varphi}} + af + b\bar{f} = g$ has a unique solution for any $g \in C^\infty(T^2; \mathbb{C})$ then the torus is isolated and persistent. But the linearization we studied in theorem 19. $\square$

<u>**3**</u>°) Note that in holomorphic bundle with $\lambda^n = 1$ one can find tori $f(T^2)$ with $f \neq 0$ of the type $T^2(2\pi kn, \nu l)$, which cover zero section torus $T^2 = T^2(2\pi, \nu)$. But in a fixed homology class all PH-tori are of the same holomorphic type:

Lemma 21. *Let $\tilde{T}_1^2, \tilde{T}_2^2 \subset E$ be two PH-tori in a linear almost complex bundle $E \to T^2$. If they are homologous then they are biholomorphic.*

Proof. First consider tori in the homology class of the zero section T^2. As was shown before theorem 19 the projection is a diffeomorphism. Since in linear almost complex bundles $E \to T^2$ the projection is an almost complex mapping, its restriction is a biholomorphism of the tori: $\tilde{T}_1^2 \simeq T^2 \simeq \tilde{T}_2^2$. The general case $[\tilde{T}_i^2] = k[T^2]$ follows from the case $k = 1$ by means of a k-covering. $\square$

If we do not demand the bundle condition the opposite situation can occur: in example [A1]§27 a neighborhood of the torus is foliated by holomorphic tori of different holomorphic type. Similar situation occurs also in almost complex case and invariants of section 1 can be nontrivial:

Example. Consider a foliation $f_\alpha : T^2 \to \mathcal{O}$ of 4-dim neighborhood of some torus. Introduce the structure J in horizontal directions so that all the tori T_α^2 are pseudoholomorphic but nonequivalent (the parameter ν is changing). Choose the structure J on the normals D_φ so that the transports Φ_γ are not holomorphic (§3.3). Define J globally by the product formula. Then the distribution Π^2 is nonintegrable and we get the distribution $L^1 = \Pi^3 \cap T(T^2)$ (possibly with singularities).

<u>**4**</u>°) Note that the tori in the same homology class can occur both in families and discretely.

Example. Let (T^4, J_0) be the standard complex torus, i.e. quotient of $\mathbb{C}^2$ by the lattice $\mathbb{Z}^4$. Consider a PH-torus $T_0^2 \subset T^4$. It is possible to perturb J_0 in a neighborhood $\mathcal{O}$ of T_0^2 so that the new structure J is isomorphic to the model structure near the torus $T^2(2\pi, \nu)$ given by (3) in a neighborhood $\mathcal{O}' \subset \mathcal{O}$ and $J = J_0$ outside $\mathcal{O}$. Then there is 2-parametric family of PH-tori of $[T_0^2]$-homology class outside $\mathcal{O}$ and a unique PH-torus T_0^2 inside $\mathcal{O}'$.

$\underline{\mathbf{5}}°$) Note that for PH-torus $\mathcal{C} = T^2 \subset (M^4, J)$ with $N_J|_a = 0$ for all $a \in T^2$ the normal bundle $N_{\mathcal{C}}M$ is holomorphic.

Proposition 22. *If the Nijenhuis tensor vanishes along a PH-torus and the pair (λ, ν), characterizing the holomorphic bundle $N_{\mathcal{C}}M$, is nonresonant then small neighborhood $\mathcal{O}$ of this torus cannot be foliated by $T^2(2\pi, \nu)$-tori.*

Proof. Actually if there is a PH-foliation by tori then the linearization of this foliation determines a holomorphic foliation of the normal bundle which is impossible by [A1]§27. □

$\underline{\mathbf{6}}°$) An intermediate condition on foliation between holomorphic and pseudoholomorphic is that it be pseudoholomorphic with complex transports. Proposition 18 implies

Proposition 23. *If the distribution Π^2 in a neighborhood $\mathcal{O} \supset T^2$ is not integrable or is integrable with noncompact leaves, then $\mathcal{O}$ cannot be foliated by PH-tori with complex transports.*

Note that in the considerations above we needed to fix a holomorphic structures on the tori sought for. The last proposition does not require this.

A Normal bundle in Riemannian geometry.

The construction of structure on the normal to the submanifold bundle from §2.1 can be carried out for some other cases.

For example in Riemannian geometry there exists a unique Riemannian connection ∇. This Levi-Civita connection splits the normal bundle $N_L M$ of any submanifold $i : L \subset M$ with induced Riemannian metric $g_L = i^*(g_M)$. However the uniqueness of the connection ∇ makes the situation more rigid and for our construction we should require that L is a totally geodesic submanifold.

So we construct the metric in $N_L M$ and then again ask about relations between two tensors: Riemannian curvature R_g of the manifold M at points of $L \subset M$ and the curvature $\hat{R}_g$ for the total space of $N_L M$ at zero section $L \subset N_L M$. In general there are no relations, but for some parts of the tensors there is.

Namely consider the curvature of the normal bundle $R^\perp$, which is the curvature tensor of the normal connection $\nabla^\perp$ given by the orthogonal decomposition in $TM|_L = TL \oplus N_L M$, $W = W_\parallel + W_\perp$. Note that $R^\perp(X, Y) = \hat{R}_g(X, Y)$ for $X, Y \in TL$ and the left-hand side is not defined for others X, Y.

Let $\mathrm{II} : TL \otimes TL \to N_L M$ be the second quadratic form of L and $A : TL \otimes N_L M \to TL$ be the shape (Peterson) operator given by $g(A(X,V),Y) = g(\mathrm{II}(X,Y),V)$, $X,Y \in TL$, $V \in N_L M$. Then if we denote R^L the curvature of the Levi-Civita connection of L we have the following famous equations:

$$[R_g(X,Y)Z]_{\|} = R^L(X,Y)Z + A(Y,\mathrm{II}(X,Z)) - A(X,\mathrm{II}(Y,Z)) \qquad \text{(Gauss eq.)},$$
$$[R_g(X,Y)Z]_{\perp} = (\nabla_Y \mathrm{II})(X,Z) - (\nabla_X \mathrm{II})(Y,Z) \qquad \text{(Codazzi-Maynardi eq.)},$$
$$[R_g(X,Y)V]_{\perp} = R^{\perp}(X,Y)V + \mathrm{II}(X,A(Y,V)) - \mathrm{II}(Y,A(X,V)) \qquad \text{(Ricci eq.)},$$

where $X,Y,Z \in TL$, $V \in N_L M$.

In particular when L is totally geodesic $\mathrm{II} = 0$ and $A = 0$, so that the equations above mean $R_g(X,Y) = \hat{R}(X,Y)$ for $X,Y \in TL$ at the points of L. However there are no relations if we allow X,Y to be arbitrary from TM.

So horizontal parts of the both curvatures R_g and $\hat{R}_g$ coincide. Note that in the almost complex case this is trivially so because these horizontal parts vanish.

Question: What happens in other geometries – conformal, projective etc? There are also notions of normal connections ([N]) but on the normal spaces considered as bundles not manifolds.

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Note on figures: if they are not visible look at www.math.uit.no/seminar/preprints.html

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