Invariant characterization of Liouville metrics and polynomial integrals

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Abstract

A criterion in terms of differential invariants for a metric on a surface to be Liouville is established. Moreover, in this paper we completely solve in invariant terms the local mobility problem of a 2D metric, considered by Darboux: How many quadratic in momenta integrals does the geodesic flow of a given metric possess? The method is also applied to recognition of other polynomial integrals of geodesic flows. [1]

Introduction

The problem of recognizing by a metric, how many integrals admits its geodesic flow is classical. In this paper we study locally metrics on surfaces. We will look for the integrals analytic in momenta.

By Whittaker theorem [W] existence of such an integral is equivalent to existence of an integral polynomial in momenta. Note that locally geodesic flows are integrable, but the corresponding integrals are usually analytic only on $T^*M \setminus M$. So in general polynomial integrability requires certain conditions even locally.

The integrals of degree one in momenta correspond to surfaces of revolution, locally $ds^2 = f(x)(dx^2 + dy^2)$. It is an easy fact that if such integrals exist, then there are either one (generically) or three (space form). We provide a precise criterion for determining existence of a local linear integral (Killing vector field).

The next interesting case concerns geodesic flows with quadratic in momenta integrals. They correspond to Liouville metrics. The local analytic form of such metrics near a generic point is well-known [D, B]: $ds^2 = (f(x)+h(y))(dx^2+dy^2)$ and a metric has an additional quadratic integral iff it can be transformed into such a form.

However no criterion, when the metric is Liouville has been previously obtained, except for the paper [Su] (the first, rather unsuccessful attempt was done in [V]). However the criterion of this work was not explicit. Neither did it contain invariant formulae, making it difficult even to decide how many differential invariants characterize Liouville metric and which order they have.

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As the main result of this paper we resolve the classical problem of recognition for Liouville metrics and provide an explicit criterion written via a basis of scalar differential invariants of the metric.

Moreover we shall determine the number of quadratic integrals, which coincides with the degree of mobility of a Riemannian metric on a surface. It can be 6 (space form), 4 (case characterized in [D] though not through differential invariants; note that in classical works the Hamiltonian is disregarded, so that there are 5, 3 etc integrals in their way of counting), 3 (the case studied by Kœning [Koe]), 2 (general Liouville form) or 1 (metrics with no additional quadratic integrals). Each of these cases will be characterized via an invariant condition written in terms of differential invariants of the Riemannian metric.

The method of our study is the Cartan's prolongation-projection method: we write the system of PDEs for existence of a quadratic integral and subsequently calculate the compatibility conditions. If they are trivial, the system is compatible and we stop. Otherwise we add new equations, the space of solutions (which is a finite-dimensional linear space from the beginning – the system is of finite type) shrinks and we continue.

For effectiveness of the method we should have explicit formulas for compatibility conditions, but they are given by the result of [KL_2].

The procedure stops in several steps because finally we arrive to only one possible quadratic integral, which is just the Hamiltonian, an obvious integral of the geodesic flow. The prolongation-projection scheme usually is characterized by the rapidly growing complexity with each step. It is also true in our problem, but in this case we manage to arrive to the very end of the method and to establish the solvability criterion.

The problem of invariant characterization of Liouville metrics was initiated in paper [KL_3] in a collaboration with V.Lychagin as an application of our general compatibility criterion. The results are repeated in a revised form in sections 3-4. Moreover the general idea of solution to the problem was sketched there, but the complete answer appears here for the first time.

Let us also indicate that the solution of the problem presented here is expressed via scalar differential invariants and we especially care to minimize the number/order of the invariants.

At the end of the paper we discuss the problem of higher degree integrals and make some claims and conjectures about dimension of the space $\mathscr{I}_n$ of polynomial in momenta integrals F of $\deg F = n$.

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1. PDEs and Prolongation-Projection scheme

In this section we deduce the basic system of equations for polynomial integrability, discuss compatibility criterions and formulate the general scheme of investigating solvability.

This paper deals mostly with local existence problem, so that whenever opposite is not explicitly stated all statements should be assumed local in M. Moreover we impose the usual regularity assumption. Regular points form a nonempty open set, but it does not need to be of full measure in the C^∞ case (in analytic case regular points are generic) and such pathological examples exist even in $\mathbb{R}^2$.

Let (x, y) be local coordinates on M^2 and p_x, p_y be the corresponding momenta on T^*M. Writing the metric $ds^2 = g_{ij}dx^i dx^j$ we express Hamiltonian of the geodesic flow as $H = g^{11}p_x^2 + 2g^{12}p_x p_y + g^{22}p_y^2$, where the matrix g^{ij} is inverse to the matrix g_{ij} of the metric g.

A homogeneous term of an integral is obviously an integral, and so we study a function $F_n = \sum_{i+j=n} u_{ij}(x, y)p_x^i p_y^j$ on T^*M. Involutivity condition, i.e. vanishing of the Poisson bracket $\{H, F_n\} = 0$, is equivalent to $(n+2)$ equations $E_1 = 0, \dots, E_{n+2} = 0$ on $(n+1)$ unknown function $u_{n0}(x, y), \dots, u_{0n}(x, y)$.

This system $\mathcal{E}$ is of generalized complete intersection type studied in [KL$_3$]. The compatibility criterion developed there states that the system is formally integrable iff the following multi-bracket vanishes:

$$E_{n+3} = \{E_1, \dots, E_{n+2}\} \ (\operatorname{mod} E_1, \dots, E_{n+2}) = 0. \tag{1}$$

For linear differential operators this is defined as follows.

Let $E_i(u) = \sum_{j=0}^n E_i^j u_{j(n-j)}$ be representation of the vector operator in components $E_i = (E_i^0, \dots, E_i^n)$ (E_i^j are scalar differential operators). Then the multi-bracket equals (in this formula $m = n+1$)

$$\{E_1, \dots, E_{m+1}\} = \frac{1}{m!} \sum_{\alpha \in \mathbf{S}_m, \beta \in \mathbf{S}_{m+1}} (-1)^\alpha (-1)^\beta \, E_{\beta(1)}^{\alpha(0)} \cdot E_{\beta(2)}^{\alpha(1)} \cdots E_{\beta(m)}^{\alpha(m-1)} \cdot E_{\beta(m+1)}.$$

Reduction modulo the system in (1) means the following. Orders of differential operators E_i are 1 and order of the multi-bracket $\{E_1, \dots, E_{n+2}\}$ is (no greater than) $(n+1)$. We prolong the system $\mathcal{E} = \{E_1 = 0, \dots, E_{n+2} = 0\}$ to the order $(n+1)$ (such prolongation exists!), i.e. take the space of all linear combinations $\sum_{i=1}^{n+2} \nabla_i E_i$ with linear scalar differential operators ∇_i of $\operatorname{ord} \nabla_i \le n$, and consider the class of the multi-bracket in the quotient space (see [KL$_2$] for details).

Now the system $\mathcal{E}$ is of finite type, i.e. has no complex characteristics (we refer the reader for this and further notions from geometry of PDEs to [KLV, KL$_4$]), so if compatibility condition (1) is satisfied, the system is locally integrable.

If this condition is not satisfied we add the equation $E_{n+3} = 0$ to the system $\mathcal{E}$ and continue with solvability investigation of the new prolonged system $\mathcal{E}'$. Prolongation means addition of derivatives of the generators. But if their combination (differential corollary) drops in order (projection), it is the compatibility condition, which should be added to the system. Thus we get new equations $\mathcal{E}''$ etc, until we stabilize at a system $\bar{\mathcal{E}}$ in a finite number of steps (Cartan-Kuranishi theorem).

Denoting $T = T_{(x,y)}M$ the model tangent space for independent variables and $N \simeq \mathbb{R}^{n+2}(u_{i(n-i)})$ the space of dependent variables, the system $\mathcal{E}$ has symbols $g_k \subset S^kT^* \otimes N$ (respectively $\mathcal{E}'$ has symbols g'_k etc). Codimension of this linear subspace $(n+2)(k+1) - \dim g_k$ equals the number of independent equations in the prolonged system of order k.

Cohomology $H^{k-1,2}(\mathcal{E})$ of the Spencer δ-complex

$$0 \to g_{k+1} \xrightarrow{\delta} g_k \otimes T^* \xrightarrow{\delta} g_{k-1} \otimes \Lambda^2 T^* \to 0$$

at the last term is the space of compatibility conditions ([KLV, KL$_4$]). For the initial system $\mathcal{E}$ the only non-zero second Spencer δ-cohomology is $H^{n,2}(\mathcal{E}) \simeq \mathbb{R}^1$ and the only compatibility condition is the above reduced multi-bracket E_{n+3}. Thus $\mathcal{E}' = \{E_1 = 0, \dots, E_{n+3} = 0\}$. Further compatibility conditions (for $\mathcal{E}'$ etc) will be indicated along with prolongation-projection process.

Let $\mathscr{J}_n = \{(u_{i(n-i)})\}$ denotes the solution space of the system $\mathcal{E}$. This space is linear and $\dim \mathscr{J}_n = \sum \dim \bar{g}_k$. In particular this is smaller than $\sum \dim g_k$ and since the latter quantity strictly decreases during prolongation-projection, the method stops in a finite number of steps giving either non-trivial locally integrable system of PDEs (solvability) or no solution result.

2. Differential invariants of a Riemannian metric

Here we describe the algebra $\mathfrak{A}$ of scalar differential invariants of a Riemannian metric g on a two-dimensional surface M. It is well-known that the first such invariant occurs in order 2 and is given by the scalar curvature K.

Denote $\operatorname{grad} K$ be the g-gradient of the curvature and let $\operatorname{sgrad} K = J_0 \operatorname{grad} K$ be its rotation by $\pi/2$ (one needs to fix orientation, which is possible as we treat (M, g) locally; alternatively we can square those invariants, which have undetermined sign). There are two invariant differentiations $\mathfrak{L}_{\operatorname{grad} K}$ and $\mathfrak{L}_{\operatorname{sgrad} K}$ ($\mathfrak{L}$ is the Lie derivative). These differentiations and K do not generate $\mathfrak{A}$, but if we also allow commutators they do.

There are precisely $(k-1-\delta_{k3})$ functionally independent differential invariants of order k for $k > 0$. Let us briefly explain why (cf. [T]). Consider the jet-space of Riemannian metrics $J^k(S^2_+T^*M)$. Fibers of the projections $\pi_{k,k-1} : J^k \to J^{k-1}$ have dimensions $3(k+1)$, where as usual $J^{-1} = M$.

The pseudogroup $\mathrm{Diff}_{\mathrm{loc}}(M)$ acts naturally on the jet-spaces. Its action is transitive up to 1st jets. Indeed, the action is clearly transitive on the base and let us consider the jets of the stabilizer of $x \in M$, i.e. the differential group $G^k_x = J^k_{x,x}(M, M)$. G^1_x acts transitively on $J^0_x(S^2_+T^*M)$ with one-dimensional stabilizer $O(2)$.

The action of G^2_x on $J^1_x(S^2_+T^*M)$ is transitive as well, but the action of G^3_x on $J^2_x(S^2_+T^*M)$ is not (though it has dimensional freedom to be!). Codimension of a generic orbit is 1 and curvature K is the only invariant.

The stabilizer disappears only on the next step and starting from $k = 3$ the actions of G^{k+1}_x on $J^k_x(S^2_+T^*M)$ are free. Thus since $\dim \operatorname{Ker}(G^{k+1}_x \to G^k_x) =$

$2(k+2)$, codimension of the generic orbit of $\operatorname{Ker}(G_x^{k+1}\to G_x^k)$ on $\pi_{k,k-1}^{-1}(*)$ is equal to $(k-1-\delta_{k3})$.

Let us establish a basis in the space of invariants.

For a tensor T denote $d_\nabla^{\otimes k}T = d_\nabla(d_\nabla^{\otimes(k-1)}T)$ the iterated covariant derivative of the tensor T ($d_\nabla^{\otimes 2}$ differs from d_∇^2, which is equal to multiplication by the curvature tensor). In particular, we obtain the forms $d_\nabla^{\otimes i}K \in C^\infty(\otimes^i T^*M)$.

Since K is a scalar function, $d_\nabla K = dK$. The next differential is symmetric, because we consider metric (Levi-Civita) connection:

Lemma 1. *$d_\nabla^{\otimes 2}F \in C^\infty(S^2T^*M)\ \forall F \in C^\infty(M)$ iff connection ∇ is symmetric.*

Proof. Since $d_\nabla^{\otimes 2}F(\xi,\eta) = (\nabla_\xi dF)(\eta) = \nabla_\xi[\eta(F)] - [\nabla_\xi\eta](F)$, we have:

$$d_\nabla^{\otimes 2}F(\xi,\eta) - d_\nabla^{\otimes 2}F(\eta,\xi) = \bigl(L_{[\xi,\eta]} - (\nabla_\xi\eta - \nabla_\eta\xi)\bigr)(F) = T_\nabla(\eta,\xi)(F),$$

where T_∇ is the torsion tensor. In coordinates this is expressed via Christoffel symbols as $(d_\nabla^{\otimes 2}F)_{ij} = F_{ij} - \Gamma_{ij}^l F_l$, where $F_\sigma = \frac{\partial^{|\sigma|}F}{\partial x^\sigma}$ are the partial derivatives. □

The next differential $d_\nabla^{\otimes 3}F \in \Omega^1 M \otimes S^2\Omega^1 M$, but this (and higher) tensors are fully symmetric iff the metric is flat:

Lemma 2. *Let $T_\nabla = 0$. Then $d_\nabla^{\otimes 3}F \in C^\infty(S^3T^*M)\ \forall F \in C^\infty(M)$ iff $R_\nabla = 0$.*

Proof. $d_\nabla^{\otimes 3}F(\xi,\eta,\theta) = (\nabla_\xi\nabla_\eta dF - \nabla_{\nabla_\xi\eta}dF)(\theta)$, whence

$$d_\nabla^{\otimes 3}F(\xi,\eta,\theta) - d_\nabla^{\otimes 3}F(\eta,\xi,\theta) = \bigl(([\nabla_\xi,\nabla_\eta] - \nabla_{[\xi,\eta]})dF\bigr)(\theta) = R_\nabla(\xi,\eta)^* dF(\theta)$$

and the result follows. □

Now to fix a basis in invariants of order $i = 2 + l$ we consider the form $d_\nabla^{\otimes l}K$ and denote (in non-flat case the order is essential!)

$$I_{ij} = d_\nabla^{\otimes l}K(\underbrace{\operatorname{grad}K,\dots,\operatorname{grad}K}_{l-j},\underbrace{\operatorname{sgrad}K,\dots,\operatorname{sgrad}K}_{j}).$$

If we change the order, the expression will be changed by a lower order differential invariant. We will not use it and so omit the details, but for instance

$$d_\nabla^{\otimes 3}K(\operatorname{sgrad}K,\operatorname{sgrad}K,\operatorname{grad}K) - d_\nabla^{\otimes 3}K(\operatorname{grad}K,\operatorname{sgrad}K,\operatorname{sgrad}K) = |\operatorname{grad}K|^2.$$

The first invariants are: $I_2 = K$ and $I_3 = |\nabla K|^2$ (the index refers to the order of differential invariant). Starting from $i = 4$ there are $l + 1 = i - 1$ invariants I_{ij} and we re-enumerate the index j by letters (so we write I_{4a} instead of I_{40}, I_{5d} instead of I_{53} etc). For instance $I_{4b} = d_\nabla^{\otimes 2}K(\operatorname{grad}K,\operatorname{sgrad}K)$.

The two approaches to describe the algebra $\mathfrak{A}$ of differential invariants, one via the basic invariant I_2 with two invariant differentiations and another one via the basis I_{ij} are closely related: the former is obtained from the latter via the Lie-Tresse approach [Tr]. Namely let say I_2, I_3 be chosen as a basis, and $\hat{\partial}/\hat{\partial}_{I_2}, \hat{\partial}/\hat{\partial}_{I_3}$ be the corresponding Tresse derivatives (see [KL$_1$]). In local coordinates (x^1, x^2) they can be expressed as

$$\hat{\partial}/\hat{\partial}_{I_2} = \Delta^{-1}\bigl(\mathcal{D}_2(I_3)\mathcal{D}_1 - \mathcal{D}_1(I_3)\mathcal{D}_2\bigr), \quad \hat{\partial}/\hat{\partial}_{I_3} = \Delta^{-1}\bigl(\mathcal{D}_1(I_2)\mathcal{D}_2 - \mathcal{D}_2(I_2)\mathcal{D}_1\bigr),$$

where $\mathcal{D}_i$ are total derivatives [KLV] and $\Delta = \mathcal{D}_1(I_2)\mathcal{D}_2(I_3) - \mathcal{D}_2(I_2)\mathcal{D}_1(I_3)$ is the determinant (basis requirement above means $\Delta \not\equiv 0$). Then the two invariant differentiations $\nabla_1 = \mathfrak{L}_{\text{grad}\,K}$ and $\nabla_2 = \mathfrak{L}_{\text{sgrad}\,K}$ equal

$$\nabla_1 = I_3 \cdot \hat{\partial}/\hat{\partial}_{I_2} + 2I_{4a} \cdot \hat{\partial}/\hat{\partial}_{I_3}, \quad \nabla_2 = 2I_{4b} \cdot \hat{\partial}/\hat{\partial}_{I_3}.$$

Relation to the other side constitutes an infinite sequence of identities:

$$\nabla_1 I_2 = I_3,\ \nabla_2 I_2 = 0, \quad \nabla_1 I_3 = 2I_{4a},\ \nabla_2 I_3 = 2I_{4b},$$
$$\nabla_1 I_{4a} = I_{5a} + \frac{2(I_{4a}^2 + I_{4b}^2)}{I_3},\ \nabla_2 I_{4a} = I_{5b} + \frac{2I_{4b}(I_{4a} + I_{4c})}{I_3},$$
$$\nabla_1 I_{4b} = I_{5b} + \frac{I_{4b}(I_{4a} + I_{4c})}{I_3},\ \nabla_2 I_{4b} = I_{5c} + \frac{I_{4c}^2 - I_{4a}I_{4c} + 2I_{4b}^2}{I_3} + I_2 I_3^2,$$
$$\nabla_1 I_{4c} = I_{5c} + \frac{2(I_{4a}I_{4c} - I_{4b}^2)}{I_3},\ \nabla_2 I_{4c} = I_{5d}, \qquad \dots$$

They can be obtained successively with the help of the commutation rule for invariant differentiations:

$$[\text{grad}\,K, \text{sgrad}\,K] = -\frac{2I_{4b}}{I_3}\,\text{grad}\,K + \frac{I_{4a} - I_{4c}}{I_3}\,\text{sgrad}\,K.$$

3. Linear integrals

A Riemannian metric g on a surface M^2 possesses a Killing vector field iff it has the following local form near the point, where the field does not vanish: $ds^2 = g_{11}(x)dx^2 + 2g_{12}(x)dxdy + g_{22}(x)dy^2$, so that (M^2, g) is a surface of revolution. How to recognize such a metric?

Let us write the metric locally in isothermal hyperbolic coordinates (possibly over $\mathbb{C}$): $ds^2 = e^{\lambda(x,y)}dxdy$. If the metric is positive definite (not pseudo-Riemannian), one should rather write $e^\lambda dzd\bar{z}$ and this complexification pop-ups as follows: while the gradient of a function K equals $(2e^{-\lambda}K_y, 2e^{-\lambda}K_x)$, the skew-gradient is $(2\mathrm{i}\,e^{-\lambda}K_y, -2\mathrm{i}\,e^{-\lambda}K_x)$! Moreover we shall encounter i as a factor at some coefficients below, but this does not lead to contradiction: vanishing of these coefficients turns out to be a real condition.

In [KL$_3$] we chose the general form, but since the answer will be expressed in differential invariants, the choice is not essential.

Function $F_1 = up_x + vp_y$ is an integral of the geodesic flow iff the following 3 linear PDEs (coefficients of $\{H, F_1\}$) are satisfied:

$$u_y = 0, \qquad u_x + v_y + u\lambda_x + v\lambda_y = 0, \qquad v_x = 0.$$

Denote them by E_1, E_2, E_3 respectively. This system $\mathcal{E}$ has symbols: $\dim g_0 = 2$, $\dim g_1 = 1$, $\dim g_2 = 0$. The compatibility condition is given by the relation

$$E_4 = \{E_1, E_2, E_3\} \ (\mathrm{mod}\, E_1, E_2, E_3) = 0.$$

In general case the bracket should have order 2 in pure form and 1 after reduction, but in our case E_4 is of order 0 and equals:

$$E_4 = \tfrac{1}{2} e^{\lambda} (K_x u + K_y v),$$

where K is the Gaussian curvature. Thus compatibility condition means (M^2, g) is a spatial form: $K = \mathrm{const}$. This is the case, when $\dim \mathscr{I}_1 = 3$.

If K is non-constant, to study solvability we add the equation $E_4 = 0$ to the system. To describe the new system $\mathcal{E}'$ we let $u = K_y w$, $v = -K_x w$ and obtain the following system on one function $w(x, y)$:

$$\begin{pmatrix} 0 & K_y & K_{yy} \\ -K_x & 0 & -K_{xx} \\ K_y & -K_x & \lambda_x K_y - \lambda_y K_x \end{pmatrix} \cdot \begin{bmatrix} w_x \\ w_y \\ w \end{bmatrix} = 0.$$

In order to have solutions the determinant of this matrix should vanish. It equals $-\frac{\mathrm{i}}{4} e^{2\lambda} I_{4b}$. Given this condition we can drop one equation and transform the system to the form

$$(\log |K_x\, w|)_x = 0, \qquad (\log |K_y\, w|)_y = 0.$$

Its solvability is equivalent to a 3rd order relation on the curvature, which can be expressed as the condition $I_3(I_{5b} + I_{5d}) = 2 I_{4b}(I_{4a} + I_{4c})$. However when $I_{4b} = 0$, then $I_{5b} = 0$ and we obtain:

Theorem 1. $\dim \mathscr{I}_1 = 3$ *iff* $K = \mathrm{const}$ *(i.e.* $I_3 = 0$*) and* $\dim \mathscr{I}_1 = 1$ *iff*

$$I_{4b} = 0,\ I_{5d} = 0.$$

Otherwise there exist no local Killing vector fields.

Remark 1. *This and further statements hold only near regular points (here this means $dK \neq 0$). Indeed in non-analytic case there exist pathological counterexamples. For instance for any $\varepsilon > 0$ it is possible to construct a C^∞-metric on the disk $D^2(1)$ satisfying $I_{4b} = I_{5d} = 0$, such that the set of regular points (where a Killing field exist) has Lebesgue measure $< \varepsilon$.*

We can reformulate this criterion as vanishing of the differential invariants $\mathrm{Jac}_g(K, |\nabla K|^2)$ and $\mathrm{Jac}_g(K, \Delta_g K)$, where $\mathrm{Jac}_g(F, G) = dF \wedge dG\big(\frac{\mathrm{grad}\, K}{|\nabla K|}, \frac{\mathrm{sgrad}\, K}{|\nabla K|}\big)$ is the Jacobian and $\Delta_g F = \mathrm{Tr}_g[d_\nabla^{\otimes 2} F]$ is the Laplacian. Indeed we have: $\Delta_g K = (I_{4a} + I_{4c})/I_3$, so the claim follows from:

$$\mathrm{Jac}_g(K, |\nabla K|^2) = 2 I_{4b}, \qquad \mathrm{Jac}_g(K, \Delta_g K) = \frac{I_{5b} + I_{5d}}{I_3}$$

(note that $I_{4b} = 0$ implies $I_{5b} = 0$).

Remark 2. *Some classical criteria for existence of local (global implications follow) Killing fields are contained in [Nij, Nom], but they are neither explicit conditions on the metric g nor finitely formulated. Our criterion in the form of dependence of* $|\nabla K|$ *and* $\Delta_g K$ *on* K *is implicitly contained in [D].*

4. More than 3 quadratic integrals

We turn now to characterization of Liouville metrics. We will again use isometric hyperbolic coordinates, $H = e^{-\lambda} p_x p_y$, which does not restrict generality.

The function $F_2 = u(x,y)p_x^2 + 2v(x,y)p_xp_y + w(x,y)p_y^2$ is a quadratic integral of the geodesic flow iff the following system $\mathcal{E}$ is satisfied:

$$u_y = 0, \quad u_x + 2v_y + 2u\lambda_x + 2v\lambda_y = 0, \quad 2v_x + w_y + 2v\lambda_x + 2w\lambda_y = 0, \quad w_x = 0.$$

Denote the equations respectively by E_1, E_2, E_3, E_4. The compatibility condition can be expressed via the multi-bracket

$$E_5' = \{E_1, E_2, E_3, E_4\} \ (\operatorname{mod} E_1, E_2, E_3, E_4) = 0.$$

Even though it might be expected from the general theory that E_5' has order 2, in our case it has order 1. Divided by $2e^{\lambda}$ it equals to

$$E_5 = 5K_xv_y - 5K_yv_x - (K_{xx} - \lambda_xK_x)u + 5(\lambda_yK_x - \lambda_xK_y)v + (K_{yy} - \lambda_yK_y)w.$$

Thus the system $\mathcal{E}$ is formally integrable iff $K = \text{const}$. In this case $\dim g_0 = 3, \dim g_1 = 2, \dim g_2 = 1$, $g_{2+i} = 0$ for $i > 0$ and the dimension of the solutions space is $\dim \mathscr{I}_2 = \sum \dim g_k = 6$. Indeed $\mathscr{I}_2 = S^2 \mathscr{I}_1$, i.e. a basis in the space of quadratic integrals is formed by pair-wise products of elements of a basis in is the space of linear integrals.

Suppose that $K \neq \text{const}$, so that E_5 is a differential relation of the first order in u, v, w. Adding $E_5 = 0$ we get the system[2] $\mathcal{E}' \subset J^1(2,3)$ of formal codimension 5.

Its symbols $g_i' \subset S^iT^* \otimes \mathbb{R}^3$ have $\dim g_0' = 3$, $\dim g_1' = 1$, $\dim g_2' = 0$ and thus the only non-zero second δ-cohomology groups[3] are $H^{0,2}(\mathcal{E}') \simeq \mathbb{R}^1$, $H^{1,2}(\mathcal{E}') \simeq \mathbb{R}^1$. There are two obstructions to compatibility – Weyl tensors $W_1' \in H^{0,2}(\mathcal{E}')$ and $W_2' \in H^{1,2}(\mathcal{E}')$. The former W_1' is proportional to

$$E_6' = K_yE_{5x} + K_xE_{5y} - \frac{5}{2}K_x^2(E_{2y} - E_{1x}) + \frac{5}{2}K_y^2(E_{3x} - E_{4y}) (\operatorname{mod} E_1, E_2, E_3, E_4, E_5).$$

Multiplying this by $5K_x$ and further simplifying modulo E_1, E_2, E_3, E_4, E_5 we obtain the following expression:

$$E_6 = \tfrac{35}{4\mathrm{i}} e^{2\lambda} I_{4b}\, v_x + Q_1\, u + \tfrac{35}{4\mathrm{i}} e^{2\lambda} \lambda_x I_{4b}\, v + Q_2\, w,$$

[2] $J^k(m,r)$ is the space of k-jets of maps $\varphi : \mathbb{R}^m \to \mathbb{R}^r$ and formal codimension of a system $\mathcal{E} \subset J^k(m,r)$ is $\sum_i \dim H^{1,i}(\mathcal{E})$, the precise number of the equations in the system [KL$_4$].

[3] The second Spencer δ-cohomology $H^{*,2} = \oplus H^{i-1,2}$ is the space that contains all compatibility conditions of the system. The latter are called Weyl tensors $W_i \in H^{i-1,2}$ (also called curvatures/torsions/structural functions). We refer to [KL$_4$] for a review.

where

$$-128\,e^{-4\lambda}K_y^3Q_1 = J_{5a} \quad\text{and}\quad 32\,e^{-3\lambda}I_3K_xQ_2 = J_{5b}$$

are differential invariants and provided $I_{4b} = 0$ (which implies $I_{5b} = 0$, see identities in §2) they reduce to $J_{5a}|_{I_{4b}=0} = J_{5b}|_{I_{4b}=0} = J_5$, where

$$J_5 = 5I_3(I_{5a} - I_{5c}) + (I_{4a} - I_{4c})(I_{4c} - 6I_{4a}) - 25I_2I_3^3$$

We see that the coefficients of E_6 (as well as that of other E_i) are not invariant (neither are real), but the condition of their vanishing is invariant (and real).

If E_6 vanishes, the system $\mathcal{E}'$ can be prolonged to the second jets, but is not yet formally integrable. Another curvature – Weyl tensor W_2' – is the obstruction to prolongation to the third and henceforth infinite jets. Since $g_2' = 0$, it is the Frobenius condition of the canonical Cartan distribution on the first prolongation $\mathcal{E}_2'$ of $\mathcal{E}_1' = \mathcal{E}'$ (but it is one equation, not three as one can guess without calculation of Spencer δ-cohomology!). Originally a (linear) function on $\mathcal{E}_2'$, it can be represented as a linear function on $\mathcal{E}'$ due to isomorphism $\pi_{2,1}' : \mathcal{E}_2' \xrightarrow{\sim} \mathcal{E}'$.

This new equation $\tilde{E}_6$ has coefficients of order 6, but they can be simplified modulo the conditions $I_{4b} = 0$, $J_5 = 0$. Indeed we can differentiate these along invariant fields ∇_1, ∇_2, see for instance the next section (this allows to express all the higher invariants I_{ij} with $i \geq 5$ through invariants of order ≤ 4). Thus the second obstruction to formal integrability W_2' is the following equation, which turns out to be a linear function on $J^0(2,3)$ (we multiply it by the factor $64e^{-3\lambda}I_3^3K_xK_y$):

$$\tilde{E}_6 = J_4 \cdot (K_x^2u - K_y^2v),$$

where

$$J_4 = 3(I_{4a} - I_{4c})(I_{4a} + 4I_{4c})I_{4c} - 15I_2I_3^3(I_{4a} + 4I_{4c}) + 25I_3^5.$$

Notice that in expression for E_6 we simplified modulo the conditions $I_{4b} = 0, J_5 = 0$. Otherwise the coefficients are complex and more complicated, and in addition there are terms with v_x and v. For instance the coefficient of v_x term is $\frac{35}{8\mathrm{i}}e^{2\lambda}(I_{5b} + I_{5d})I_3^{-1}$, but it simplifies to zero.

Since formal (=local due to finite type condition) integrability of $\mathcal{E}'$ means existence of 4 integrals of the geodesic flow, we get the following statement:

Theorem 2. *The condition of exactly 4 quadratic integrals* $\dim \mathscr{I}_2 = 4$ *can be expressed as 3 differential conditions on the metric:*

$$I_{4b} = 0, \quad J_5 = 0, \quad J_4 = 0.$$

5. Digression I: Darboux-Kœning's theorem

We can deduce now the classical theorem due to Darboux and Kœnning:

Theorem 3. *A plane metric has exactly 4 quadratic integrals iff it has exactly one linear integral and one more quadratic integral independent of the Hamiltonian and the square of the linear integral.*

To one side this was proved in [D], while to the other it was given in [Koe]. It is instructive to see the equivalence by using differential invariants only (thus below is an alternative proof of this classical theorem):

Proof. Let us suppose at first that g has 4 quadratic integrals. We may assume $K \neq \text{const}$. Thus $I_{4b} = 0$ and $J_5 = 0$. We must show $I_{5d} = 0$ (this follows easily from the condition that v_x-coefficient of $\tilde{E}_6$ vanishes, but we will show that it suffices to use only the first two conditions of Theorem 2).

Note that under condition $I_{4b} = 0$ we have: $\nabla_2 I_2 = 0$, $\nabla_2 I_3 = 0$ and $\nabla_2 I_{4a} = 0$ (see identities of §2). The latter follows from $I_{5b} = 0$ as well as from the fact that the commutator $[\nabla_1, \nabla_2]$ is proportional to ∇_2. Now equation $J_5 = 0$ can be written as

$$5I_3\nabla_1 I_{4a} - 16I_{4a}^2 + 2I_{4a}I_{4c} + 4I_{4c}^2 - 20I_2I_3^3 = 0.$$

Applying ∇_2 to this we get $2I_{5d}(I_{4a} + 4I_{4c}) = 0$, which yields either $I_{5d} = 0$ or $I_{4c} = -\frac{1}{4}I_{4a}$. The latter after application of ∇_2 gives $I_{5d} = 0$ as well.

Now suppose that g has a Killing vector field and an additional quadratic integral, so that the dimension of the space of quadratic integrals is at least 3. Since $I_{4b} = 0$, the equation E_6 is of order 0. If $E_6 \not\equiv 0$, then its prolongation adds a new first order equation to the system and the symbols satisfy: $\dim g_0 \leq 2$, $\dim g_1 = 0$, so that the space of quadratic integrals cannot have dimension greater than 2. If all the coefficients of E_6 vanish, then $J_5 = 0$. If $J_4 \neq 0$, then $\tilde{E}_6$ is non-zero and of order 0. The same calculus for dimensions of symbols and solutions space leads to contradiction. On the other hand, if $J_4 = 0$, then $\tilde{E}_6 \equiv 0$ and we have 4 quadratic integrals. □

Corollary 1. *If g possesses a Killing vector field, then its local degree of mobility* $\dim \mathscr{I}_2$ *is even: 2, 4 or 6.*

6. Digression II: On the number of invariants

Conditions $I_{4b} = 0, J_5 = 0$ do not imply $J_4 = 0$. This pair of relations for differential invariants can be considered as an overdetermined system, but it is compatible meaning they do not produce new differential relations of lower order. Actually, we showed in the previous section that the two relations imply $I_{5d} = 0$. Relations $\nabla_1 I_{4b} = 0$ gives $I_{5b} = 0$ and $\nabla_2 I_{4b} = 0$ yields $I_{5c} = (I_{4a}I_{4c} - I_{4c}^2 - I_2I_3^3)/I_3$. Then $J_5 = 0$ implies $I_{5a} = \frac{2}{5}(3I_{4a}^2 - I_{4a}I_{4c} - 2I_{4c}^2 + 10I_2I_3^3)/I_3$.

Further derivations of these identities with ∇_s yield expressions for higher differential invariants I_{ij}, $i \geq 6$, via invariants of order ≤ 4 and they agree (there are 8 equations to determine 5 invariants of order 6, 12 equations to determine 6 invariants of order 7 etc), which manifests the above mentioned compatibility.

On the other hand, under certain genericity assumption, namely $I_{4c}(2I_{4a}+3I_{4c})\neq 5I_2I_3^3$, the conditions $I_{4b}=0,\ J_4=0$ imply $J_5=0$. Indeed if we express $I_{5a},I_{5b},I_{5c},I_{5d}$ from $\nabla_1 I_{4b}=0,\nabla_2 I_{4b}=0,\nabla_1 J_4=0,\nabla_2 J_4=0$, and substitute this into J_5, the expression will have the factor J_4. Thus in this case the criterion of 4 integrals can be expressed as two differential conditions

$$I_{4b}=0,\quad J_4=0.$$

In general, however, we cannot remove the condition $J_5=0$ from Theorem 2.[4]

Example. For the metric $g=\varepsilon_1 e^{(\beta+2)x}dx^2+\varepsilon_2 e^{\beta x}dy^2$ ($\varepsilon_k=\pm1$; this is one family from the classification of [BMM]) we have (the first two identities are obvious because ∂_y is the Killing field):

$$I_{4b}=0,\quad I_{5d}=0,\quad J_5=\tfrac{1}{64}e^{-10(\beta+2)x}\beta^6(\beta-1)(\beta-6)(2+\beta)^6,$$
$$J_4=\tfrac{\varepsilon_1}{1024}e^{-15(\beta+2)x}\beta^{10}(\beta-1)(\beta+2)^9(3\beta+22).$$

Since $I_3=\frac{\varepsilon_1}{4}e^{-3(\beta+2)x}\beta^2(\beta+2)^2$, the cases $\beta=-2,0$ correspond to constant curvature. Otherwise $J_5=0$ for $\beta=1$ or $\beta=6$. In the first case $J_4=0$ and we have $\dim\mathscr{I}_2(g)=4$. But in the second case $\dim\mathscr{I}_2(g)=2$.

Note also that $J_4=0$ for $\beta=-22/3$, but then $I_{4c}(2I_{4a}+3I_{4c})=5I_2I_3^3$ and this does not imply $J_5=0$.

Remark 3. *J_4 is a forth order invariant obtained via reduction from a 6th order invariant modulo the conditions $I_{4b}=0$, $J_5=0$ and their ∇_i-prolongations. Thus its vanishing alone without $J_5=0$ has no geometrical meaning.*

7. Precisely 3 quadratic integrals

If the compatibility condition $E_6=0$ is not trivial, then we add it and get a new system $\mathcal{E}''$. In this section we consider the generic case when this new equation is of order 1 in u,v,w, i.e. $I_{4b}\neq 0$.

Then the symbol of the system $\mathcal{E}''$ is $g_1''=0$, i.e. it is of Frobenius type. Its Spencer cohomology group $H^{0,2}(\mathcal{E}'')\simeq\mathbb{R}^3$, so the obstruction to integrability – curvature tensor – W_1'' has 3 components, represented by 3 linear relations on $J^0(2,3)$. Indeed, we can express from $\mathcal{E}''$ all derivatives u_x,u_y,v_x,v_y,w_x,w_y, calculate 3 difference of pairs of mixed derivatives and substitute the derivative expressions. We get the following equations:

$$E_7=Au+Bw=0,\qquad E_7''=\bar{B}u+\bar{A}w=0,\qquad E_7'=\tfrac12(E_7'+E_7'')=0,\quad (2)$$

where A,B are certain complex differential expressions of order 6 in metric (see below). One peculiarity of (2) is absence of v. Another is that there are only two equations, not three as expected from the general theory.

[4]Indeed if the indicated inequality of forth order is an identity, we have 3 differential conditions of order 4 and so the condition $J_5=0$ can be reduced in order, but since this leads to an expression with roots, we do not provide it here.

Vanishing of E_7, E_7'' is equivalent to four real conditions $A=0, B=0$, which can be expressed via differential invariants of order 6. In the following sections we will show that $I_{4b}=0$, but $J_5\neq 0$ or $J_4\neq 0$ implies $\dim \mathscr{I}_2<3$ and so we obtain: the following criterion (note that I_{6e} does not enter the formulae):

Theorem 4. *The condition of exactly 3 quadratic integrals is equivalent to two inequalities $I_3\neq 0$, $I_{4b}\neq 0$ and 4 differential relations on the metric:*

$$\begin{aligned}I_{6a}=\frac{1}{175I_3^2I_{4b}}\bigl(&700I_3^5I_{4b}-825I_2I_3^4I_{5b}+50I_2I_3^3I_{4b}(31I_{4a}-18I_{4c})\\&+6I_{4b}(I_{4a}-I_{4c})(6I_{4a}^2+49I_{4b}^2-37I_{4a}I_{4c}+6I_{4c}^2)-25I_3^2I_{5b}(-8I_{5a}+I_{5c})\\&-5I_3(48I_{4a}^2I_{5b}-27I_{5b}I_{4c}^2+2I_{4b}I_{4c}(-11I_{5a}+46I_{5c})\\&+I_{4a}(-43I_{5a}I_{4b}-21I_{5b}I_{4c}+8I_{4b}I_{5c})+7I_{4b}^2(4I_{5b}-11I_{5d}))\bigr)\end{aligned}$$

$$\begin{aligned}I_{6b}=\frac{1}{175I_3^2I_{4b}}\bigl(&1505I_2I_3^3I_{4b}^2+72I_{4a}^2I_{4b}^2+245I_3I_{5a}I_{4b}^2+588I_{4b}^4+225I_3^2I_{5b}^2\\&+405I_3I_{4b}I_{5b}I_{4c}+72I_{4b}^2I_{4c}^2-6I_{4a}I_{4b}(55I_3I_{5b}+74I_{4b}I_{4c})-490I_3I_{4b}^2I_{5c}\bigr)\end{aligned}$$

$$\begin{aligned}I_{6c}=\frac{1}{175I_3^2I_{4b}}\bigl(&-175I_3^5I_{4b}+300I_2I_3^4I_{5b}-25I_3^2I_{5a}I_{5b}-100I_2I_3^3I_{4b}(5I_{4a}-9I_{4c})\\&-6I_{4b}(I_{4a}-I_{4c})(6I_{4a}^2+49I_{4b}^2-37I_{4a}I_{4c}+6I_{4c}^2)+200I_3^2I_{5b}I_{5c}+5I_3(6I_{4a}^2I_{5b}\\&+36I_{5b}I_{4c}^2-I_{4b}I_{4c}(I_{5a}+34I_{5c})+6I_{4a}(I_{5a}I_{4b}-7I_{5b}I_{4c}-I_{4b}I_{5c})+7I_{4b}^2(8I_{5b}-I_{5d}))\bigr)\end{aligned}$$

$$\begin{aligned}I_{6d}=\frac{1}{175I_3^2I_{4b}}\bigl(&1500I_2^2I_3^6+36I_{4a}^4+25I_3^2I_{5a}^2+245I_3I_{5a}I_{4b}^2+588I_{4b}^4+225I_3^2I_{5b}^2\\&-294I_{4a}^3I_{4c}+895I_3I_{4b}I_{5b}I_{4c}-185I_3I_{5a}I_{4c}^2+366I_{4b}^2I_{4c}^2+36I_{4c}^4+6I_{4a}^2(61I_{4b}^2\\&+86I_{4c}^2-5I_3(2I_{5a}-9I_{5c}))-225I_3^2I_{5a}I_{5c}-490I_3I_{4b}^2I_{5c}+220I_3I_{4c}^2I_{5c}+200I_3^2I_{5c}^2\\&+5I_2I_3^3(102I_{4a}^2-294I_{4a}I_{4c}+4(49I_{4b}^2+48I_{4c}^2)+I_3(-85I_{5a}+260I_{5c}))-245I_3I_{4b}I_{4c}I_{5d}\\&-I_{4a}(6I_{4c}(172I_{4b}^2+49I_{4c}^2)+5I_3(-49I_{4c}(I_{5a}-2I_{5c})+I_{4b}(164I_{5b}-49I_{5d})))\bigr)\end{aligned}$$

Remark 4. *Denoting the above four equations (i.e. l.h.s-r.h.s.) by V_1,V_2,V_3,V_4, we can write $A=(V_2+V_4)+i(V_1+V_3)$, $B=(3V_2-V_4)+i(3V_3-V_1)$.*

Example. Consider the metric

$$ds^2=(x^2+q_2(y))(dx^2+dy^2),$$

where $q_2(y)=ay^2+by+c$. This metric is in Liouville form and hence has an additional quadratic integral. We can calculate the invariants from the previous theorems to find when the space of quadratic integrals has dimension $D>2$. Here's the result according to dimension:

◃ $D = 6$ if $a = 1$ & $4c = b^2$;

◃ $D = 4$ if $a = 1$ & $4c \neq b^2$;

◃ $D = 3$ if $a = 4^{\pm 1}$ & b, c arbitrary.

Note that the integrable metric $(x^2 + 4y^2 + 1)ds^2_{\text{Eucl}}$ was found in classification of Matveev [M$_3$]. However the methods used by him are global and do not apply to local non-complete situation.

Remark 5. *One can substitute the general Liouville form $ds^2 = \Lambda \cdot (dx^2 + dy^2)$ in local conformal coordinates into the above four expressions. The result is a system of 3 PDEs of order 6 in Λ together with the equation $\Lambda_{xy} = 0$ (which simplifies the 3 PDEs a lot). This system is not of finite type (for instance because it contains the cases of 4 integrals depending on 1 function of 1 variable) and it is not formally integrable: an easy elimination reduces one PDE of order 6 to order 5. Then its prolongation yields two new PDEs of order 5, but they are too long to be treated effectively.*

In fact, normal forms of metrics with 2 additional integrals are better obtained with a different approach, see [Koe].

8. Digression III: Simplification of invariants

The four relations from Theorem 4 provide the complete set, characterizing the condition $\dim \mathscr{I}_2 = 3$, but they are not compatible in the following sense. If we deduce the differential corollaries via derivations ∇_1, ∇_2, some of them will have lower order and be simpler. Let us indicate this.

Substitution of the expressions of $I_{6a}, I_{6b}, I_{6c}, I_{6d}$ from Theorem 4 to the identity (twice: before and after derivations!)

$$\nabla_1 I_{6b} - \nabla_2 I_{6a}$$
$$= (6I_{4b}I_3^3 + 6I_2I_{5b}I_3^2 + 3I_2I_{4b}(I_{4a}+I_{4c})I_3 - 5I_{6a}I_{4b} + 4I_{4a}I_{6b} - 4I_{6b}I_{4c} + 3I_{4b}I_{6c})/I_3$$

yields us the following new relation:

$$I_{6e} = \frac{1}{13475 I_3^2 I_{4b}^3}\big(375 I_2^2(-34 I_{4a} I_{4b} + 764 I_{4c} I_{4b} + 75 I_3 I_{5b}) I_3^6 + 61250 I_{4b}^3 I_3^5 + 4500 I_{5b}^3 I_3^3 + 1125 I_{5b} I_{5c}^2 I_3^3 + 1125 I_{5a}^2 I_{5b} I_3^3 - 2250 I_{5a} I_{5b} I_{5c} I_3^3 - 10 I_2(906 I_{4b} I_{4a}^3 - 6(225 I_3 I_{5b} + 2068 I_{4b} I_{4c}) I_{4a}^2 + (6 I_{4b}(917 I_{4b}^2 + 3083 I_{4c}^2) - 5 I_3(151 I_{5a} I_{4b} + 94 I_{5c} I_{4b} - 315 I_{5b} I_{4c})) I_{4a} - 4 I_{4b} I_{4c}(4333 I_{4b}^2 + 1749 I_{4c}^2) + 1125 I_3^2 I_{5b}(I_{5a} - I_{5c}) - 5 I_3(35(22 I_{5b} + 7 I_{5d}) I_{4b}^2 - 4 I_{4c}(524 I_{5a} - 769 I_{5c}) I_{4b} + 45 I_{5b} I_{4c}^2)) I_3^3 - 450 I_{5a} I_{5b} I_{4c}^2 I_3^2 + 19300 I_{4b} I_{4c} I_{5c}^2 I_3^2 - 2800 I_{5a} I_{4b}^2 I_{5b} I_3^2 + 55900 I_{4b} I_{5b}^2 I_{4c} I_3^2 + 9500 I_{5a}^2 I_{4b} I_{4c} I_3^2 + 450 I_{5b} I_{4c}^2 I_{5c} I_3^2 - 27825 I_{4b}^2 I_{5b} I_{5c} I_3^2 - 28800 I_{5a} I_{4b} I_{4c} I_{5c} I_3^2 - 2450 I_{5a} I_{4b}^2 I_{5d} I_3^2 - 9800 I_{4b} I_{5b} I_{4c} I_{5d} I_3^2 + 2450 I_{4b}^2 I_{5c} I_{5d} I_3^2 + 45 I_{5b} I_{4c}^4 I_3 - 13600 I_{5a} I_{4b} I_{4c}^3 I_3 + 111560 I_{4b}^2 I_{5b} I_{4c}^2 I_3 + 26705 I_{4b}^4 I_{5b} I_3 + 40110 I_{5a} I_{4b}^3 I_{4c} I_3 + 15560 I_{4b} I_{4c}^3 I_{5c} I_3 - 108465 I_{4b}^3 I_{4c} I_{5c} I_3 + 45080 I_{4b}^4 I_{5d} I_3 - 15190 I_{4b}^2 I_{4c}^2 I_{5d} I_3 + 2340 I_{4b} I_{4c}^5 + 25698 I_{4b}^3 I_{4c}^3 - 1440 I_{4a}^5 I_{4b} + 53802 I_{4b}^5 I_{4c} + 60 I_{4a}^4(27 I_3 I_{5b} + 235 I_{4b} I_{4c}) + 6 I_{4a}^3(-2458 I_{4b}^3 - 6625 I_{4c}^2 I_{4b} + 10 I_3(40 I_{5a} I_{4b} + 9 I_{5c} I_{4b} - 63 I_{5b} I_{4c})) + I_{4a}^2(67644 I_{4c} I_{4b}^3 + 45300 I_{4c}^3 I_{4b} - 2700 I_3^2 I_{5b}(I_{5a} - I_{5c}) + 5 I_3(2(2476 I_{5b} + 931 I_{5d}) I_{4b}^2 + 2 I_{4c}(3148 I_{5c} - 2315 I_{5a}) I_{4b} + 549 I_{5b} I_{4c}^2)) - 2 I_{4a}(25(20 I_{4b} I_{5a}^2 + 9(I_{4b} I_{5c} - 7 I_{5b} I_{4c}) I_{5a} + 63 I_{5b} I_{4c} I_{5c} + I_{4b}(373 I_{5b}^2 + 49 I_{5d} I_{5b} - 29 I_{5c}^2)) I_3^2 - 5(2359 I_{5c} I_{4b}^3 + I_{4c}(1813 I_{5d} - 16307 I_{5b}) I_{4b}^2 - 4758 I_{4c}^2 I_{5c} I_{4b} - 63 I_{5b} I_{4c}^3 + I_{5a}(679 I_{4b}^3 + 3435 I_{4c}^2 I_{4b})) I_3 + 3 I_{4b}(4067 I_{4b}^4 + 15599 I_{4c}^2 I_{4b}^2 + 3425 I_{4c}^4))\big).$$

Using similar identities for $\nabla_1 I_{6c} - \nabla_2 I_{6b}$, $\nabla_1 I_{6d} - \nabla_2 I_{6c}$, $\nabla_1 I_{6e} - \nabla_2 I_{6d}$ and substitutions of the 6th order invariants via the lower ones, we get 3 differential relations of order 5 (but they are non-linear even in higher order basic invariants). The first of them is:

$$1500 I_2^2 I_3^6 - 5 I_2(-102 I_{4a}^2 + 294 I_{4c} I_{4a} - 6(49 I_{4b}^2 + 32 I_{4c}^2) + 5 I_3(17 I_{5a} - 52 I_{5c})) I_3^3 + 25 I_{5a}^2 I_3^2 + 275 I_{5b}^2 I_3^2 + 200 I_{5c}^2 I_3^2 - 225 I_{5a} I_{5c} I_3^2 - 175 I_{5b} I_{5d} I_3^2 + 245 I_{5a} I_{4b}^2 I_3 - 185 I_{5a} I_{4c}^2 I_3 + 1265 I_{4b} I_{5b} I_{4c} I_3 - 1225 I_{4b}^2 I_{5c} I_3 + 220 I_{4c}^2 I_{5c} I_3 - 280 I_{4b} I_{4c} I_{5d} I_3 + 36 I_{4a}^4 + 1176 I_{4b}^4 + 36 I_{4c}^4 + 438 I_{4b}^2 I_{4c}^2 - 294 I_{4a}^3 I_{4c} + 6 I_{4a}^2(73 I_{4b}^2 + 86 I_{4c}^2 - 5 I_3(2 I_{5a} - 9 I_{5c})) - I_{4a}(6 I_{4c}(246 I_{4b}^2 + 49 I_{4c}^2) + 5 I_3(I_{4b}(188 I_{5b} - 91 I_{5d}) - 49 I_{4c}(I_{5a} - 2 I_{5c}))) = 0$$

and the other two are more complicated.

Furthermore these three relations can be invariantly differentiated and then simplified with substitutions, which resembles Cartan's prolongation-projection method, though for differential invariants. In a sequel one gets "compatible" set of relations for differential invariants, but this involves consideration of cases (lots of inequalities and equalities) and will be omitted.

9. Generic case: Liouville form

Here we continue investigation of the previous section, when $I_{4b} \neq 0$. Suppose that not all equalities of the previous theorem hold. Then E_7 is a non-trivial equation. If E_7'' is independent of it, we get $u = w = 0$ and then $v = \text{const} \cdot e^{-\lambda}$, so that there exists no quadratic integral besides the Hamiltonian.

Thus for existence of an additional quadratic integral the corresponding determinant $|A|^2 - |B|^2$ should vanish (note that this implies $w = \bar{u}$, which could be predicted because integral F is real). In this case the symbols dimensions are $\dim g_0 = 2$, $\dim g_1 = 0$, so for Liouville (quadratic) integrability of the metric g the system $\mathcal{E}''' = \{E_1 = 0, \dots, E_7 = 0\}$ should be compatible.

There are precisely two compatibility conditions: $\mathcal{D}_x E_7 = 0 \bmod \mathcal{E}'''$ and $\mathcal{D}_y E_7 = 0 \bmod \mathcal{E}'''$. The reduction $\bmod \mathcal{E}'''$ can be considered here as follows: all derivatives are expressed from the first 6 equations and substituted into derivatives of E_7. Then the equations are again linear and contain only u- and w-terms. Writing linear dependence with E_7 we get vanishing of two (complex) determinants. This constitutes 4 real relations of order 7, but we write them as 2 complex relations.

In the theorem below A, B are differential invariants from (2) (expressions are given in Remark 4) and $\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3, \mathfrak{J}_4$ are some differential invariants of order 7, precise form of which is given in Appendix.

Theorem 5. *Suppose that $K \neq \text{const}$, $I_{4b} \neq 0$ and $|A|^2 + |B|^2 \neq 0$ (cases considered separately). Then the metric g is Liouville iff it satisfies one real relation of order 6: $|A|^2 = |B|^2$ and 4 real relations of order 7:*

$$B\,\mathfrak{J}_1 = A\,\mathfrak{J}_2, \qquad A\,\mathfrak{J}_3 = B\,\mathfrak{J}_4.$$

Thus the problem of invariant characterization of Liouville metrics is solved.

Remark 6. *Similar to Section 8 one can reduce in order and simplify differential relations from Theorem 5, but since the resulting minimal set is very cumbersome (collection of cases involving equalities and inequalities), it won't be discussed.*

10. Singular locus: 2 quadratic integrals

Consider now the last case $I_{4b} = 0$, but suppose that either $J_5 \neq 0$ or $J_4 \neq 0$. In this case the equation E_6 (resp. $\tilde{E}_6$) transforms into the equation (since $K \neq \text{const}$, we may assume $K_x \neq 0$ or $K_y \neq 0$; formulae below are easily adjustable to one of the cases):

$$K_x^2 u = K_y^2 w. \tag{3}$$

Prolonging this equation and using the system $\mathcal{E}' = \{E_1 = \dots = E_5 = 0\}$ we can rewrite the new system $\bar{\mathcal{E}}$ (prolongation of $\mathcal{E}''$) in the form:

$$\begin{aligned} u_x &= \Bigl(2\log\frac{K_y}{K_x}\Bigr)_x u, \quad v_x = -\lambda_x\, v + \Bigl(\log\frac{K_y}{K_x} - \lambda\Bigr)_y w, \quad w_x = 0, \\ u_y &= 0, \qquad\qquad\qquad v_y = -\lambda_y\, v + \Bigl(\log\frac{K_x}{K_y} - \lambda\Bigr)_x u, \quad w_y = \Bigl(2\log\frac{K_x}{K_y}\Bigr)_y \cdot w, \end{aligned} \tag{4}$$

considered together with (3). System (4) consists of a three pair of equations, two uncoupled and one coupled with the other two. The system is of Frobenius type. Writing compatibility conditions of $\bar{\mathcal{E}}$ modulo (3)+(4) we get 3 conditions on the system to be integrable.

These three conditions are dependent (2 conditions), but modulo the condition $I_{4b} = 0$ they collapse to only one condition $I_{5d} = 0$.

Note that the system has dimension of symbols $\dim \bar{g}_0 = 2, \dim \bar{g}_1 = 0$, so that the maximal dimension of the solution space is 2. Since the minimum is 1, we arrive to the following statement:

Theorem 6. *Let $I_{4b} = 0$, but either $J_5 \neq 0$ or $J_4 \neq 0$. Then the system is Liouville iff $I_{5d} = 0$ and in this case there exists only one additional (independent of the Hamiltonian) quadratic integral.*

Note that condition $I_{4b} = 0, I_{5d} = 0$ are characteristic for existence of local Killing vector field. Thus we conclude:

Corollary 2. *Riemannian metric g possesses a local Killing field iff $I_{4b} = 0$ and there is a quadratic integral, independent of the Hamiltonian.*

Note that if the space of such additional integrals is 1, a representative can be chosen as the square of a linear integral.

11. Liouville metrics: some global questions

Proposition 7. *Let Liouville metric on M^2 have non-constant curvature and H be the corresponding Hamiltonian. Then for any two quadratic integrals F, G such that the triple (F, G, H) is linear independent (over $\mathbb{R}$), the triple is functionally independent (in particular the integral $\{F, G\}$ is non-zero).*

Proof. Since F, G, H are quadrics in p, the only kind of functional dependence for them can be either linear or quadratic.

Assume at first that the integrals H, F, G are linear dependent over $C^\infty(M)$, i.e. $H = a \cdot F + b \cdot G$, where $a, b \in C^\infty(M)$ are non-constant. Then bracketing this with H we get $\{H, a\}F + \{H, b\}G = 0$, which would imply that F, G, H

have a common factor: $H = v \cdot w$, $F = v \cdot \zeta$, $G = v \cdot \eta$. Commutation of H and F, G, H gives

$$\{H, \zeta\} = \{v, w\}\zeta,\ \{H, \eta\} = \{v, w\}\eta,\ \{H, w\} = \{v, w\}w.$$

Substitution of $w = a \cdot \zeta + b \cdot \eta$ into the last equality yields $\{H, a\} = \{H, b\} = 0$, i.e. $a = \text{const}, b = \text{const}$, so that F, G, H are linearly dependent.

If H, F, G are linear independent over $C^\infty(M)$, but are functionally dependent, then they must satisfy a quadratic relation:

$$U^2 + V^2 = W^2, \qquad U = \zeta^2 - \eta^2,\ V = 2\,\zeta\eta,\ W = \zeta^2 + \eta^2,$$

($\zeta, \eta \in C^\infty(T^*M)$ are linear in p functions) with a constant non-degenerate transition matrix A:

$$(F, G, H) = (U, V, W) \cdot A.$$

Denoting $\Theta = \{\zeta, \eta\}$ we observe this factor in all pair-wise Poisson brackets $\{U, V\}$, $\{U, W\}$, $\{V, W\}$, so that

$$0 = \{H, F\} = (\alpha U + \beta V + \gamma W) \cdot \Theta.$$

Here α, β, γ are certain minors of the matrix A and since F and H are non-proportional, some of them are non-zero, implying $\Theta = 0$. This yields $\{H, \zeta\} = 0$ and $\{H, \eta\} = 0$. Thus we have two Killing vector fields and $K = \text{const}$. □

This proposition immediately implies the following statement, known due to Kolokoltsev and Matveev ([Kol, M_1, BMF]):

Corollary 3. *The only closed Riemannian surfaces that admit more than one additional quadratic integrals are the standard round sphere and flat torus in the oriented case and the standard projective plane in the non-oriented one.*

Proof. Indeed, if the metric has non-constant curvature and two additional integrals then the Hamiltonian flow is resonant: every trajectory is given by equation $\{H = c_1, F = c_2, G = c_3\}$ and hence is closed. Thus (M^2, g) is either S^2 or $\mathbb{R}P^2$. An additional investigation of metrics on S^2 with all the geodesics closed leads to $K = \text{const}$. The standard round sphere has 3 Killing vector fields and thus 6 quadratic integrals. They all descend to the standard projective plane.

Consider now the case of constant curvature. If a closed surface has negative constant curvature, its metric is non-integrable [Koz, P]. For positive curvature we are already done. For zero curvature we get torus or Klein bottle. Torus has 2 Killing vector fields and the symmetric square yields 3 quadratic integrals. However a flat Klein bottle has only one Killing vector field and the number of quadratic integrals (including Hamiltonian) is two. □

Global classification of Liouville metrics is discussed in [Ki, IKS, Kol, M_2]. The final classification was achieved in [M_1], see the review [BMF] for other contributions, results and references.

12. Cubic integrals

Let us consider next the case of integrals of degree 3. Again for brevity sake we bring the metric to hyperbolic conformal form $H=e^{-\lambda(x,y)}p_xp_y$. Then function $F=u(x,y)p_x^3+v(x,y)p_x^2p_y+w(x,y)p_xp_y^2+\varrho(x,y)p_y^3$ is an integral of the geodesic flow if the following equations hold:

$$u_y=0,\qquad u_x+v_y+3u\lambda_x+v\lambda_y=0,\qquad v_x+w_y+2u\lambda_x+2v\lambda_y=0,$$
$$w_x+\varrho_y+w\lambda_x+3\varrho\lambda_y=0,\qquad \varrho_x=0.$$

Denoting this system of PDEs by $\mathcal{E}=\{E_1=E_2=E_3=E_4=E_5=0\}$ we get criterion of integrability:

$$E_6=\{E_1,E_2,E_3,E_4,E_5\}\ \mathrm{mod}(E_1,E_2,E_3,E_4,E_5)=0.$$

Multiplied by $\frac{2}{15}\,e^{\lambda}$ this PDE has the form $E_6=K_xu_{xx}+K_yv_{xx}+1^{\text{st}}$ order terms, and its vanishing is equivalent to $K=\text{const}$. In this and only in this case the dimension of the solution space for $\mathcal{E}$ is $\dim\mathcal{J}_3=10$.

If $K\neq\text{const}$, we add equation E_6 and get a new system $\mathcal{E}'=\mathcal{E}\cap\{E_6=0\}$, which has the following symbols: $\dim g_0=4,\dim g_1=3,\dim g_2=1,\dim g_3=0$. Thus its solution space has dimension at most 8. The only Spencer second δ-cohomology groups are: $H^{1,2}(\mathcal{E}')\simeq\mathbb{R}^1$ and $H^{2,2}(\mathcal{E}')\simeq\mathbb{R}^1$. Thus the Weyl tensor has two components W_2 and W_3. The first can be obtained as follows.

Prolongation of E_6 to 3rd jets together with the system $\mathcal{E}$ yields 17 third order PDEs, while there's 16 third order differential monomials. Elimination gives the following equation of order 2:

$$\begin{aligned}E_7=&-4320\,\mathrm{i}e^{\lambda}K_x^2I_3^4I_{4b}\,v_{xx}+64I_3K_x^5\big(12(7I_{4a}-9\,\mathrm{i}I_{4b}-\\&-2I_{4c})(I_{4a}-2\,\mathrm{i}I_{4b}-I_{4c})+5I_3(72I_2I_3^2-14I_{5a}+29\,\mathrm{i}I_{5b}+16I_{5c}-\mathrm{i}I_{5d})\big)\,u_x\\&-80\,\mathrm{i}e^{\lambda}K_x^2I_3^3\big((I_{5b}+I_{5d})K_x+162I_3I_{4b}\lambda_x\big)\,v_x\\&-4e^{2\lambda}K_xI_3^3\big(12(7I_{4a}-9\,\mathrm{i}I_{4b}-2I_{4c})(I_{4a}+2\,\mathrm{i}I_{4b}-I_{4c})\\&+5I_3(72I_2I_3^2-14I_{5a}-29\,\mathrm{i}I_{5b}+16I_{5c}+\mathrm{i}I_{5d})\big)\,w_x+0^{\text{th}}\text{ order terms}=0.\end{aligned}$$

Thus vanishing of E_7 implies $I_{4b}=0$ and $I_{5d}=0$, so that there is a Killing vector field. Moreover further investigation of coefficients gives $J_5=0$ and

$$50I_3^5+5I_2I_3^3(I_{4a}+4I_{4c})-I_{4c}(I_{4a}^2+3I_{4a}I_{4c}-4I_{4c}^2)=0.$$

This latter condition (notice the expression is similar to J_4, but different) leads however to contradiction: The conditions $I_{4b}=0,I_{5d}=0,J_5=0$ allows to express all invariants of order ≥ 5 through invariants I_2,I_3,I_{4a},I_{4c}. Applying ∇_1 to the above expression yields thus 3 polynomial equations on I_{4a},I_{4b}, which are compatible only with $I_3=0$. Thus we get:

Theorem 8. *If a metric g has non-constant curvature, then* $\dim\mathcal{J}_3\leq 7$.

In fact, we can continue and consider the system $\mathcal{E}'' = \mathcal{E}' \cap \{E_7 = 0\}$. It has symbols with $\dim g_0 = 4, \dim g_1 = 3, \dim g_2 = 0$. The non-zero second Spencer δ-cohomology group is $H^{1,2}(\mathcal{E}'') \simeq \mathbb{R}^3$ and the compatibility is the Frobenius condition on the second jets, which leads (by vanishing of coefficients) to a complicated overdetermined polynomial system on differential invariants of order ≤ 5 (higher order are expressed via these). Computer investigation indicates incompatibility, implying strict inequality in the above theorem.

In a similar way we can continue prolongation-projection method for 6 more times. Finally we arrive to high order (and highly non-linear in $I_{\sigma\tau}$) differential invariants, which express existence of at least one cubic integral. Intermediate steps give more invariants, describing super-integrable cases, but it is rather complicated to decide what is the precise number of the conditions (because there are relations via derivations ∇_1, ∇_2).

Moreover it seems that $\dim \mathscr{I}_3$ can be neither 6 nor 5, i.e. the next realized dimension of $\mathscr{I}_3$ after 10 is 4! However this has no proof so far (as well as the fact that this implies 4 quadratic integrals). To the reverse side we have:

Theorem 9. *If* $\dim \mathscr{I}_2 = 4$, *then* $\dim \mathscr{I}_3 = 4$.

Proof. We will exploit the following statement, which can be derived from the works of V.Matveev:

Lemma 3. *If two metrics g, $\bar{g}$ are projectively equivalent, then for any $k \ge 1$:* $\dim \mathscr{I}_k(g) = \dim \mathscr{I}_k(\bar{g})$.

Actually the statement holds for any $n = \dim M$ (to a certain extent this can be found in [TM] for $k = 2$, but the case of general k is similar): A local diffeomorphism $\varphi : (M, g) \to (\overline{M}, \bar{g})$ is a projective transformation iff the map

$$F \mapsto \bar{F} = (\det G)^{-\frac{k}{n+1}} \cdot \varphi_*(F)$$

is the isomorphism $\mathcal{J}_k(g) \simeq \mathcal{J}_k(\bar{g})$ $\forall k$. Here $\varphi_*(F) := (\varphi^{-1})^*(F) \circ \varphi$, $\varphi_* g$ is defined similarly and

$$G = \sharp^{\bar{g}} \circ \flat_{\varphi_* g} : T\overline{M} \to T\overline{M},$$

where $\flat_g : TM \to T^*M$, $\sharp^{\bar{g}} : T^*\overline{M} \to T\overline{M}$ are the natural morphisms of shifting indices.

Remark 7. *Denote $\mathcal{G}(g)$ the space of metrics geodesically equivalent to g ($\varphi =$* Id *above). Then according to* [TM] $\mathcal{G}(g) \simeq \mathcal{J}_2(g)$ *with the equivalence being given by*

$$\bar{g} \mapsto I = (\det G)^{\frac{2}{n+1}} \cdot \bar{g}.$$

By the results of [BMM] (now again $n = 2$) any (pseudo-) Riemannian metric g with $\dim \mathscr{I}_2(g) = 4$ is projectively equivalent to a metric of the family

$$g_0 = e^{3x}\, dx^2 + \sigma e^x\, dy^2 \;\simeq\; x \cdot ds_0^2, \qquad \sigma \neq 0.$$

Here ds_0^2 is the standard Euclidean or Minkovsky metric on $\mathbb{R}^2(x, y)$.

By the above remark and Lemma 3 it is enough to investigate $\mathscr{I}_3(g)$ for $g = g_0$ only. Since the latter representative for g_0 has the simplest form, this is an easy investigation and the result is dim $\mathscr{I}_3(g_0) = 4$. □

13. Higher order integrals

When we pass to integrals of degree n, as we noted at the beginning, the system $\mathcal{E}$ is given by $(n+2)$ equations on $(n+1)$ unknowns. This system is of finite type and is a generalized complete intersection, so by theorem C of [KL$_3$] its solution space has dimension $\frac{1}{2}(n+1)(n+2)$ iff the compatibility condition $K = \text{const}$ holds. Otherwise the dimension drops at least by 2:

Proposition 10. *If* $\dim \mathscr{I}_n \geq \frac{n^2+3n}{2}$*, then the inequality is strict and* $K =$ const.

Proof. Indeed, since the equations in the system $\mathcal{E}$ are linear of the first order $E_i = (\nabla_i^1 + \nabla_i^0)(\mathbf{u})$ and have constant coefficient of first order terms ∇_i^1, order of the multi-bracket $E_{n+3} = \{E_1, \dots, E_{n+2}\}$ drops (compared to expected $(n+1)$ in general) and becomes n in pure form and $(n-1)$ after reduction by equations E_i. So if the new equation is not zero, the symbols of the new system satisfy: $\dim g_i' \leq n+1-i$ for $i < n-1$ and $\dim g_{n-1}' = 1$, $\dim g_n' = 0$. Then $\dim \mathrm{Sol}(\mathcal{E}) \leq \sum \dim g_i' < \frac{n^2+3n}{2}$. □

Further steps of prolongation-projection generalize Darboux-Kœning theorem, but are more complicated. To understand this let us give more details on the integrals for the metric $g_0 = x \cdot (dx^2 + dy^2)$ from the previous section.

The Killing form is $K = q$ (with p, q being the momenta dual to x, y) and the 4 quadratic integrals are:

$$H = \frac{p^2+q^2}{x}, \quad K^2 = q^2, \quad F = yH - 2p\,K, \quad G = y^2H - 4(yp - xq)K.$$

The latter integral can be considered as the additional integral from the Kœning's theorem because $\{K, G\} = 2F$, but $\{K, F\} = H$ (with a slight difference in generators, these relations were also observed in [KKW]).

For cubic integrals we have: $\mathcal{J}_3(g_0) = \mathcal{J}_1(g_0) \cdot \mathcal{J}_2(g_0) = \langle HK, K^3, KF, KG \rangle$. In fact, Poisson brackets of $\mathcal{J}_2(g_0)$ give nothing new: $\{G, F\} = 16K^3$.

For integrals of higher degree we have: $\mathcal{J}_k(g_0) = \mathcal{J}_2(g_0) \cdot \mathcal{J}_{k-2}(g_0)$ for $k > 2$. There are however relations, which are generated by precisely 1 relation in degree 4: $HG - F^2 = 4K^4$. Thus for $d_k(g) = \dim \mathcal{J}_k(g)$ we have:

$$d_{2k}(g_0) = d_{2k+1}(g_0) = \dim S^k \mathcal{J}_2(g_0) - \dim S^{k-2} \mathcal{J}_2(g_0). \tag{5}$$

This implies:

Theorem 11. *If* $d_2(g) = 4$*, then* $d_{2k}(g) = d_{2k+1}(g) = (k+1)^2$.

Proof. Indeed, by the same argument as in the last proof $d_2(g) = 4$ implies $d_n(g) = d_n(g_0)$ and the latter quantity for $n = 2k$ or $2k + 1$ due to formula (5) equals $\binom{k+3}{3} - \binom{k+1}{3} = (k + 1)^2$. □

It is natural to expect that the cases from the last theorem are the next in prolongation-projection method after the space forms:

> **Conjecture.** If $K \neq \text{const}$, then $d_n(g) \leq ([\frac{n}{2}]+1)^2$ and the equality is attained for metrics with $d_2(g) = 4$.

For $n = 2$ this obviously holds, for $n = 3$ we supported this by arguments in the previous section, while for $n \geq 4$ this seems to be hardly treated via successive prolongation-projection scheme.

One is tempted to suggest a kind of monotonicity as an approach, i.e. that $d_2(g) < d_2(h)$ could imply $d_n(g) < d_n(h)$ for two metrics g, h, but this would be wrong. For instance there are Liouville metrics with $d_2(g) = 2, d_3(g) = 0$, but there are other metrics, for which the cubic integrals are the simplest polynomial integrals: $d_2(h) = 1, d_3(h) = 1$. Indeed, according to [Te] there are metrics g_k such that $d_{2k+1}(g_k) \geq 1$, while for $i \leq k$: $d_{2i-1}(g_k) = 0$, $d_{2i}(g_k) = 1$ (the latter is nonzero because Hamiltonian is always an integral).

This was proved in the loc.sit. paper via a simple calculation, but it also follows from our approach, because the criterion of existence of non-trivial integrals of degree n (i.e. $\mathcal{J}_n \neq 0$ for odd n and $d_n > 1$ for even n) is given by a criterion via differential invariants of order, which is monotonic in n. In particular, we can arrange $d_{2k+1}(g_k) = 1$ for the above sequence.

A Long formulae

Below are the expressions for the seventh order differential invariants involved in Theorem 5. The calculations are performed using MATHEMATICA[5].

[5] Copy of the notebook with detailed computations is available from the author.

$$
\begin{aligned}
\mathfrak{J}_1 = {} & 142500{I_2}^3{I_3}^9+216{I_{4a}}^6-125{I_3}^3{I_{5a}}^3+6125\mathrm{i}{I_3}^6I_{5a}I_{4b}-875\mathrm{i}{I_3}^3I_{5a}I_{6a}I_{4b}+350{I_3}^2{I_{5a}}^2{I_{4b}}^2\\
& -6125{I_3}^3I_{7a}{I_{4b}}^2-133525\mathrm{i}{I_3}^5{I_{4b}}^3-25725\mathrm{i}{I_3}^2I_{6a}{I_{4b}}^3+23030I_3I_{5a}{I_{4b}}^4+98784{I_{4b}}^6+1125\mathrm{i}{I_3}^3{I_{5a}}^2I_{5b}\\
& -73500{I_3}^6I_{4b}I_{5b}+14000{I_3}^3I_{6a}I_{4b}I_{5b}+48825\mathrm{i}{I_3}^2I_{5a}{I_{4b}}^2I_{5b}-12250\mathrm{i}I_3{I_{4b}}^4I_{5b}-16250{I_3}^3I_{5a}{I_{5b}}^2\\
& +51275{I_3}^2{I_{4b}}^2{I_{5b}}^2+36000\mathrm{i}{I_3}^3{I_{5b}}^3+5250{I_3}^3I_{5a}I_{4b}I_{6b}-26950{I_3}^2{I_{4b}}^3I_{6b}-45500\mathrm{i}{I_3}^3I_{4b}I_{5b}I_{6b}\\
& +12250\mathrm{i}{I_3}^3{I_{4b}}^2I_{7b}+36{I_{4a}}^5(-5\mathrm{i}I_{4b}-126I_{4c})-1350\mathrm{i}{I_3}^2{I_{5a}}^2I_{4b}I_{4c}-67375{I_3}^5{I_{4b}}^2I_{4c}+2800{I_3}^2I_{6a}{I_{4b}}^2I_{4c}\\
& +51905\mathrm{i}I_3I_{5a}{I_{4b}}^3I_{4c}+76440\mathrm{i}{I_{4b}}^5I_{4c}-24350{I_3}^2I_{5a}I_{4b}I_{5b}I_{4c}+171185I_3{I_{4b}}^3I_{5b}I_{4c}+85300\mathrm{i}{I_3}^2I_{4b}{I_{5b}}^2I_{4c}\\
& -42175\mathrm{i}{I_3}^2{I_{4b}}^2I_{6b}I_{4c}+2700{I_3}^2{I_{5a}}^2{I_{4c}}^2-25725\mathrm{i}{I_3}^5I_{4b}{I_{4c}}^2+175\mathrm{i}{I_3}^2I_{6a}I_{4b}{I_{4c}}^2-22490I_3I_{5a}{I_{4b}}^2{I_{4c}}^2\\
& +120288{I_{4b}}^4{I_{4c}}^2-3950\mathrm{i}{I_3}^2I_{5a}I_{5b}{I_{4c}}^2+90105\mathrm{i}I_3{I_{4b}}^2I_{5b}{I_{4c}}^2+6750{I_3}^2{I_{5b}}^2{I_{4c}}^2+11200{I_3}^2I_{4b}I_{6b}{I_{4c}}^2\\
& +13070\mathrm{i}I_3I_{5a}I_{4b}{I_{4c}}^3+28470\mathrm{i}{I_{4b}}^3{I_{4c}}^3+71440I_3I_{4b}I_{5b}{I_{4c}}^3-13315I_3I_{5a}{I_{4c}}^4+34122{I_{4b}}^2{I_{4c}}^4+25245\mathrm{i}I_3I_{5b}{I_{4c}}^4\\
& +2340\mathrm{i}I_{4b}{I_{4c}}^5+2556{I_{4c}}^6+6{I_{4a}}^4(942{I_{4b}}^2+635\mathrm{i}I_{4b}I_{4c}+4715{I_{4c}}^2-90I_3(I_{5a}-3\mathrm{i}I_{5b}-8I_{5c}))+3000{I_3}^3{I_{5a}}^2I_{5c}\\
& -30625\mathrm{i}{I_3}^6I_{4b}I_{5c}+875\mathrm{i}{I_3}^3I_{6a}I_{4b}I_{5c}+11550{I_3}^2I_{5a}{I_{4b}}^2I_{5c}-112210I_3{I_{4b}}^4I_{5c}-5750\mathrm{i}{I_3}^3I_{5a}I_{5b}I_{5c}\\
& -61075\mathrm{i}{I_3}^2{I_{4b}}^2I_{5b}I_{5c}+19750{I_3}^3{I_{5b}}^2I_{5c}+7000{I_3}^3I_{4b}I_{6b}I_{5c}+16700\mathrm{i}{I_3}^2I_{5a}I_{4b}I_{4c}I_{5c}-120505\mathrm{i}I_3{I_{4b}}^3I_{4c}I_{5c}\\
& +96800{I_3}^2I_{4b}I_{5b}I_{4c}I_{5c}-30950{I_3}^2I_{5a}{I_{4c}}^2I_{5c}+1315I_3{I_{4b}}^2{I_{4c}}^2I_{5c}+53650\mathrm{i}{I_3}^2I_{5b}{I_{4c}}^2I_{5c}-35470\mathrm{i}I_3I_{4b}{I_{4c}}^3I_{5c}\\
& +18320I_3{I_{4c}}^4I_{5c}-17875{I_3}^3I_{5a}{I_{5c}}^2-11900{I_3}^2{I_{4b}}^2{I_{5c}}^2+29125\mathrm{i}{I_3}^3I_{5b}{I_{5c}}^2-39850\mathrm{i}{I_3}^2I_{4b}I_{4c}{I_{5c}}^2\\
& +30700{I_3}^2{I_{4c}}^2{I_{5c}}^2+15000{I_3}^3{I_{5c}}^3+25{I_2}^2{I_3}^6(2298{I_{4a}}^2-6790{I_{4b}}^2+6I_{4a}(-785\mathrm{i}I_{4b}-1701I_{4c})+3900\mathrm{i}I_{4b}I_{4c}\\
& +7908{I_{4c}}^2+5I_3(-383I_{5a}+561\mathrm{i}I_{5b}+1888I_{5c}))+2625\mathrm{i}{I_3}^3I_{5a}I_{4b}I_{6c}+26950\mathrm{i}{I_3}^2{I_{4b}}^3I_{6c}-17500{I_3}^3I_{4b}I_{5b}I_{6c}\\
& -62300{I_3}^2{I_{4b}}^2I_{4c}I_{6c}-25025\mathrm{i}{I_3}^2I_{4b}{I_{4c}}^2I_{6c}-27125\mathrm{i}{I_3}^3I_{4b}I_{5c}I_{6c}-6\mathrm{i}{I_{4a}}^3(365{I_{4b}}^3-11213\mathrm{i}{I_{4b}}^2I_{4c}+3615I_{4b}{I_{4c}}^2\\
& -10395\mathrm{i}{I_{4c}}^3+5I_3(3I_{5a}(13I_{4b}+84\mathrm{i}I_{4c})+14I_{4c}(19I_{5b}-109\mathrm{i}I_{5c})+I_{4b}(-678\mathrm{i}I_{5b}+31I_{5c}+392\mathrm{i}I_{5d})))\\
& -6125\mathrm{i}{I_3}^2I_{5a}{I_{4b}}^2I_{5d}+17150\mathrm{i}I_3{I_{4b}}^4I_{5d}-51450{I_3}^2{I_{4b}}^2I_{5b}I_{5d}+22050{I_3}^2I_{5a}I_{4b}I_{4c}I_{5d}-62965I_3{I_{4b}}^3I_{4c}I_{5d}\\
& -39200\mathrm{i}{I_3}^2I_{4b}I_{5b}I_{4c}I_{5d}+7350\mathrm{i}I_3{I_{4b}}^2{I_{4c}}^2I_{5d}-21560I_3I_{4b}{I_{4c}}^3I_{5d}+18375\mathrm{i}{I_3}^2{I_{4b}}^2I_{5c}I_{5d}-39200{I_3}^2I_{4b}I_{4c}I_{5c}I_{5d}\\
& +8575{I_3}^2{I_{4b}}^2{I_{5d}}^2+8750{I_3}^3I_{5a}I_{4b}I_{6d}+31850{I_3}^2{I_{4b}}^3I_{6d}-14000\mathrm{i}{I_3}^3I_{4b}I_{5b}I_{6d}+37625\mathrm{i}{I_3}^2{I_{4b}}^2I_{4c}I_{6d}\\
& -14000{I_3}^2I_{4b}{I_{4c}}^2I_{6d}-21000{I_3}^3I_{4b}I_{5c}I_{6d}+5I_2{I_3}^3(1296{I_{4a}}^4-11025\mathrm{i}{I_3}^5I_{4b}+33614{I_{4b}}^4+6{I_{4a}}^3(-575\mathrm{i}I_{4b}\\
& -2534I_{4c})+29995\mathrm{i}{I_{4b}}^3I_{4c}+43420{I_{4b}}^2{I_{4c}}^2+12480\mathrm{i}I_{4b}{I_{4c}}^3+14316{I_{4c}}^4+2{I_{4a}}^2(12295{I_{4b}}^2+10665\mathrm{i}I_{4b}I_{4c}\\
& +20418{I_{4c}}^2-30I_3(36I_{5a}-59\mathrm{i}I_{5b}-239I_{5c}))-I_{4a}(17535\mathrm{i}{I_{4b}}^3+58410{I_{4b}}^2I_{4c}+30360\mathrm{i}I_{4b}{I_{4c}}^2+41244{I_{4c}}^3\\
& +10I_3(I_{5a}(-165\mathrm{i}I_{4b}-1267I_{4c})+7I_{4c}(249\mathrm{i}I_{5b}+734I_{5c})+3I_{4b}(1818I_{5b}+440\mathrm{i}I_{5c}-441I_{5d})))+5I_3(I_{5a}(4375{I_{4b}}^2\\
& +470\mathrm{i}I_{4b}I_{4c}-2102{I_{4c}}^2)+926{I_{4c}}^2(3\mathrm{i}I_{5b}+8I_{5c})+2I_{4b}I_{4c}(5984I_{5b}+325\mathrm{i}I_{5c}-2548I_{5d})+35\mathrm{i}{I_{4b}}^2(115I_{5b}+139\mathrm{i}I_{5c}\\
& -49I_{5d}))+25{I_3}^2(36{I_{5a}}^2+35\mathrm{i}I_{6a}I_{4b}+678{I_{5b}}^2-112I_{4b}I_{6b}+I_{5a}(-118\mathrm{i}I_{5b}-478I_{5c})+650\mathrm{i}I_{5b}I_{5c}+932{I_{5c}}^2\\
& -301\mathrm{i}I_{4b}I_{6c}-448I_{4b}I_{6d}))+{I_{4a}}^2(-7350\mathrm{i}{I_3}^5I_{4b}+6(7308{I_{4b}}^4+5975\mathrm{i}{I_{4b}}^3I_{4c}+31787{I_{4b}}^2{I_{4c}}^2+5835\mathrm{i}I_{4b}{I_{4c}}^3\\
& +9915{I_{4c}}^4)+5I_3(I_{5a}(1277{I_{4b}}^2+32\mathrm{i}I_{4b}I_{4c}-5363{I_{4c}}^2)+2423{I_{4c}}^2(3\mathrm{i}I_{5b}+8I_{5c})+2I_{4b}I_{4c}(12162I_{5b}+1489\mathrm{i}I_{5c}\\
& -5733I_{5d})+\mathrm{i}{I_{4b}}^2(5176I_{5b}+1102\mathrm{i}I_{5c}-1715I_{5d}))+150{I_3}^2(3{I_{5a}}^2+7\mathrm{i}I_{6a}I_{4b}+130{I_{5b}}^2-42I_{4b}I_{6b}+I_{5a}(-18\mathrm{i}I_{5b}\\
& -48I_{5c})+46\mathrm{i}I_{5b}I_{5c}+143{I_{5c}}^2-21\mathrm{i}I_{4b}I_{6c}-70I_{4b}I_{6d}))+12250\mathrm{i}{I_3}^3{I_{4b}}^2I_{7d}-6125\mathrm{i}{I_3}^2{I_{4b}}^3I_{6e}\\
& +14700{I_3}^2{I_{4b}}^2I_{4c}I_{6e}+I_{4a}(1225{I_3}^5I_{4b}(130I_{4b}+27\mathrm{i}I_{4c})-6\mathrm{i}(980{I_{4b}}^5-35756\mathrm{i}{I_{4b}}^4I_{4c}+16355{I_{4b}}^3{I_{4c}}^2\\
& -27203\mathrm{i}{I_{4b}}^2{I_{4c}}^3+3215I_{4b}{I_{4c}}^4-3941\mathrm{i}{I_{4c}}^5)+5I_3(I_{5a}(-3731\mathrm{i}{I_{4b}}^3-379{I_{4b}}^2I_{4c}-2412\mathrm{i}I_{4b}{I_{4c}}^2+6622{I_{4c}}^3)\\
& +14{I_{4c}}^3(-789\mathrm{i}I_{5b}-1054I_{5c})-7{I_{4b}}^3(2361I_{5b}-1513\mathrm{i}I_{5c}-1449I_{5d})+{I_{4b}}^2I_{4c}(-26097\mathrm{i}I_{5b}+239I_{5c}+5145\mathrm{i}I_{5d})\\
& +2I_{4b}{I_{4c}}^2(-17272I_{5b}+2151\mathrm{i}I_{5c}+6713I_{5d}))+25{I_3}^2(-2832\mathrm{i}I_{4b}{I_{5b}}^2+1547\mathrm{i}{I_{4b}}^2I_{6b}+{I_{5a}}^2(44\mathrm{i}I_{4b}-126I_{4c})\\
& +7I_{6a}I_{4b}(9I_{4b}-7\mathrm{i}I_{4c})-1050{I_{5b}}^2I_{4c}-196I_{4b}I_{6b}I_{4c}-2092I_{4b}I_{5b}I_{5c}-2422\mathrm{i}I_{5b}I_{4c}I_{5c}-26\mathrm{i}I_{4b}{I_{5c}}^2\\
& -2086I_{4c}{I_{5c}}^2+1232{I_{4b}}^2I_{6c}+1127\mathrm{i}I_{4b}I_{4c}I_{6c}+588\mathrm{i}I_{4b}I_{5b}I_{5d}+1078I_{4b}I_{5c}I_{5d}-2I_{5a}(7I_{4c}(-19\mathrm{i}I_{5b}-109I_{5c})\\
& +I_{4b}(53I_{5b}+9\mathrm{i}I_{5c}+196I_{5d}))+35\mathrm{i}{I_{4b}}^2I_{6d}+980I_{4b}I_{4c}I_{6d}-343{I_{4b}}^2I_{6e}))+6125{I_3}^3{I_{4b}}^2I_{7e}.
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{J}_2 = {} & 142500 I_2{}^3 I_3{}^9 + 216 I_{4a}{}^6 - 125 I_3{}^3 I_{5a}{}^3 + 2625\mathrm{i} I_3{}^6 I_{5a} I_{4b} - 875\mathrm{i} I_3{}^3 I_{5a} I_{6a} I_{4b} - 4550 I_3{}^2 I_{5a}{}^2 I_{4b}{}^2 \\
& - 6125 I_3{}^3 I_{7a} I_{4b}{}^2 + 126175\mathrm{i} I_3{}^5 I_{4b}{}^3 + 13475\mathrm{i} I_3{}^2 I_{6a} I_{4b}{}^3 - 23030 I_3 I_{5a} I_{4b}{}^4 - 98784 I_{4b}{}^6 + 625\mathrm{i} I_3{}^3 I_{5a}{}^2 I_{5b} \\
& - 66500 I_3{}^6 I_{4b} I_{5b} + 14000 I_3{}^3 I_{6a} I_{4b} I_{5b} - 32375\mathrm{i} I_3{}^2 I_{5a} I_{4b}{}^2 I_{5b} - 58310\mathrm{i} I_3 I_{4b}{}^4 I_{5b} - 19750 I_3{}^3 I_{5a} I_{5b}{}^2 \\
& - 1225 I_3{}^2 I_{4b}{}^2 I_{5b}{}^2 - 36000\mathrm{i} I_3{}^3 I_{5b}{}^3 + 8750 I_3{}^3 I_{5a} I_{4b} I_{6b} - 2450 I_3{}^2 I_{4b}{}^3 I_{6b} + 42000\mathrm{i} I_3{}^3 I_{4b} I_{5b} I_{6b} \\
& - 12250\mathrm{i} I_3{}^3 I_{4b}{}^2 I_{7b} + 36 I_{4a}{}^5 (19\mathrm{i} I_{4b} - 126 I_{4c}) - 1450\mathrm{i} I_3{}^2 I_{5a}{}^2 I_{4b} I_{4c} - 12075 I_3{}^5 I_{4b}{}^2 I_{4c} + 2800 I_3{}^2 I_{6a} I_{4b}{}^2 I_{4c} \\
& - 14455\mathrm{i} I_3 I_{5a} I_{4b}{}^3 I_{4c} - 175224\mathrm{i} I_{4b}{}^5 I_{4c} - 24350 I_3{}^2 I_{5a} I_{4b} I_{5b} I_{4c} - 58275 I_3 I_{4b}{}^3 I_{5b} I_{4c} - 95100\mathrm{i} I_3{}^2 I_{4b} I_{5b}{}^2 I_{4c} \\
& + 14525\mathrm{i} I_3{}^2 I_{4b}{}^2 I_{6b} I_{4c} + 2700 I_3{}^2 I_{5a}{}^2 I_{4c}{}^2 + 23975\mathrm{i} I_3{}^5 I_{4b} I_{4c}{}^2 + 175\mathrm{i} I_3{}^2 I_{6a} I_{4b} I_{4c}{}^2 - 21370 I_3 I_{5a} I_{4b}{}^2 I_{4c}{}^2 \\
& + 5544 I_{4b}{}^4 I_{4c}{}^2 + 6750\mathrm{i} I_3{}^2 I_{5a} I_{5b} I_{4c}{}^2 - 176695\mathrm{i} I_3 I_{4b}{}^2 I_{5b} I_{4c}{}^2 + 450 I_3{}^2 I_{5b}{}^2 I_{4c}{}^2 + 10500 I_3{}^2 I_{4b} I_{6b} I_{4c}{}^2 \\
& + 15210\mathrm{i} I_3 I_{5a} I_{4b} I_{4c}{}^3 - 85842\mathrm{i} I_{4b}{}^3 I_{4c}{}^3 + 14740 I_3 I_{4b} I_{5b} I_{4c}{}^3 - 13315 I_3 I_{5a} I_{4c}{}^4 + 23034 I_{4b}{}^2 I_{4c}{}^4 - 25875\mathrm{i} I_3 I_{5b} I_{4c}{}^4 \\
& - 7884\mathrm{i} I_{4b} I_{4c}{}^5 + 2556 I_{4c}{}^6 + 25 I_2{}^2 I_3{}^6 (2298 I_{4a}{}^2 - 770 I_{4b}{}^2 + I_{4a} (2890\mathrm{i} I_{4b} - 10206 I_{4c}) - 9780\mathrm{i} I_{4b} I_{4c} + 7908 I_{4c}{}^2 \\
& - 5 I_3 (383 I_{5a} + 351\mathrm{i} I_{5b} - 1888 I_{5c})) + 6 I_{4a}{}^4 (774 I_{4b}{}^2 - 2245\mathrm{i} I_{4b} I_{4c} + 4715 I_{4c}{}^2 - 30 I_3 (3 I_{5a} - 5\mathrm{i} I_{5b} - 24 I_{5c})) \\
& + 3000 I_3{}^3 I_{5a}{}^2 I_{5c} + 21875\mathrm{i} I_3{}^6 I_{4b} I_{5c} + 875\mathrm{i} I_3{}^3 I_{6a} I_{4b} I_{5c} + 26250 I_3{}^2 I_{5a} I_{4b}{}^2 I_{5c} + 112210 I_3 I_{4b}{}^4 I_{5c} \\
& + 5750\mathrm{i} I_3{}^3 I_{5a} I_{5b} I_{5c} + 44625\mathrm{i} I_3{}^2 I_{4b}{}^2 I_{5b} I_{5c} + 16250 I_3{}^3 I_{5b}{}^2 I_{5c} + 3500 I_3{}^3 I_{4b} I_{6b} I_{5c} + 14800\mathrm{i} I_3{}^2 I_{5a} I_{4b} I_{4c} I_{5c} \\
& + 101675\mathrm{i} I_3 I_{4b}{}^3 I_{4c} I_{5c} + 48500 I_3{}^2 I_{4b} I_{5b} I_{4c} I_{5c} - 30950 I_3{}^2 I_{5a} I_{4c}{}^2 I_{5c} + 54515 I_3 I_{4b}{}^2 I_{4c}{}^2 I_{5c} \\
& - 57150\mathrm{i} I_3{}^2 I_{5b} I_{4c}{}^2 I_{5c} + 2010\mathrm{i} I_3 I_{4b} I_{4c}{}^3 I_{5c} + 18320 I_3 I_{4c}{}^4 I_{5c} - 17875 I_3{}^3 I_{5a} I_{5c}{}^2 - 21700 I_3{}^2 I_{4b}{}^2 I_{5c}{}^2 \\
& - 30875\mathrm{i} I_3{}^3 I_{5b} I_{5c}{}^2 + 11150\mathrm{i} I_3{}^2 I_{4b} I_{4c} I_{5c}{}^2 + 30700 I_3{}^2 I_{4c}{}^2 I_{5c}{}^2 + 15000 I_3{}^3 I_{5c}{}^3 - 875\mathrm{i} I_3{}^3 I_{5a} I_{4b} I_{6c} \\
& + 2450\mathrm{i} I_3{}^2 I_{4b}{}^3 I_{6c} - 10500 I_3{}^3 I_{4b} I_{5b} I_{6c} - 7000 I_3{}^2 I_{4b}{}^2 I_{4c} I_{6c} + 24675\mathrm{i} I_3{}^2 I_{4b} I_{4c}{}^2 I_{6c} + 25375\mathrm{i} I_3{}^3 I_{4b} I_{5c} I_{6c} \\
& - 6 I_{4a}{}^3 (-3067\mathrm{i} I_{4b}{}^3 + 8161 I_{4b}{}^2 I_{4c} - 12365\mathrm{i} I_{4b} I_{4c}{}^2 + 10395 I_{4c}{}^3 + 5 I_3 (I_{5a} (87\mathrm{i} I_{4b} - 252 I_{4c}) + 14 I_{4c} (-15\mathrm{i} I_{5b} \\
& + 109 I_{5c}) + I_{4b} (818 I_{5b} - 353\mathrm{i} I_{5c} - 392 I_{5d}))) - 6825\mathrm{i} I_3{}^2 I_{5a} I_{4b}{}^2 I_{5d} + 53410\mathrm{i} I_3 I_{4b}{}^4 I_{5d} - 15750 I_3{}^2 I_{4b}{}^2 I_{5b} I_{5d} \\
& + 22050 I_3{}^2 I_{5a} I_{4b} I_{4c} I_{5d} - 45045 I_3 I_{4b}{}^3 I_{4c} I_{5d} + 44100\mathrm{i} I_3{}^2 I_{4b} I_{5b} I_{4c} I_{5d} + 33950\mathrm{i} I_3 I_{4b}{}^2 I_{4c}{}^2 I_{5d} \\
& - 21560 I_3 I_{4b} I_{4c}{}^3 I_{5d} - 5425\mathrm{i} I_3{}^2 I_{4b}{}^2 I_{5c} I_{5d} - 39200 I_3{}^2 I_{4b} I_{4c} I_{5c} I_{5d} + 8575 I_3{}^2 I_{4b}{}^2 I_{5d}{}^2 + 8750 I_3{}^3 I_{5a} I_{4b} I_{6d} \\
& - 2450 I_3{}^2 I_{4b}{}^3 I_{6d} + 17500\mathrm{i} I_3{}^3 I_{4b} I_{5b} I_{6d} - 9975\mathrm{i} I_3{}^2 I_{4b}{}^2 I_{4c} I_{6d} - 14000 I_3{}^2 I_{4b} I_{4c}{}^2 I_{6d} - 21000 I_3{}^3 I_{4b} I_{5c} I_{6d} \\
& + 5 I_2 I_3{}^3 (1296 I_{4a}{}^4 + 2275\mathrm{i} I_3{}^5 I_{4b} - 15974 I_{4b}{}^4 + 6\mathrm{i} I_{4a}{}^3 (281 I_{4b} + 2534\mathrm{i} I_{4c}) - 44681\mathrm{i} I_{4b}{}^3 I_{4c} - 10816 I_{4b}{}^2 I_{4c}{}^2 \\
& - 41376\mathrm{i} I_{4b} I_{4c}{}^3 + 14316 I_{4c}{}^4 + 2 I_{4a}{}^2 (12827 I_{4b}{}^2 - 16699\mathrm{i} I_{4b} I_{4c} + 20418 I_{4c}{}^2 - 30 I_3 (36 I_{5a} + 3\mathrm{i} I_{5b} - 239 I_{5c})) \\
& + 5 I_3 (I_{5a} (4151 I_{4b}{}^2 + 1266\mathrm{i} I_{4b} I_{4c} - 2102 I_{4c}{}^2) + 2 I_{4c}{}^2 (-1683\mathrm{i} I_{5b} + 3704 I_{5c}) + 2 I_{4b} I_{4c} (4178 I_{5b} - 3783\mathrm{i} I_{5c} - 2548 I_{5d}) \\
& - 7\mathrm{i} I_{4b}{}^2 (985 I_{5b} - 803\mathrm{i} I_{5c} - 615 I_{5d})) + I_{4a} (53221\mathrm{i} I_{4b}{}^3 - 30438 I_{4b}{}^2 I_{4c} + 73088\mathrm{i} I_{4b} I_{4c}{}^2 - 41244 I_{4c}{}^3 \\
& + 10 I_3 (I_{5a} (-263\mathrm{i} I_{4b} + 1267 I_{4c}) + 7 I_{4c} (243\mathrm{i} I_{5b} - 734 I_{5c}) + I_{4b} (-5258 I_{5b} + 1908\mathrm{i} I_{5c} + 1323 I_{5d}))) + 25 I_3{}^2 (36 I_{5a}{}^2 \\
& + 35\mathrm{i} I_{6a} I_{4b} + 762 I_{5b}{}^2 - 252 I_{4b} I_{6b} + I_{5a} (6\mathrm{i} I_{5b} - 478 I_{5c}) - 678\mathrm{i} I_{5b} I_{5c} + 932 I_{5c}{}^2 + 231\mathrm{i} I_{4b} I_{6c} - 448 I_{4b} I_{6d})) \\
& + I_{4a}{}^2 (-3150\mathrm{i} I_3{}^5 I_{4b} + 6 (1904 I_{4b}{}^4 - 29341\mathrm{i} I_{4b}{}^3 I_{4c} + 17339 I_{4b}{}^2 I_{4c}{}^2 - 19765\mathrm{i} I_{4b} I_{4c}{}^3 + 9915 I_{4c}{}^4) \\
& + 5 I_3 (I_{5a} (2621 I_{4b}{}^2 + 2936\mathrm{i} I_{4b} I_{4c} - 5363 I_{4c}{}^2) + I_{4c}{}^2 (-8235\mathrm{i} I_{5b} + 19384 I_{5c}) + 2 I_{4b} I_{4c} (10482 I_{5b} - 6767\mathrm{i} I_{5c} \\
& - 5733 I_{5d}) + I_{4b}{}^2 (-14024\mathrm{i} I_{5b} - 3622 I_{5c} + 6685\mathrm{i} I_{5d})) + 150 I_3{}^2 (3 I_{5a}{}^2 + 7\mathrm{i} I_{6a} I_{4b} + 158 I_{5b}{}^2 - 70 I_{4b} I_{6b} + I_{5a} (-10\mathrm{i} I_{5b} - \\
& 48 I_{5c}) - 46\mathrm{i} I_{5b} I_{5c} + 143 I_{5c}{}^2 + 7\mathrm{i} I_{4b} I_{6c} - 70 I_{4b} I_{6d})) - 12250\mathrm{i} I_3{}^3 I_{4b}{}^2 I_{7d} - 11025\mathrm{i} I_3{}^2 I_{4b}{}^3 I_{6e} + 14700 I_3{}^2 I_{4b}{}^2 I_{4c} I_{6e} \\
& + I_{4a} (175 I_3{}^5 I_{4b} (834 I_{4b} - 119\mathrm{i} I_{4c}) + 6 (17444\mathrm{i} I_{4b}{}^5 + 5572 I_{4b}{}^4 I_{4c} + 46581\mathrm{i} I_{4b}{}^3 I_{4c}{}^2 - 13791 I_{4b}{}^2 I_{4c}{}^3 \\
& + 10845\mathrm{i} I_{4b} I_{4c}{}^4 - 3941 I_{4c}{}^5) + 5 I_3 (I_{5a} (6321\mathrm{i} I_{4b}{}^3 - 1947 I_{4b}{}^2 I_{4c} - 5456\mathrm{i} I_{4b} I_{4c}{}^2 + 6622 I_{4c}{}^3) + 14 I_{4c}{}^3 (855\mathrm{i} I_{5b} \\
& - 1054 I_{5c}) - 2 I_{4b} I_{4c}{}^2 (9502 I_{5b} - 5507\mathrm{i} I_{5c} - 6713 I_{5d}) + 3 I_{4b}{}^2 I_{4c} (17421\mathrm{i} I_{5b} + 173 I_{5c} - 6125\mathrm{i} I_{5d}) \\
& + 7 I_{4b}{}^3 (575 I_{5b} - 2415\mathrm{i} I_{5c} + 197 I_{5d})) + 25 I_3{}^2 (3224\mathrm{i} I_{4b} I_{5b}{}^2 - 1281\mathrm{i} I_{4b}{}^2 I_{6b} + I_{5a}{}^2 (68\mathrm{i} I_{4b} - 126 I_{4c}) \\
& + 7 I_{6a} I_{4b} (9 I_{4b} - 7\mathrm{i} I_{4c}) - 966 I_{5b}{}^2 I_{4c} - 1700 I_{4b} I_{5b} I_{5c} + 2562\mathrm{i} I_{5b} I_{4c} I_{5c} + 334\mathrm{i} I_{4b} I_{5c}{}^2 - 2086 I_{4c} I_{5c}{}^2 + 700 I_{4b}{}^2 I_{6c} \\
& - 1029\mathrm{i} I_{4b} I_{4c} I_{6c} + 2 I_{5a} (7 I_{4c} (-15\mathrm{i} I_{5b} + 109 I_{5c}) + I_{4b} (17 I_{5b} - 201\mathrm{i} I_{5c} - 196 I_{5d})) - 784\mathrm{i} I_{4b} I_{5b} I_{5d} + 1078 I_{4b} I_{5c} I_{5d} \\
& - 301\mathrm{i} I_{4b}{}^2 I_{6d} + 980 I_{4b} I_{4c} I_{6d} - 343 I_{4b}{}^2 I_{6e})) + 6125 I_3{}^3 I_{4b}{}^2 I_{7e}.
\end{aligned}
$$

$$\begin{aligned}
\mathfrak{J}_3 ={}& 142500 I_2^3 I_3^9 + 216 I_{4a}^6 - 125 I_3^3 I_{5a}^3 - 30625\mathrm{i} I_3^6 I_{5a} I_{4b} + 2625\mathrm{i} I_3^3 I_{5a} I_{6a} I_{4b} + 4550 I_3^2 I_{5a}^2 I_{4b}^2 \\
&+ 6125 I_3^3 I_{7a} I_{4b}^2 - 341775\mathrm{i} I_3^5 I_{4b}^3 - 13475\mathrm{i} I_3^2 I_{6a} I_{4b}^3 - 24990 I_3 I_{5a} I_{4b}^4 - 98784 I_{4b}^6 - 3375\mathrm{i} I_3^3 I_{5a}^2 I_{5b} \\
&+ 147000 I_3^6 I_{4b} I_{5b} - 17500 I_3^3 I_{6a} I_{4b} I_{5b} + 29225\mathrm{i} I_3^2 I_{5a} I_{4b}^2 I_{5b} - 108290\mathrm{i} I_3 I_{4b}^4 I_{5b} + 24250 I_3^3 I_{5a} I_{5b}^2 \\
&- 31675 I_3^2 I_{4b}^2 I_{5b}^2 + 36000\mathrm{i} I_3^3 I_{5b}^3 - 14000 I_3^3 I_{5a} I_{4b} I_{6b} + 46550 I_3^2 I_{4b}^3 I_{6b} - 64750\mathrm{i} I_3^3 I_{4b} I_{5b} I_{6b} \\
&+ 24500\mathrm{i} I_3^3 I_{4b}^2 I_{7b} + 36\mathrm{i} I_{4a}^5 (29 I_{4b} + 126\mathrm{i} I_{4c}) + 12450\mathrm{i} I_3^2 I_{5a}^2 I_{4b} I_{4c} + 314825 I_3^5 I_{4b}^2 I_{4c} - 15750 I_3^2 I_{6a} I_{4b}^2 I_{4c} \\
&+ 91105\mathrm{i} I_3 I_{5a} I_{4b}^3 I_{4c} - 22344\mathrm{i} I_{4b}^5 I_{4c} - 23200 I_3^2 I_{5a} I_{4b} I_{5b} I_{4c} - 116305 I_3 I_{4b}^3 I_{5b} I_{4c} + 8300\mathrm{i} I_3^2 I_{4b} I_{5b}^2 I_{4c} \\
&- 43575\mathrm{i} I_3^2 I_{4b}^2 I_{6b} I_{4c} + 2700 I_3^2 I_{5a}^2 I_{4c}^2 + 91875\mathrm{i} I_3^5 I_{4b} I_{4c}^2 - 12775\mathrm{i} I_3^2 I_{6a} I_{4b} I_{4c}^2 - 67700 I_3 I_{5a} I_{4b}^2 I_{4c}^2 \\
&- 147336 I_{4b}^4 I_{4c}^2 + 36350\mathrm{i} I_3^2 I_{5a} I_{5b} I_{4c}^2 - 99945\mathrm{i} I_3 I_{4b}^2 I_{5b} I_{4c}^2 - 64350 I_3^2 I_{5b}^2 I_{4c}^2 + 39550 I_3^2 I_{4b} I_{6b} I_{4c}^2 \\
&+ 18680\mathrm{i} I_3 I_{5a} I_{4b} I_{4c}^3 - 105342\mathrm{i} I_{4b}^3 I_{4c}^3 - 55910 I_3 I_{4b} I_{5b} I_{4c}^3 - 13315 I_3 I_{5a} I_{4c}^4 + 4314 I_{4b}^2 I_{4c}^4 - 31635\mathrm{i} I_3 I_{5b} I_{4c}^4 \\
&- 12564\mathrm{i} I_{4b} I_{4c}^5 + 2556 I_{4c}^6 - 25 I_2^2 I_3^6 (-2298 I_{4a}^2 + 6 I_{4a}(-3475\mathrm{i} I_{4b} + 1701 I_{4c}) + 4(4270 I_{4b}^2 + 5235\mathrm{i} I_{4b} I_{4c} \\
&- 1977 I_{4c}^2) + 5 I_3 (383 I_{5a} + 2271\mathrm{i} I_{5b} - 1888 I_{5c})) - 6 I_{4a}^4 (-534 I_{4b}^2 + 3515\mathrm{i} I_{4b} I_{4c} - 4715 I_{4c}^2 + 90 I_3 (I_{5a} + 9\mathrm{i} I_{5b} - 8 I_{5c})) \\
&+ 3000 I_3^3 I_{5a}^2 I_{5c} + 116375\mathrm{i} I_3^6 I_{4b} I_{5c} - 14875\mathrm{i} I_3^3 I_{6a} I_{4b} I_{5c} - 26250 I_3^2 I_{5a} I_{4b}^2 I_{5c} + 148470 I_3 I_{4b}^4 I_{5c} \\
&+ 41750\mathrm{i} I_3^3 I_{5a} I_{5b} I_{5c} - 63525\mathrm{i} I_3^2 I_{4b}^2 I_{5b} I_{5c} - 83750 I_3^3 I_{5b}^2 I_{5c} + 50750 I_3^3 I_{4b} I_{6b} I_{5c} - 20350\mathrm{i} I_3^2 I_{5a} I_{4b} I_{4c} I_{5c} \\
&- 32795\mathrm{i} I_3 I_{4b}^3 I_{4c} I_{5c} + 21450 I_3^2 I_{4b} I_{5b} I_{4c} I_{5c} - 30950 I_3^2 I_{5a} I_{4c}^2 I_{5c} + 95455 I_3 I_{4b}^2 I_{4c}^2 I_{5c} - 92350\mathrm{i} I_3^2 I_{5b} I_{4c}^2 I_{5c} \\
&- 15670\mathrm{i} I_3 I_{4b} I_{4c}^3 I_{5c} + 18320 I_3 I_{4c}^4 I_{5c} - 17875 I_3^3 I_{5a} I_{5c}^2 + 21700 I_3^2 I_{4b}^2 I_{5c}^2 - 62875\mathrm{i} I_3^3 I_{5b} I_{5c}^2 \\
&+ 20150\mathrm{i} I_3^2 I_{4b} I_{4c} I_{5c}^2 + 30700 I_3^2 I_{4c}^2 I_{5c}^2 + 15000 I_3^3 I_{5c}^3 - 20125\mathrm{i} I_3^3 I_{5a} I_{4b} I_{6c} + 75950\mathrm{i} I_3^2 I_{4b}^3 I_{6c} \\
&+ 77000 I_3^3 I_{4b} I_{5b} I_{6c} + 25200 I_3^2 I_{4b}^2 I_{4c} I_{6c} + 40775\mathrm{i} I_3^2 I_{4b} I_{4c}^2 I_{6c} + 56875\mathrm{i} I_3^3 I_{4b} I_{5c} I_{6c} - 36750 I_3^3 I_{4b}^2 I_{7c} \\
&- 6125\mathrm{i} I_3^2 I_{5a} I_{4b}^2 I_{5d} + 65170\mathrm{i} I_3 I_{4b}^4 I_{5d} + 29400 I_3^2 I_{4b}^2 I_{5b} I_{5d} + 22050 I_3^2 I_{5a} I_{4b} I_{4c} I_{5d} + 7595 I_3 I_{4b}^3 I_{4c} I_{5d} \\
&+ 61250\mathrm{i} I_3^2 I_{4b} I_{5b} I_{4c} I_{5d} + 63210\mathrm{i} I_3 I_{4b}^2 I_{4c}^2 I_{5d} - 21560 I_3 I_{4b} I_{4c}^3 I_{5d} + 6125\mathrm{i} I_3^2 I_{4b}^2 I_{5c} I_{5d} - 39200 I_3^2 I_{4b} I_{4c} I_{5c} I_{5d} \\
&+ 8575 I_3^2 I_{4b}^2 I_{5d}^2 + 6 I_{4a}^3 (3317\mathrm{i} I_{4b}^3 - 3321 I_{4b}^2 I_{4c} + 19595\mathrm{i} I_{4b} I_{4c}^2 - 10395 I_{4c}^3 + 5 I_3 (I_{5a}(89\mathrm{i} I_{4b} + 252 I_{4c}) \\
&+ 14 I_{4c}(127\mathrm{i} I_{5b} - 109 I_{5c}) + I_{4b}(522 I_{5b} + 219\mathrm{i} I_{5c} + 392 I_{5d}))) + 8750 I_3^3 I_{5a} I_{4b} I_{6d} - 66150 I_3^2 I_{4b}^3 I_{6d} \\
&+ 29750\mathrm{i} I_3^3 I_{4b} I_{5b} I_{6d} - 17325\mathrm{i} I_3^2 I_{4b}^2 I_{4c} I_{6d} - 14000 I_3^2 I_{4b} I_{4c}^2 I_{6d} - 21000 I_3^3 I_{4b} I_{5c} I_{6d} + 5 I_2 I_3^3 (1296 I_{4a}^4 \\
&+ 47775\mathrm{i} I_3^5 I_{4b} - 15974 I_{4b}^4 + 6\mathrm{i} I_{4a}^3 (2943 I_{4b} + 2534\mathrm{i} I_{4c}) + 42007\mathrm{i} I_{4b}^3 I_{4c} - 102974 I_{4b}^2 I_{4c}^2 - 67008\mathrm{i} I_{4b} I_{4c}^3 \\
&+ 14316 I_{4c}^4 - 2 I_{4a}^2 (22927 I_{4b}^2 + 47787\mathrm{i} I_{4b} I_{4c} - 20418 I_{4c}^2 + 30 I_3 (36 I_{5a} + 275\mathrm{i} I_{5b} - 239 I_{5c})) - 5 I_3 (I_{5a}(5481 I_{4b}^2 \\
&- 32\mathrm{i} I_{4b} I_{4c} + 2102 I_{4c}^2) + 2 I_{4c}^2 (4755\mathrm{i} I_{5b} - 3704 I_{5c}) + 7\mathrm{i} I_{4b}^2 (131 I_{5b} + 503\mathrm{i} I_{5c} - 1155 I_{5d}) + 2 I_{4b} I_{4c} (6955 I_{5b} \\
&+ 4391\mathrm{i} I_{5c} + 2548 I_{5d})) + I_{4a}(-21007\mathrm{i} I_{4b}^3 + 142428 I_{4b}^2 I_{4c} + 144924\mathrm{i} I_{4b} I_{4c}^2 - 41244 I_{4c}^3 + 10 I_3 (I_{5a}(-761\mathrm{i} I_{4b} \\
&+ 1267 I_{4c}) + 7 I_{4c}(915\mathrm{i} I_{5b} - 734 I_{5c}) + I_{4b}(7290 I_{5b} + 5206\mathrm{i} I_{5c} + 1323 I_{5d}))) + 25 I_3^2 (36 I_{5a}^2 - 203\mathrm{i} I_{6a} I_{4b} - 1446 I_{5b}^2 \\
&+ 854 I_{4b} I_{6b} + I_{5a}(550\mathrm{i} I_{5b} - 478 I_{5c}) - 2342\mathrm{i} I_{5b} I_{5c} + 932 I_{5c}^2 + 1099\mathrm{i} I_{4b} I_{6c} - 448 I_{4b} I_{6d})) + I_{4a}^2 (36750\mathrm{i} I_3^5 I_{4b} \\
&- 6(56 I_{4b}^4 + 32091\mathrm{i} I_{4b}^3 I_{4c} + 6741 I_{4b}^2 I_{4c}^2 + 31435\mathrm{i} I_{4b} I_{4c}^3 - 9915 I_{4c}^4) - 5 I_3 (I_{5a}(5105 I_{4b}^2 + 3932\mathrm{i} I_{4b} I_{4c} + 5363 I_{4c}^2) \\
&+ I_{4c}^2 (24747\mathrm{i} I_{5b} - 19384 I_{5c}) + \mathrm{i} I_{4b}^2 (3364 I_{5b} + 8206\mathrm{i} I_{5c} - 9947 I_{5d}) + 2 I_{4b} I_{4c} (9648 I_{5b} + 5881\mathrm{i} I_{5c} + 5733 I_{5d})) \\
&+ 150 I_3^2 (3 I_{5a}^2 - 21\mathrm{i} I_{6a} I_{4b} - 194 I_{5b}^2 + 112 I_{4b} I_{6b} + 6\mathrm{i} I_{5a}(9 I_{5b} + 8\mathrm{i} I_{5c}) - 334\mathrm{i} I_{5b} I_{5c} + 143 I_{5c}^2 + 161\mathrm{i} I_{4b} I_{6c} \\
&- 70 I_{4b} I_{6d})) - 24500\mathrm{i} I_3^3 I_{4b}^2 I_{7d} - 23275\mathrm{i} I_3^2 I_{4b}^3 I_{6e} + 14700 I_3^2 I_{4b}^2 I_{4c} I_{6e} + I_{4a}(-1225 I_3^5 I_{4b}(262 I_{4b} + 105\mathrm{i} I_{4c}) \\
&+ 6(15484\mathrm{i} I_{4b}^5 + 33012 I_{4b}^4 I_{4c} + 40331\mathrm{i} I_{4b}^3 I_{4c}^2 + 8809 I_{4b}^2 I_{4c}^3 + 17275\mathrm{i} I_{4b} I_{4c}^4 - 3941 I_{4c}^5) + 5 I_3 (I_{5a}(-4711\mathrm{i} I_{4b}^3 \\
&+ 25945 I_{4b}^2 I_{4c} - 338\mathrm{i} I_{4b} I_{4c}^2 + 6622 I_{4c}^3) + 14 I_{4c}^3 (1527\mathrm{i} I_{5b} - 1054 I_{5c}) + \mathrm{i} I_{4b}^2 I_{4c} (25353 I_{5b} + 26897\mathrm{i} I_{5c} - 27489 I_{5d}) \\
&+ 7 I_{4b}^3 (93 I_{5b} - 13\mathrm{i} I_{5c} - 567 I_{5d}) + 2 I_{4b} I_{4c}^2 (13673 I_{5b} + 6791\mathrm{i} I_{5c} + 6713 I_{5d})) + 25 I_3^2 (-732\mathrm{i} I_{4b} I_{5b}^2 + 903\mathrm{i} I_{4b}^2 I_{6b} \\
&+ I_{5a}^2 (-118\mathrm{i} I_{4b} - 126 I_{4c}) + 7 I_{6a} I_{4b} (15 I_{4b} + 91\mathrm{i} I_{4c}) + 3738 I_{5b}^2 I_{4c} - 2254 I_{4b} I_{6b} I_{4c} + 452 I_{4b} I_{5b} I_{5c} + 5698\mathrm{i} I_{5b} I_{4c} I_{5c} \\
&- 6\mathrm{i} I_{4b} I_{5c}^2 - 2086 I_{4c} I_{5c}^2 - 1148 I_{4b}^2 I_{6c} - 2597\mathrm{i} I_{4b} I_{4c} I_{6c} + 2 I_{5a}(7 I_{4c}(-127\mathrm{i} I_{5b} + 109 I_{5c}) + I_{4b}(229 I_{5b} + 307\mathrm{i} I_{5c} \\
&- 196 I_{5d})) - 1470\mathrm{i} I_{4b} I_{5b} I_{5d} + 1078 I_{4b} I_{5c} I_{5d} - 7\mathrm{i} I_{4b}^2 I_{6d} + 980 I_{4b} I_{4c} I_{6d} - 343 I_{4b}^2 I_{6e})) + 6125 I_3^3 I_{4b}^2 I_{7e}.
\end{aligned}$$

$$
\begin{aligned}
\mathfrak{J}_4 = {} & 142500{I_2}^3{I_3}^9+216{I_{4a}}^6-125{I_3}^3{I_{5a}}^3-27125\mathrm{i}{I_3}^6I_{5a}I_{4b}+2625\mathrm{i}{I_3}^3I_{5a}I_{6a}I_{4b}-350{I_3}^2{I_{5a}}^2{I_{4b}}^2 \\
&+6125{I_3}^3I_{7a}{I_{4b}}^2+15925\mathrm{i}{I_3}^5{I_{4b}}^3+25725\mathrm{i}{I_3}^2I_{6a}{I_{4b}}^3+24990I_3I_{5a}{I_{4b}}^4+98784{I_{4b}}^6-2875\mathrm{i}{I_3}^3{I_{5a}}^2I_{5b} \\
&-7000{I_3}^6I_{4b}I_{5b}-10500{I_3}^3I_{6a}I_{4b}I_{5b}-63175\mathrm{i}{I_3}^2I_{5a}{I_{4b}}^2I_{5b}+37730\mathrm{i}I_3{I_{4b}}^4I_{5b}+11750{I_3}^3I_{5a}{I_{5b}}^2 \\
&+50225{I_3}^2{I_{4b}}^2{I_{5b}}^2-36000\mathrm{i}{I_3}^3{I_{5b}}^3-10500{I_3}^3I_{5a}I_{4b}I_{6b}-17150{I_3}^2{I_{4b}}^3I_{6b}+36750\mathrm{i}{I_3}^3I_{4b}I_{5b}I_{6b} \\
&+36\mathrm{i}{I_{4a}}^5(5I_{4b}+126\mathrm{i}I_{4c})+12550\mathrm{i}{I_3}^2{I_{5a}}^2I_{4b}I_{4c}-44275{I_3}^5{I_{4b}}^2I_{4c}-14350{I_3}^2I_{6a}{I_{4b}}^2I_{4c}-55615\mathrm{i}I_3I_{5a}{I_{4b}}^3I_{4c} \\
&-76440\mathrm{i}{I_{4b}}^5I_{4c}+19400{I_3}^2I_{5a}I_{4b}I_{5b}I_{4c}+169995I_3{I_{4b}}^3I_{5b}I_{4c}-86700\mathrm{i}{I_3}^2I_{4b}{I_{5b}}^2I_{4c}+64925\mathrm{i}{I_3}^2{I_{4b}}^2I_{6b}I_{4c} \\
&+2700{I_3}^2{I_{5a}}^2{I_{4c}}^2+42175\mathrm{i}{I_3}^5I_{4b}{I_{4c}}^2-12775\mathrm{i}{I_3}^2I_{6a}I_{4b}{I_{4c}}^2+10900I_3I_{5a}{I_{4b}}^2{I_{4c}}^2+120288{I_{4b}}^4{I_{4c}}^2 \\
&+25650\mathrm{i}{I_3}^2I_{5a}I_{5b}{I_{4c}}^2-69665\mathrm{i}I_3{I_{4b}}^2I_{5b}{I_{4c}}^2+57150{I_3}^2{I_{5b}}^2{I_{4c}}^2-10150{I_3}^2I_{4b}I_{6b}{I_{4c}}^2+16540\mathrm{i}I_3I_{5a}I_{4b}{I_{4c}}^3 \\
&-28470\mathrm{i}{I_{4b}}^3{I_{4c}}^3+117430I_3I_{4b}I_{5b}{I_{4c}}^3-13315I_3I_{5a}{I_{4c}}^4+34122{I_{4b}}^2{I_{4c}}^4+19485\mathrm{i}I_3I_{5b}{I_{4c}}^4-2340\mathrm{i}I_{4b}{I_{4c}}^5 \\
&+2556{I_{4c}}^6-25{I_2}^2{I_3}^6(-2298{I_{4a}}^2+2I_{4a}(-6625\mathrm{i}I_{4b}+5103I_{4c})+12(1085{I_{4b}}^2+605\mathrm{i}I_{4b}I_{4c}-659{I_{4c}}^2) \\
&+5I_3(383I_{5a}+1359\mathrm{i}I_{5b}-1888I_{5c}))-6{I_{4a}}^4(-942{I_{4b}}^2+635\mathrm{i}I_{4b}I_{4c}-4715{I_{4c}}^2+30I_3(3I_{5a}+23\mathrm{i}I_{5b}-24I_{5c})) \\
&+3000{I_3}^3{I_{5a}}^2I_{5c}+63875\mathrm{i}{I_3}^6I_{4b}I_{5c}-14875\mathrm{i}{I_3}^3I_{6a}I_{4b}I_{5c}-11550{I_3}^2I_{5a}{I_{4b}}^2I_{5c}-148470I_3{I_{4b}}^4I_{5c} \\
&+30250\mathrm{i}{I_3}^3I_{5a}I_{5b}I_{5c}+72975\mathrm{i}{I_3}^2{I_{4b}}^2I_{5b}I_{5c}+47750{I_3}^3{I_{5b}}^2I_{5c}-1750{I_3}^3I_{4b}I_{6b}I_{5c}-18450\mathrm{i}{I_3}^2I_{5a}I_{4b}I_{4c}I_{5c} \\
&+50225\mathrm{i}I_3{I_{4b}}^3I_{4c}I_{5c}+22950{I_3}^2I_{4b}I_{5b}I_{4c}I_{5c}-30950{I_3}^2I_{5a}{I_{4c}}^2I_{5c}-89825I_3{I_{4b}}^2{I_{4c}}^2I_{5c}+18450\mathrm{i}{I_3}^2I_{5b}{I_{4c}}^2I_{5c} \\
&-53150\mathrm{i}I_3I_{4b}{I_{4c}}^3I_{5c}+18320I_3{I_{4c}}^4I_{5c}-17875{I_3}^3I_{5a}{I_{5c}}^2+11900{I_3}^2{I_{4b}}^2{I_{5c}}^2-2875\mathrm{i}{I_3}^3I_{5b}{I_{5c}}^2 \\
&-30850\mathrm{i}{I_3}^2I_{4b}I_{4c}{I_{5c}}^2+30700{I_3}^2{I_{4c}}^2{I_{5c}}^2+15000{I_3}^3{I_{5c}}^3-16625\mathrm{i}{I_3}^3I_{5a}I_{4b}I_{6c}-46550\mathrm{i}{I_3}^2{I_{4b}}^3I_{6c} \\
&-49000{I_3}^3I_{4b}I_{5b}I_{6c}-34300{I_3}^2{I_{4b}}^2I_{4c}I_{6c}-8925\mathrm{i}{I_3}^2I_{4b}{I_{4c}}^2I_{6c}+4375\mathrm{i}{I_3}^3I_{4b}I_{5c}I_{6c}+12250{I_3}^3{I_{4b}}^2I_{7c} \\
&-5425\mathrm{i}{I_3}^2I_{5a}{I_{4b}}^2I_{5d}+5390\mathrm{i}I_3{I_{4b}}^4I_{5d}-65100{I_3}^2{I_{4b}}^2I_{5b}I_{5d}+22050{I_3}^2I_{5a}I_{4b}I_{4c}I_{5d}-18165I_3{I_{4b}}^3I_{4c}I_{5d} \\
&-22050\mathrm{i}{I_3}^2I_{4b}I_{5b}I_{4c}I_{5d}+36610\mathrm{i}I_3{I_{4b}}^2{I_{4c}}^2I_{5d}-21560I_3I_{4b}{I_{4c}}^3I_{5d}+29925\mathrm{i}{I_3}^2{I_{4b}}^2I_{5c}I_{5d}-39200{I_3}^2I_{4b}I_{4c}I_{5c}I_{5d} \\
&+8575{I_3}^2{I_{4b}}^2{I_{5d}}^2+6{I_{4a}}^3(365\mathrm{i}{I_{4b}}^3-11213{I_{4b}}^2I_{4c}+3615\mathrm{i}I_{4b}{I_{4c}}^2-10395{I_{4c}}^3+5I_3(I_{5a}(137\mathrm{i}I_{4b}+252I_{4c}) \\
&+14I_{4c}(93\mathrm{i}I_{5b}-109I_{5c})+I_{4b}(-146I_{5b}-165\mathrm{i}I_{5c}+392I_{5d})))+8750{I_3}^3I_{5a}I_{4b}I_{6d}+36750{I_3}^2{I_{4b}}^3I_{6d} \\
&-1750\mathrm{i}{I_3}^3I_{4b}I_{5b}I_{6d}+30275\mathrm{i}{I_3}^2{I_{4b}}^2I_{4c}I_{6d}-14000{I_3}^2I_{4b}{I_{4c}}^2I_{6d}-21000{I_3}^3I_{4b}I_{5c}I_{6d}+5I_2{I_3}^3(1296{I_{4a}}^4 \\
&+34475\mathrm{i}{I_3}^5I_{4b}-1666{I_{4b}}^4+6\mathrm{i}{I_{4a}}^3(2087I_{4b}+2534\mathrm{i}I_{4c})-37093\mathrm{i}{I_{4b}}^3I_{4c}+37470{I_{4b}}^2{I_{4c}}^2-13152\mathrm{i}I_{4b}{I_{4c}}^3 \\
&+14316{I_{4c}}^4-2{I_{4a}}^2(-5925{I_{4b}}^2+20423\mathrm{i}I_{4b}I_{4c}-20418{I_{4c}}^2+30I_3(36I_{5a}+213\mathrm{i}I_{5b}-239I_{5c}))+5I_3(I_{5a}(175{I_{4b}}^2 \\
&-764\mathrm{i}I_{4b}I_{4c}-2102{I_{4c}}^2)+2{I_{4c}}^2(-1683\mathrm{i}I_{5b}+3704I_{5c})+2I_{4b}I_{4c}(4787I_{5b}-283\mathrm{i}I_{5c}-2548I_{5d})-7\mathrm{i}{I_{4b}}^2(939I_{5b} \\
&-445\mathrm{i}I_{5c}-295I_{5d}))+I_{4a}(37093\mathrm{i}{I_{4b}}^3-36920{I_{4b}}^2I_{4c}+41476\mathrm{i}I_{4b}{I_{4c}}^2-41244{I_{4c}}^3+10I_3(I_{5a}(-333\mathrm{i}I_{4b}+1267I_{4c}) \\
&+7I_{4c}(423\mathrm{i}I_{5b}-734I_{5c})+I_{4b}(-2122I_{5b}+1978\mathrm{i}I_{5c}+1323I_{5d})))+25{I_3}^2(36{I_{5a}}^2-203\mathrm{i}I_{6a}I_{4b}+6{I_{5b}}^2+322I_{4b}I_{6b} \\
&+I_{5a}(426\mathrm{i}I_{5b}-478I_{5c})-1014\mathrm{i}I_{5b}I_{5c}+932{I_{5c}}^2+567\mathrm{i}I_{4b}I_{6c}-448I_{4b}I_{6d}))+{I_{4a}}^2(32550\mathrm{i}{I_3}^5I_{4b}+6(7308{I_{4b}}^4 \\
&-5975\mathrm{i}{I_{4b}}^3I_{4c}+31787{I_{4b}}^2{I_{4c}}^2-5835\mathrm{i}I_{4b}{I_{4c}}^3+9915{I_{4c}}^4)-5I_3(I_{5a}(1985{I_{4b}}^2+6836\mathrm{i}I_{4b}I_{4c}+5363{I_{4c}}^2) \\
&+{I_{4c}}^2(9243\mathrm{i}I_{5b}-19384I_{5c})-2I_{4b}I_{4c}(10944I_{5b}+2375\mathrm{i}I_{5c}-5733I_{5d})+\mathrm{i}{I_{4b}}^2(7108I_{5b}+5590\mathrm{i}I_{5c}-1547I_{5d})) \\
&+150{I_3}^2(3{I_{5a}}^2-21\mathrm{i}I_{6a}I_{4b}-94{I_{5b}}^2+84I_{4b}I_{6b}+I_{5a}(46\mathrm{i}I_{5b}-48I_{5c})-242\mathrm{i}I_{5b}I_{5c}+143{I_{5c}}^2+133\mathrm{i}I_{4b}I_{6c}-70I_{4b}I_{6d})) \\
&-18375\mathrm{i}{I_3}^2{I_{4b}}^3I_{6e}+14700{I_3}^2{I_{4b}}^2I_{4c}I_{6e}-I_{4a}(175{I_3}^5I_{4b}(22I_{4b}+427\mathrm{i}I_{4c})-6\mathrm{i}(980{I_{4b}}^5+35756\mathrm{i}{I_{4b}}^4I_{4c} \\
&+16355{I_{4b}}^3{I_{4c}}^2+27203\mathrm{i}{I_{4b}}^2{I_{4c}}^3+3215I_{4b}{I_{4c}}^4+3941\mathrm{i}{I_{4c}}^5)+5I_3(I_{5a}(-7693\mathrm{i}{I_{4b}}^3+295{I_{4b}}^2I_{4c}-2706\mathrm{i}I_{4b}{I_{4c}}^2 \\
&-6622{I_{4c}}^3)+14{I_{4c}}^3(117\mathrm{i}I_{5b}+1054I_{5c})+2I_{4b}{I_{4c}}^2(22249I_{5b}-3435\mathrm{i}I_{5c}-6713I_{5d})+7{I_{4b}}^3(3067I_{5b}+1925\mathrm{i}I_{5c} \\
&-2309I_{5d})+{I_{4b}}^2I_{4c}(-19041\mathrm{i}I_{5b}-4775I_{5c}+3969\mathrm{i}I_{5d}))+25{I_3}^2(-3868\mathrm{i}I_{4b}{I_{5b}}^2+2597\mathrm{i}{I_{4b}}^2I_{6b}+7I_{6a}I_{4b}(33I_{4b} \\
&-91\mathrm{i}I_{4c})+1722{I_{5b}}^2I_{4c}+98I_{4b}I_{6b}I_{4c}+2{I_{5a}}^2(71\mathrm{i}I_{4b}+63I_{4c})+3268I_{4b}I_{5b}I_{5c}-714\mathrm{i}I_{5b}I_{4c}I_{5c}+366\mathrm{i}I_{4b}{I_{5c}}^2 \\
&+2086I_{4c}{I_{5c}}^2-2352{I_{4b}}^2I_{6c}+441\mathrm{i}I_{4b}I_{4c}I_{6c}-2I_{5a}(7I_{4c}(-93\mathrm{i}I_{5b}+109I_{5c})+I_{4b}(367I_{5b}+499\mathrm{i}I_{5c}-196I_{5d})) \\
&+98\mathrm{i}I_{4b}I_{5b}I_{5d}-1078I_{4b}I_{5c}I_{5d}-329\mathrm{i}{I_{4b}}^2I_{6d}-980I_{4b}I_{4c}I_{6d}+343{I_{4b}}^2I_{6e}))+6125{I_3}^3{I_{4b}}^2I_{7e}.
\end{aligned}
$$

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