INVOLUTIVITY OF FIELD EQUATIONS

BORIS KRUGLIKOV

Abstract. We prove involutivity of Einstein and Einstein-Maxwell equations by calculating the Spencer cohomology of these systems. Relation with Cartan method is traced in details. Basic implications through Cartan-Kähler theory are derived.

Introduction

The race for deriving basic equations of special and general relativity in competition with respectively Poincaré and Hilbert is a dramatic story in Einstein's most creative period, providing reach material for history of science [W, LMP, Eb]. Various opinions on these rivalries almost draw a line between physical and mathematical comprehension.

On the other hand correspondence of E.Cartan and A.Einstein [CE], which lasted for 3 years, is a happy story and a good example of profound interrelations. It mainly concerns discussion of known and other possible field equations, which satisfy the basic requirement of the new Cartan theory of involutivity. During the whole period Cartan calls Einstein nothing but 'Cher et illustre Maître', while the response ranges from 'Dear Colleage' to 'My very dear M. Cartan'.

The two great scholars had never written together, though had coherent publications (in successive pages) in 1930. It is interesting that though the fundamental equations of general relativity were written in 1915, and some very important particular solutions were found right after this, not so much was known on their solution space before Cartan's contribution.

Even more this concerns the unified field equations. The reason is probably, as noted in [BCG^3], that 'they are a highly overdetermined system'. Cartan approached indeed through his theory of exterior differential systems [C$_1$], namely by calculating Cartan characters and verifying that they pass the Cartan test. This is quite an involved work.

In this paper we prove involutivity of the field equations using the formal theory of differential equations [S]: we calculate all Spencer cohomology of the system and check their vanishing in the prescribed range (together with vanishing of the structure tensor). By Serre's contribution to [GS] this is equivalent to Cartan test. Since chasing diagrams is considered nowadays standard, this turns out to be a reasonable path.

We do our calculations for both Einstein and Einstein-Maxwell system, leaving aside other possible field equations which can be treated similarly. We relate our calculations to those of Cartan which is not obvious, since the two theories – Cartan and Spencer – though accepted being equivalent, are not in direct correspondence. Finally we derive some simple but important implications using the Cartan-Kähler theorem.

Let us mention that in his papers [C$_2$] Cartan mostly considers the so-called unified field theory based on distant parallelism[1], which corresponds to Einstein system with non-zero torsion, so that the number of differential equations in the system is $16 + 6 = 22$, not 10 or $10 + 6 = 16$ as in the usual Einstein and Einstein-Maxwell equations. Involutivity of these two latter do not follow from involutivity of the former upon a specification. In addition, Cartan arguments by exhibiting relations between the equations but not proving they are all. The formal theory approach, adapted here, provides both rigorous and economic way to prove involutivity.

1. Background: jets, Spencer cohomology and all this

We will consider here only the theory of systems of PDEs of the same order k. The general theory, developed in [KL$_1$, KL$_3$], shall be useful for other purposes.

Thus let $\mathcal{E} = \mathcal{E}_k \subset J^k\pi$ be a submanifold in the space of k-jets of sections of a bundle $\pi : E \to M$, subject to certain regularity assumptions, which include the claim that $\pi_{k,k-1} : \mathcal{E}_k \to J^{k-1}\pi$

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[1]Specific references are vol.II p. 1199–1229 and vol.III-1 p.549–611.

is submersion. We let $\mathcal{E}_l = J^l\pi$ for $l < k$ and $\mathcal{E}_l = \mathcal{E}_k^{(l-k)}$ for $l > k$, where the latter space is the prolongation defined as

$$\mathcal{E}_k^{(l-k)} = \{[s]_x^l \in J^l\pi \,:\, j^k(s) \text{ is tangent to } \mathcal{E}_k \text{ at } [s]_x^k \text{ with order } (l-k)\}.$$

Equation $\mathcal{E}$ is called formally integrable if all the projections $\pi_{l,l-1} : \mathcal{E}_l \to \mathcal{E}_{l-1}$ are submersions.

Let us denote by N the tangent space to the fiber of π and by T the tangent space to M. Then the symbol spaces $g_t \subset S^tT^* \otimes N$ are the kernels of $d\pi_{t,t-1} : T\mathcal{E}_t \to T\mathcal{E}_{t-1}$. We obviously have $g_t = S^tT^* \otimes N$ for $t < k$ and the space g_k is determined by the equation, however the higher index spaces are difficult to calculate without knowledge of formal integrability.

Instead one considers formal prolongations defined as $g_t = (g_k \otimes S^{t-k}T^*) \cap (S^tT^* \otimes N)$ for $t > k$. These symbols are united into the Spencer δ-complex

$$0 \to g_t \to g_{t-1} \otimes T^* \xrightarrow{\delta} g_{t-2} \otimes \Lambda^2T^* \xrightarrow{\delta} \cdots \xrightarrow{\delta} g_{t-i} \otimes \Lambda^iT^* \to \dots \tag{1}$$

with morphisms δ being the symbols of the de Rham operator. The cohomology at the term $g_{t-i} \otimes \Lambda^iT^*$ is denoted by $H^{t-i,i}(\mathcal{E})$ and is called Spencer δ-cohomology of $\mathcal{E}$.

Formal theory of PDEs describes obstructions to formal integrability as elements $W_t \in H^{t-1,2}(\mathcal{E})$, called curvature, torsion, structure functions or Weyl tensors. Their vanishing is equivalent to formal integrability (and in certain cases to local integrability).

Symbolic system $g = \oplus g_t$ is called involutive if $H^{i,j}(g) = 0$ for all $i \neq k-1$ and $i + j > 0$. This is equivalent to fulfillment of Cartan test for the corresponding EDS (which in turn means a PDE system of the 1st order).

Equation $\mathcal{E}$ is called involutive if its symbolic system is involutive and in addition the only obstruction W_k vanishes. Thus involutive systems are formally integrable.

Advantage of involutive systems is that compatibility conditions should be calculated only at one order, while in general they exist in different places and one shall carry the whole prolongation-projection method through [KLV, KL$_3$, S]. Fortunately many equations of mathematical physics are involutive and we are going to prove this for relativity equations.

2. Einstein equations

We run the setup very briefly, referring to plentiful books on differential geometry and relativity for details ([B] is an excellent choice).

Let M be a (four-dimensional) manifold, g pseudo-Riemannian tensor (for relativity: of Lorentz signature (1,3)) with Ricci tensor Ric and scalar curvature R, Λ a cosmological constant and T the energy-momenta tensor. The Einstein equations [E, H] are:

$$\operatorname{Ric} - \tfrac{1}{2}R\,g + \Lambda\,g = T. \tag{2}$$

We will assume in this section that T is a given traceless tensor, so that it is a part of data and only the metric g is unknown (further on we'll treat the case, when T is a part of unknowns entering the equations).

Bianchi identity implies $\delta_g T = 0$, where $\delta_g : C^\infty(T^*M) \to \Omega^1 M$ is the divergence operator[2]. This is the first order PDE and so system (2) is not involutive unless $T = 0$. Thus in what follows in this section we'll concentrate on the vacuum case[3]: $T = 0$.

Tracing (2) by g yields $4\Lambda = R$, so that the Einstein equation $\mathcal{E}$ is equivalent to

$$\operatorname{Ric} = \tilde{T} = T + \Lambda\,g.$$

To understand this equation we need to study solvability of the Ricci operator[4]

$$\operatorname{Ric} : C^\infty(S^2_+T^*) \to C^\infty(S^2T^*),$$

which gives rise to the sequence of operators $\phi_{\text{Ric}} : J^{k+2}(S^2_+T^*) \to J^k(S^2T^*)$ with symbols $\sigma^{(k)}_{\text{Ric}} : S^{k+2}T^* \otimes S^2T^* \to S^kT^* \otimes S^2T^*$, described in [DT, Ga, B]. Symbols of the Einstein equations are precisely

$$g_{k+2} = \operatorname{Ker}(\sigma^{(k)}_{\text{Ric}}),\ k \geq 0,$$

[2]Should not be confused with Spencer δ-differential.

[3]DeTurck's idea [DT] is to use covariance of the left hand side $G[g]$ of (2) and change the equation to $G[g] = \varphi^*T$, where $\varphi : M \to M$ is a diffeomorphism, so that T is given while (g,φ) unknown. This system (coupled with compatibility $\delta_{\varphi_* g}T = 0$) is already involutive for any non-degenerate T (it is a system of mixed orders in the sense of [KL$_1$]). The proof is similar, but a bit more involved.

[4]Here $T = TM$ is the tangent bundle to M and T^* is the cotangent bundle. No confusion with energy-momenta tensor because from now on the latter vanishes (and also we work only with cotangent space).

and we let $g_0 = S^2T^*$, $g_1 = T^* \otimes S^2T^*$.

We calculate the Spencer cohomology of $\mathcal{E}$ (2) by constructing resolutions to the symbols of the Ricci operator. The first Spencer complex is exact. The second Spencer complex includes into the commutative diagram, implying $H^{1,1}(\mathcal{E}) = S^2T$ and $H^{2-i,i}(\mathcal{E}) = 0$ for $i \neq 1$:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & g_2 & \longrightarrow & g_1 \otimes T^* & \longrightarrow & g_0 \otimes \Lambda^2 T^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^2T^* \otimes S^2T^* & \longrightarrow & T^* \otimes S^2T^* \otimes T^* & \longrightarrow & S^2T^* \otimes \Lambda^2 T^* & \longrightarrow & 0 \\
 & & \sigma_{\mathrm{Ric}} \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^2T^* & \longrightarrow & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

In what follows we shorten S^kT^* to S^k, and use similar notation Λ^k for readability of the diagrams. The third Spencer complex includes into the commutative diagram, implying $H^{1,2}(\mathcal{E}) = T^*$ and $H^{3-i,i}(\mathcal{E}) = 0$ for $i \neq 2$:

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & g_3 & \longrightarrow & g_2 \otimes T^* & \longrightarrow & g_1 \otimes \Lambda^2 & \longrightarrow & g_0 \otimes \Lambda^3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^3 \otimes S^2 & \longrightarrow & S^2 \otimes S^2 \otimes T^* & \longrightarrow & T^* \otimes S^2 \otimes \Lambda^2 & \longrightarrow & S^2 \otimes \Lambda^3 & \longrightarrow & 0 \\
 & & \sigma^{(1)}_{\mathrm{Ric}} \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T^* \otimes S^2 & \longrightarrow & S^2 \otimes T^* & \longrightarrow & 0 & & 0 & & \\
 & & \sigma_{\delta_g} \downarrow & & \downarrow & & & & & & \\
0 & \longrightarrow & T^* & \longrightarrow & 0 & & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & 0 & & & & & & & &
\end{array}
$$

The next commutative diagram is already exact:

$$
\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & g_4 & \longrightarrow & g_3 \otimes T^* & \longrightarrow & g_2 \otimes \Lambda^2 & \longrightarrow & g_1 \otimes \Lambda^3 & \longrightarrow & g_0 \otimes \Lambda^4 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^4 \otimes S^2 & \longrightarrow & S^3 \otimes S^2 \otimes T^* & \longrightarrow & S^2 \otimes S^2 \otimes \Lambda^2 & \longrightarrow & T^* \otimes S^2 \otimes \Lambda^3 & \longrightarrow & S^2 \otimes \Lambda^4 & \longrightarrow & 0 \\
 & & \sigma^{(2)}_{\mathrm{Ric}} \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^2 \otimes S^2 & \longrightarrow & T^* \otimes S^2 \otimes T^* & \longrightarrow & S^2 \otimes \Lambda^2 & \longrightarrow & 0 & & 0 & & \\
 & & \sigma^{(1)}_{\delta_g} \downarrow & & \downarrow & & \downarrow & & & & & & \\
0 & \longrightarrow & T^* \otimes T^* & \longrightarrow & T^* \otimes T^* & \longrightarrow & 0 & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & & & \\
 & & 0 & & 0 & & & & & & & &
\end{array}
$$

and this extends to the commutative diagram for any $k \geq 0$, with exact rows and columns:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & g_{k+4} & \to & g_{k+3}\otimes T^* & \to & g_{k+2}\otimes\Lambda^2 & \to & g_{k+1}\otimes\Lambda^3 & \to & g_k\otimes\Lambda^4 & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S^{k+4}\otimes S^2 & \to & S^{k+3}\otimes S^2\otimes T^* & \to & S^{k+2}\otimes S^2\otimes\Lambda^2 & \to & S^{k+1}\otimes S^2\otimes\Lambda^3 & \to & S^k\otimes S^2\otimes\Lambda^4 & \to & 0\\
 & & \sigma_{\text{Ric}}^{(k+2)}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S^{k+2}\otimes S^2 & \to & S^{k+1}\otimes S^2\otimes T^* & \to & S^k\otimes S^2\otimes\Lambda^2 & \to & S^{k-1}\otimes S^2\otimes\Lambda^3 & \to & S^{k-2}\otimes S^2\otimes\Lambda^4 & \to & 0\\
 & & \sigma_{\delta_g}^{(k+1)}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S^{k+1}\otimes T^* & \to & S^k\otimes T^*\otimes T^* & \to & S^{k-1}\otimes T^*\otimes\Lambda^2 & \to & S^{k-2}\otimes T^*\otimes\Lambda^3 & \to & S^{k-3}\otimes T^*\otimes\Lambda^4 & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

Since $H^{k,l}(\mathcal{E}) = 0$ for $k \geq 2$ and all l, the symbolic system $g = \oplus g_k$ of $\mathcal{E}$ is involutive. To prove involutivity of PDE system $\mathcal{E}$ it is thus enough to check that the actual compatibility conditions belonging to $H^{1,2}(\mathcal{E})$ vanish. But these elements, as follows from calculation of their symbols, coincide with Bianchi relations and thus their equality to zero holds identically.

Notice that we don't use specific features of Lorentz geometry, and our arguments work generally (though the diagrams become larger). We have proved:

Theorem 1. *The only nonzero Spencer δ-cohomology of Einstein equation (2)$_{T=0}$ in any dimension and signature are*

$$H^{0,0}(\mathcal{E}) \simeq S^2T^*,\ H^{1,1}(\mathcal{E}) \simeq S^2T^*,\ H^{1,2}(\mathcal{E}) \simeq T^*.$$

The Einstein equation $\mathcal{E}$ is involutive.

3. Relation with Cartan approach

In Cartan theory involutivity is checked via Cartan characters, which are defined as follows. Consider a symbolic system g of order k:

$$g_i = S^iT^*\otimes N,\ i<k; \qquad g_k \subset S^kT^*\otimes N; \qquad g_i = g_k^{(i-k)} \subset S^iT^*\otimes N,\ i>k.$$

Let $0 = V_n^* \subset V_{n-1}^* \subset \cdots \subset V_0^* = T^*$ be a generic complete flag (difference of dimensions is 1; $n = \dim T = \dim M$). By definition

$$s_i = \dim(g_k \cap S^kV_{i-1}^*\otimes N) - \dim(g_k \cap S^kV_i^*\otimes N), \quad 1 \leq i \leq n$$

(the sequence monotonically decreases), so that $\dim g_k = s_1 + \cdots + s_n$.

Cartan test [C_2, Ma, BCG^3] claims that symbolic system g is involutive iff

$$\dim g_{k+1} = s_1 + 2s_2 + \cdots + ns_n.$$

In this case we also have $\dim g_{k+1} = s_1' + \cdots + s_n'$, where $s_i' = s_i + \cdots + s_n$ are the Cartan characters for the prolongation, i.e.

$$s_i' = \dim(g_{k+1} \cap S^{k+1}V_{i-1}^*\otimes N) - \dim(g_{k+1} \cap S^{k+1}V_i^*\otimes N), \quad 1 \leq i \leq n.$$

Thus we can calculate the dimensions of symbol spaces via Cartan characters:

$$\dim g_l = m\binom{n+l-1}{l},\ l<k, \qquad \dim g_l = \sum_{i=1}^n \binom{l-k+i-1}{i-1} s_i,\ i \geq k, \tag{3}$$

where $m = \dim N$, which we also denote by s_0.

Let us relate Cartan characters and Spencer δ-cohomology for an involutive system of pure order k. The only nontrivial dimensions of the latter are $h_0 = \dim H^{0,0} = m$ and $h_i = \dim H^{k-1,i}$, $1 \leq i \leq n$. To one side the relation is given by

Proposition 1. *The numbers $(h_0,\dots,h_n)$ are expressed through $(s_0,\dots,s_n)$ as: $h_0 = s_0$ and*

$$h_l = (-1)^l \sum_{j=1}^n s_j \sum_{i=0}^{l-1} (-1)^i \binom{n}{i}\binom{l+j-i-2}{j-1} + (-1)^l s_0 \sum_{i=l}^n (-1)^i \binom{n}{i}\binom{k+l+n-i-2}{n-1}$$

for $\quad 1 \leq i \leq n.$

Proof. Due to involutivity the Euler characteristic of Spencer complex (1) equals $(-1)^{t-k+1}h_{t-k+1}$ for $t \geq k$, zero for $0 < t < k$ and h_0 for $t = 0$. Calculating it directly as $\sum_{i=0}^{n}(-1)^i \dim g_{t-i}\binom{n}{i}$ and using (3) we get the result. □

The relations above are invertible, but we obtain the inverse formula from another idea.

Proposition 2. *The numbers $(s_0, \dots, s_n)$ are expressed through $(h_0, \dots, h_n)$ in triangular way: $s_0 = h_0$ and*

$$s_l = \binom{n+k-l-1}{k-1}h_0 + \sum_{i=n-l+1}^{n}(-1)^{n-l-i}\binom{i-1}{n-l}h_i, \qquad l = 1, \dots, n.$$

Proof. For $l \gg 1$ the expression $H_{\mathcal{E}}(l) = \sum_{i \leq l} \dim g_i$ is a polynomial, called Hilbert polynomial of g and so $\dim g_z = H_{\mathcal{E}}(z) - H_{\mathcal{E}}(z-1)$ is a polynomial too (for large integers $z = l$ and we extend it to the space of all $z \in \mathbb{C}$).

The Hilbert polynomial can be computed through the standard resolution of the symbolic module g^* [Gr, KL$_2$] and we get:

$$\dim g_z = \sum_{i,j}(-1)^j \dim H^{i,j}(g) \cdot \binom{z+n-i-j-1}{n-1}$$
$$= h_0\binom{z+n-1}{n-1} - h_1\binom{z+n-k-1}{n-1} + h_2\binom{z+n-k-2}{n-1} - \dots \quad (4)$$

On the other hand from (3) we have the following expression:

$$\dim g_z = \sum_{l=1}^{n} s_l \binom{z+l-k-1}{l-1}. \quad (5)$$

Comparing (4) to (5) we obtain the result: At first substitute $z = k-1$ and get[5]

$$s_1 = h_0\binom{n+k-2}{n-1} - h_n,$$

then calculate difference derivative by z, substitute $z = k-2$ and get the formula

$$s_2 = h_0\binom{n+k-3}{n-2} - h_{n-1} + h_n\binom{n-1}{n-2}$$

and so on. □

Remark 1. *To see that formulae of proposition 2 invert these of proposition 1 is not completely trivial: one must use certain combinatorial identities.*

Now let us apply the result to Einstein vacuum equations (we restrict to the physical dimension $n = 4$, but due to previous formulae the general case is easily restored). As we calculated in the previous section

$$h_0 = 10,\ h_1 = 10,\ h_2 = 4,\ h_3 = h_4 = 0.$$

Thus proposition 2 implies that the Cartan characters are

$$s_1 = 40,\ s_2 = 30,\ s_3 = 16,\ s_4 = 4.$$

This calculation can be independently verified in MAPLE with(DifferentialGeometry): [A].

In particular, the Cartan genre is 4 and the Cartan integer is 4, i.e. the general (analytic) solution of the Einstein vacuum equations depends on 4 functions of 4 arguments. This is indeed so due to covariance: the group $\mathrm{Diff}_{\mathrm{loc}}(M)$ acts on $\mathcal{E}$ as symmetries.

We can calculate the Hilbert polynomial of the Einstein equation

$$H_{\mathcal{E}}(z) = 10 + 22z + \frac{89z^2}{6} + 3z^3 + \frac{z^4}{6}.$$

The first dimensions of the symbol spaces are:

$$\dim g_0 = 10,\ \dim g_1 = 40,\ \dim g_2 = 90,\ \dim g_3 = 164,\ \dim g_4 = 266,\ \dim g_5 = 400, \dots$$

(this in particular shows that direct calculation can be costly). The Cartan test works as follows:

$$s_1 + 2s_2 + 3s_3 + 4s_4 = 164 = \dim g_3.$$

[5] One shall be careful: in this substitution $\dim g_z$ is understood as analytic continuation (5), because the actual value of $\dim g_{k-1}$ could be different; on the other hand studying the large values l one gets the same result.

4. Einstein-Maxwell equations

These equation extends (2) in the sense that the energy-momenta tensor is prescribed as electromagnetic tensor. Denote by J the current density. Einstein-Maxwell equations have the following form[6]:

$$\mathrm{Ric}-\tfrac{1}{2}R\,g=(F^2)_0,\quad dF=0,\quad \delta_gF=J. \tag{6}$$

Here the tensor F in the first equation is viewed as a (1,1)-tensor (an operator field) via the metric, and $(F^2)_0=F^2-\frac{1}{4}\,\mathrm{Tr}(F^2)$ is the traceless part of its square, while F in the latter equations is a 2-form, and $\delta_g=\pm*\,d*:\Omega^2M\to\Omega^1M$ is the Hodge codifferential[7].

In order not to deal with involutivity of systems of PDEs of different orders (the theory developed in [KL$_1$]), we can re-write the system as a pure 2nd order system by introducing the potential $A\in\Omega^1M$, $F=dA$:

$$\mathrm{Ric}-\tfrac{1}{2}R\,g=(dA\circ dA)_0,\quad \delta_g(dA)=J. \tag{7}$$

Both systems (6) and (7) have the following compatibility condition: $\delta_gJ=0$ of order 1 in g. Thus they are not involutive unless $J=0$. This we shall assume at the end of this section[8].

Let us study at first the pure Maxwell equation (with known g), written as a 2nd order system with operator $\Box=\delta_g\circ d$:

$$\Box(A)=J. \tag{8}$$

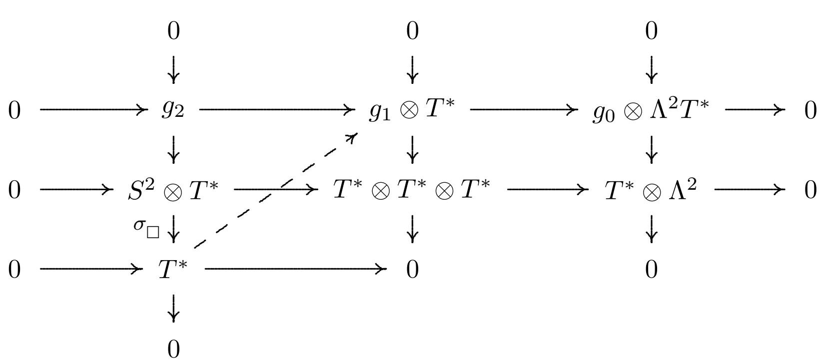
The symbol of this operator equals

$$\sigma_\Box:S^2T^*\otimes T^*\to T^*,\qquad Q\otimes p\mapsto p\lrcorner Q-\mathrm{Tr}(Q)p,$$

where in the first term to the right dualization $T^*\stackrel{g}{\simeq}T$ is used and in the second the trace is taken w.r.t. g. Thus the symbol is epimorphic, while its prolongations are not, since they have left divisor of zero:

$$\sigma^{(k-1)}_{\delta_g}\circ\sigma^{(k)}_\Box=0\quad\text{for}\quad \sigma_{\delta_g}:T^*\otimes T^*\to\mathbb{R},\quad q\otimes p\mapsto g(p,q).$$

The symbol of δ_g is however epimorphic together with all its prolongations and so we get the sequence of commutative diagrams with all rows and columns exact except for the top (Spencer δ-complex) and the bottom rows:



[6]We set the cosmological constant $\Lambda=0$, which does not restrict mathematics (can be incorporated back without destroying any conclusion), but agrees with physical observations.

[7]Not to be confused with Spencer δ-differential or (symmetric) divergence δ_g.

[8]Similar to DeTurck trick[3], we can change in the case $J\neq0$ the Einstein-Maxwell system to the following:

$$\mathrm{Ric}-\tfrac{1}{2}R\,g=(F^2)_0,\quad dF=0,\quad \delta_gF=\varphi^*J.$$

for the unknown (g,φ). Since for the diffeomorphism φ the equation is underdetermined, the above system (coupled with compatibility $\delta_{\varphi_*g}J=0$) is involutive provided J is non-vanishing. This leads to solvability (contrary to compatibility) of (6). However we will not discuss this result here.

This implies $H^{1,1} \simeq T^*$. The next complex

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & g_3 & \longrightarrow & g_2 \otimes T^* & \longrightarrow & g_1 \otimes \Lambda^2 & \longrightarrow & g_0 \otimes \Lambda^3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S^3 \otimes T^* & \longrightarrow & S^2 \otimes T^* \otimes T^* & \to & T^* \otimes T^* \otimes \Lambda^2 & \longrightarrow & T^* \otimes \Lambda^3 & \to & 0 \\
 & & \sigma_{\square}^{(1)} \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T^* \otimes T^* & \longrightarrow & T^* \otimes T^* & \longrightarrow & 0 & & 0 & & \\
 & & \sigma_{\delta_g} \downarrow & & \downarrow & & & & & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & 0 & & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & 0 & & & & & & & &
\end{array}
$$

yields $H^{1,2} \simeq \mathbb{R}$. Further complexes are already exact. Here's the next one:

$$
\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & g_4 & \longrightarrow & g_3 \otimes T^* & \longrightarrow & g_2 \otimes \Lambda^2 & \longrightarrow & g_1 \otimes \Lambda^3 & \longrightarrow & g_0 \otimes \Lambda^4 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S^4 \otimes T^* & \longrightarrow & S^3 \otimes T^* \otimes T^* & \to & S^2 \otimes T^* \otimes \Lambda^2 & \to & T^* \otimes T^* \otimes \Lambda^3 & \longrightarrow & T^* \otimes \Lambda^4 & \to & 0 \\
 & & \sigma_{\square}^{(2)} \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S^2 \otimes T^* & \longrightarrow & T^* \otimes T^* \otimes T^* & \longrightarrow & T^* \otimes \Lambda^2 & \longrightarrow & 0 & & 0 & & \\
 & & \sigma_{\delta_g}^{(1)} \downarrow & & \downarrow & & \downarrow & & & & & & \\
0 & \longrightarrow & T^* & \longrightarrow & T^* & \longrightarrow & 0 & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & & & \\
 & & 0 & & 0 & & & & & & & &
\end{array}
$$

and one can easily prolong. Since $\dim H^{*,2} = 1$, there is only one compatibility condition and from its symbol one easily identifies: $\delta_g J = 0$ (which comes from Hodge identity $\delta_g^2 = 0$). Thus we get:

Theorem 2. *The only nonzero Spencer δ-cohomology of Maxwell equation (8) in any dimension and signature are*

$$H^{0,0} \simeq T^*, \; H^{1,1} \simeq T^*, \; H^{1,2} \simeq \mathbb{R}.$$

The Maxwell equation is involutive (given that compatibility $\delta_g J = 0$ holds).

Now we can study Einstein-Maxwell equation with no external sources: $J = 0$ (in the presence of sources approach of footnote[8] shall apply).

The key observation is that this system is weakly uncoupled, meaning that its symbol splits into the sum of symbols of Einstein and Maxwell equations. Thus Spencer cohomology becomes the direct sum, and the compatibility condition for the system $\mathcal{EM}$ (7) is the union of two respective compatibility conditions (Bianchi and Hodge identities).

Theorem 3. *The only nonzero Spencer δ-cohomology of source-free Einstein-Maxwell equation $(7)_{J=0}$ in any dimension and signature are*

$$H^{0,0}(\mathcal{EM}) \simeq S^2T^* \oplus T^*, \; H^{1,1}(\mathcal{EM}) \simeq S^2T^* \oplus T^*, \; H^{1,2}(\mathcal{EM}) \simeq T^* \oplus \mathbb{R}.$$

This Einstein-Maxwell equation $\mathcal{EM}$ is involutive.

Couple of remark to properly place this results are of order.

Remark 2. *With Einstein-Maxwell system (6) we get $H^{0,0} = g_0 \simeq S^2 \oplus \Lambda^2 = T^* \otimes T^*$, which correspond to Rainich "already unified field theory"* [R] *(so that we don't have bosonic or fermionic parts, but just tensors). The other cohomology do not sum (since belong to different* *bi-grades) but unite and we get that the only non-vanishing Spencer cohomology of (6) are:*

$$H^{0,0}\simeq T^*\otimes T^*,\ H^{0,1}\simeq T^*\oplus\Lambda^3T^*,\ H^{1,1}\simeq S^2T^*,$$
$$H^{0,2}\simeq \mathbb{R}\oplus\Lambda^4T^*,\ H^{1,2}\simeq T^*,$$
$$H^{0,3}\simeq\Lambda^5T^*,\ H^{0,4}\simeq\Lambda^6T^*,\dots$$

Notice that for $n=4$ the latter line disappears.

Remark 3. *Following Rainich* [R] *Einstein-Maxwell equations are equivalent to the system*

$$(\mathrm{Ric}^2)_0=0,\ R=0,$$

where Ric *is viewed as operator and $L_0=L-\frac14\operatorname{Tr}(L)$ is the traceless part of an operator L. Though relation between the two systems is non-local (but rather simple integral), involutivity holds for them simultaneously (however Spencer cohomology vary, as well as Cartan numbers). The latter system, though more compact[9], is fully non-linear and is more complicated.*

5. Conclusion

There are other fields equations describing various physically relevant energy-momentum tensors, like pure radiation field and perfect fluid. Their investigation follows the proposed pattern.

Let us deduce several corollaries of the involutivity. They are based on the Cartan-Kähler theorem claiming that a formally integrable analytic system has local solutions. Since involutivity implies formal integrability, we conclude

Theorem 4. *Let j_0^kg be a jet of metric ($1<k<\infty$), which satisfies $(k-2)$-jet of the vacuum Einstein equations (2)$_{T=0}$. Then there exists a local analytic solution g of this equation with the prescribed jet j_0^kg at the point $0\in M$.*

In particular, if a Riemann tensor Riem_0 *at the point is given, which satisfies the obvious algebraic compatibility conditions with a metric $g_0\in S^2T_0^*M$ through (2), then there exists an analytic solution to the vacuum Einstein equations with the given initial data (g_0,Riem_0).*

This is a variation on Gasqui's theorem[10] [Ga]. If we consider non-vacuum Einstein equation (2), then we get similarly variation on DeTurck theorem [DT] using the footnote[3] provided that the tensor T is non-degenerate and analytic.

Turning now to Einstein-Maxwell equation (6) or (7) we arrive at

Theorem 5. *Let j_0^kg, j_0^kF be k-jets of a metric and an analytic 2-form, which are related by (jets of) source-free Einstein-Maxwell equation (7)$_{J=0}$ or (6)$_{J=0}$. Then there exists a local analytic solution (g,F) of this equation with the prescribed jet (j_0^kg,j_0^kF).*

In particular, if a metric $g_0\in S^2T_0^*M$, a Riemann tensor Riem_0 *and a 2-form $F_0\in\Lambda^2T_0^*M$ at the point $0\in M$ are given, which satisfy the algebraic compatibility conditions through the first equation of (6), then there exists an analytic solution to the source-free Einstein-Maxwell equations with the given initial data $(g_0,\mathrm{Riem}_0,F_0)$.*

We can extend this theorem by including electromagnetic source, getting a similar statement for the general Einstein-Maxwell equation (7) provided that J is non-vanishing and analytic.

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Institute of Mathematics and Statistics, University of Tromsø, Tromsø 90-37, Norway.
E-mail address: boris.kruglikov@uit.no