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# Geodesic Webs and PDE Systems of Euler Equations

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**Abstract**

We find necessary and sufficient conditions for the foliation defined by level sets of a function $f(x_1, ..., x_n)$ to be totally geodesic in a torsion-free connection and apply them to find the conditions for $d$-webs of hypersurfaces to be geodesic, and in the case of flat connections, for $d$-webs ($d \geq n+1$) of hypersurfaces to be hyperplanar webs. These conditions are systems of generalized Euler equations, and for flat connections we give an explicit construction of their solutions.

## 1 Introduction

In this paper we study necessary and sufficient conditions for the foliation defined by level sets of a function to be totally geodesic in a torsion-free connection on a manifold and find necessary and sufficient conditions for webs of hypersurfaces to be geodesic. These conditions has the form of a second-order PDE system for web functions. The system has an infinite pseudogroup of symmetries and the factorization of the system with respect to the pseudogroup leads us to a first-order PDE system. In the planar case (cf. [1]), the system coincides with the classical Euler equation and therefore can be solved in a constructive way. We provide a method to solve the system in arbitrary dimension and flat connection.

## 2 Geodesic Foliations and Flex Equations

Let $M^n$ be a smooth manifold of dimension $n$. Let vector fields $\partial_1, ..., \partial_n$ form a basis in the tangent bundle, and let $\omega^1, .., \omega^n$ be the dual basis. Then

$$[\partial_i, \partial_j] = \sum_k c_{ij}^k \partial_k$$

for some functions $c_{ij}^k \in C^\infty(M)$, and

$$d\omega^k + \sum_{i<j} c_{ij}^k \omega^i \wedge \omega^j = 0.$$

Let $\nabla$ be a linear connection in the tangent bundle, and let $\Gamma_{ij}^k$ be the Christoffel symbols of second type. Then

$$\nabla_i(\partial_j) = \sum_k \Gamma_{ij}^k \partial_k,$$

where $\nabla_i \stackrel{\text{def}}{=} \nabla_{\partial_i}$, and

$$\nabla_i(\omega^k) = -\sum_j \Gamma_{ij}^k \omega^j.$$

In [1] we proved the following result.

**Theorem 1** *The foliation defined by the level sets of a function* $f(x_1, ..., x_n)$ *is totally geodesic in a torsion-free connection* $\nabla$ *if and only if the function* $f$ *satisfies the following system of PDEs:*

$$\frac{\partial_i(f_i)}{f_i f_i} - \frac{\partial_i(f_j) + \partial_j(f_i)}{f_i f_j} + \frac{\partial_j(f_j)}{f_j f_j} = \sum_k \left( \Gamma_{ii}^k \frac{f_k}{f_i f_i} + \Gamma_{jj}^k \frac{f_k}{f_j f_j} - (\Gamma_{ij}^k + \Gamma_{ji}^k) \frac{f_k}{f_i f_j} \right) \tag{1}$$

*for all* $i < j, i, j = 1, ..., n$; *here* $f_i = \dfrac{\partial f}{\partial x_i}$.

We call such a system a *flex system.*

Note that conditions (1) can be used to obtain necessary and sufficient conditions for a $d$-web formed by the level sets of the functions $f_\alpha(x_1, \ldots, x_n), \alpha = 1, \ldots, d$, to be a *geodesic $d$-web,* i.e., to have the leaves of all its foliations to be totally geodesic: one should apply conditions (1) to the all web functions $f_\alpha, \alpha = 1, \ldots, d$,

## 2.1 Geodesic Webs on Manifolds of Constant Curvature

In what follows, we shall use the following definition.

**Definition 2** *We call by* $(\text{Flex } f)_{ij}$ *the following function:*

$$(\text{Flex } f)_{ij} = f_j^2 f_{ii} - 2 f_i f_j f_{ij} + f_i^2 f_{jj},$$

*where* $i, j = 1, ..., n$, $f_i = \dfrac{\partial f}{\partial x_i}$ *and* $f_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$.

It is easy to see that $(\text{Flex } f)_{ij} = (\text{Flex } f)_{ji}$, and $(\text{Flex } f)_{ii} = 0$.

**Proposition 3** *Let* $(\mathbb{R}^n, g)$ *be a manifold of constant curvature with the metric tensor*

$$g = \frac{dx_1^2 + ... + dx_n^2}{\left(1 + \kappa\left(x_1^2 + ... + x_n^2\right)\right)^2},$$

*where* $\kappa$ *is a constant. Then the level sets of a function* $f(x_1, ..., x_n)$ *are geodesics of the metric* $g$ *if and only if the function* $f$ *satisfies the following PDE system:*

$$(\text{Flex } f)_{ij} = \frac{2\kappa\left(f_i^2 + f_j^2\right)}{1 + \kappa\left(x_1^2 + ... + x_n^2\right)} \sum_k x_k f_k \tag{2}$$

*for all* $i, j$.

**Proof.** To prove formula (2), first note that the components of the metric tensor $g$ are

$$g_{ii} = b^2, \; g_{ij} = 0, \;\; i \neq j,$$

where

$$b = \frac{1}{1 + \kappa\left(x_1^2 + ... + x_n^2\right)}.$$

It follows that

$$g^{ii} = g_{ii}^{-1}, \; g^{ij} = 0, \;\; i \neq j.$$

We compute $\Gamma^i_{jk}$ using the classical formula

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right) \tag{3}$$

and get

$$\Gamma^k_{ii} = 2\kappa x_k b, \; k \neq i; \; \Gamma^i_{ii} = -2\kappa x_i b; \; \Gamma^k_{ij} = 0, \; i, j \neq k, \; i \neq j;$$
$$\Gamma^i_{ij} = -2\kappa x_j b, \; i \neq j; \; \Gamma^j_{ij} = -2\kappa x_i b, \;\; i \neq j.$$

Substituting these values of $\Gamma^i_{jk}$ into the right-hand side of formula (1), we get formula (2). ■

Note that if $n = 2$, then PDE system (2) reduces to the single equation

$$\text{Flex } f = \frac{2\kappa\left(x_1 f_1 + x_2 f_2\right)\left(f_1^2 + f_2^2\right)}{1 + \kappa\left(x_1^2 + x_2^2\right)},$$

where $\text{Flex } f = (\text{Flex } f)_{12}$.

This formula coincides with the corresponding formula in [1].

We rewrite formula (2) as follows:

$$\frac{(\text{Flex } f)_{ij}}{f_i^2 + f_j^2} = 2\kappa b \sum_k x_k f_k. \tag{4}$$

The left-hand side of equation (4) does not depend on $i$ and $j$. Thus we have

$$\frac{(\mathrm{Flex}\ f)_{ij}}{f_i^2+f_j^2}=\frac{(\mathrm{Flex}\ f)_{kl}}{f_k^2+f_l^2}$$

for any $i, j, k$, and $l$.

It follows that *if*

$$(\mathrm{Flex}\ f)_{ij}=0 \tag{5}$$

*for some fixed $i$ and $j$, then* (5) *holds for any $i$ and $j$.*

In other words, one has the following result.

**Theorem 4** *Let $W$ be a geodesic $d$-web on the manifold $(\mathbb{R}^n, g)$ given by web-functions $\left\{f^1, ...., f^d\right\}$ such that $(f_k^a)^2+(f_l^a)^2\neq 0$ for all $a=1,...,d$ and $k,l=1,2...,n$. Assume that the intersections of $W$ with the planes $(x_{i_0}, x_{j_0})$, for given $i_0$ and $j_0$, are linear planar $d$-webs. Then the intersection of $W$ with arbitrary planes $(x_i, x_j)$ are linear webs too.*

## 2.2 Geodesic Webs on Hypersurfaces in $\mathbb{R}^n$

**Proposition 5** *Let $(M, g)\subset\mathbb{R}^n$ be a hypersurface defined by an equation $x_n=u(x_1,...,x_{n-1})$ with the induced metric $g$ and the Levi-Civita connection $\nabla$. Then the foliation defined by the level sets of a function $f(x_1,...,x_{n-1})$ is totally geodesic in the connection $\nabla$ if and only if the function $f$ satisfies the following system of PDEs:*

$$(\mathrm{Flex}\ f)_{ij}=\frac{u_1f_1+...+u_{n-1}f_{n-1}}{1+u_1^2+...+u_{n-1}^2}(f_j^2u_{ii}-2f_if_ju_{ij}+f_i^2u_{jj}). \tag{6}$$

**Proof.** To prove formula (6), note that the metric induced by a surface $x_n=u(x_1,\dots,x_{n-1})$ is

$$g=ds^2=\sum_{k=1}^{n-1}(1+u_k^2)dx_k^2+2\sum_{i,j=1(i\neq j)}^{n-1}u_iu_jdx_idx_j.$$

Thus the metric tensor $g$ has the following matrix:

$$(g_{ij})=\begin{pmatrix} 1+u_1^2 & u_1u_2 & \dots & u_1u_{n-1} \\ u_2u_1 & 1+u_2^2 & \dots & u_2u_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ u_1 & u_{n-1}u_2 & \dots & 1+u_{n-1}^2, \end{pmatrix}$$

and the inverse tensor $g^{-1}$ has the matrix

$$(g^{ij}) = \frac{1}{1+\sum\limits_{k=1}^{n-1}(1+u_k^2)} \begin{pmatrix} \sum\limits_{k=2}^{n-1}(1+u_k^2) & -u_1u_2 & \dots & -u_1u_{n-1} \\ -u_2u_1 & \sum\limits_{k=1(k\neq 2)}^{n-1}(1+u_k^2) & \dots & -u_2u_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -u_{n-1}u_1 & -u_{n-1}u_2 & \dots & \sum\limits_{k=1}^{n-2}(1+u_k^2) \end{pmatrix}.$$

Computing $\Gamma^i_{jk}$ by formula (3), we find that

$$\Gamma^k_{ij} = \frac{u_k u_{ij}}{1+\sum\limits_{k=1}^{n-1}(1+u_k^2)}.$$

Applying these formulas to the right-hand side of (1), we get formula (6). ■

We rewrite equation (6) in the form

$$\frac{(\text{Flex } f)_{ij}}{f_j^2 u_{ii} - 2f_i f_j u_{ij} + f_i^2 u_{jj}} = \frac{u_1 f_1 + ... + u_n f_n}{1+u_1^2+...+u_n^2}. \tag{7}$$

It follows that the left-hand side of (7) does not depend on $i$ and $j$, i.e., we have

$$\frac{(\text{Flex } f)_{ij}}{f_j^2 u_{ii} - 2f_i f_j u_{ij} + f_i^2 u_{jj}} = \frac{(\text{Flex } f)_{kl}}{f_l^2 u_{kk} - 2f_k f_l u_{kl} + f_k^2 u_{ll}}$$

for any $i, j, k$ and $l$. This means that *if*

$$(\text{Flex } f)_{ij} = 0$$

*for some fixed $i$ and $j$, then*

$$(\text{Flex } f)_{kl} = 0$$

*for any $k$ and $l$.*

In other words, we have a result similar to the result in Theorem 4.

**Theorem 6** *Let $W$ be a geodesic $d$-web on the hypersurface $(M, g)$ given by web functions $\{f^1, ...., f^d\}$ such that $(f_j^a)\, u_{ii} - 2f_i^a f_j^a u_{ij} + (f_i^a)^2 u_{jj} \neq 0$, for all $a = 1, ..., d$ and $k, l = 1, 2..., n$. Assume that the intersections of $W$ with the planes $(x_{i_0}, x_{j_0})$, for given $i_0$ and $j_0$, are linear planar $d$-webs. Then the intersection of $W$ with arbitrary planes $(x_i, x_j)$ are linear webs too.*

# 3 Hyperplanar Webs

In this section we consider hyperplanar geodesic webs in $\mathbb{R}^n$ endowed with a flat linear connection $\nabla$.

In what follows, we shall use coordinates $x_1, \ldots, x_n$ in which the Christoffel symbols $\Gamma^i_{jk}$ of $\nabla$ vanish.

The following theorem gives us a criterion for a web of hypersurfaces to be hyperplanar.

**Theorem 7** *Suppose that a d-web of hypersurfaces, $d \geq n+1$, is given locally by web functions $f_\alpha(x_1, \ldots, x_n), \alpha = 1, ..., d$. Then the web is hyperplanar if and only if the web functions satisfy the following PDE system:*

$$(\text{Flex } f)_{st} = 0, \tag{8}$$

*for all $s < t = 1, ..., n$.*

**Proof.** For the proof, one should apply Theorem 1 to all foliations of the web. ■

In order to integrate the above PDEs system, we introduce the functions

$$A_s = \frac{f_s}{f_{s+1}}, \quad s = 1, ..., n-1,$$

and the vector fields

$$X_s = \frac{\partial}{\partial x_s} - A_s \frac{\partial}{\partial x_{s+1}}, \ s = 1, ..., n-1.$$

Then the system can be written as

$$X_s \left( A_t \right) = 0,$$

where $s, t = 1, .., n-1$.

Note that

$$[X_s, X_t] = 0$$

if the function $f$ is a solution of (8).

Hence, the vector fields $X_1, ..., X_{n-1}$ generate a completely integrable $(n-1)$-dimensional distribution, and the functions $A_1, ..., A_{n-1}$ are the first integrals of this distribution.

Moreover, the definition of the functions $A_s$ shows that

$$X_s(f) = 0, \ s = 1, ..., n-1,$$

also.

As a result, we get that

$$A_s = \Phi_s \left( f \right), \quad s = 1, ..., n-1,$$

for some functions $\Phi_s$.

In these terms, we get the following system of equations for $f$:

$$\frac{\partial f}{\partial x_s} = \Phi_s(f) \frac{\partial f}{\partial x_{s+1}}, \ \ s = 1, ..., n-1,$$

or

$$\frac{\partial f}{\partial x_s} = \Psi_s(f) \frac{\partial f}{\partial x_n}, \ s = 1, ..., n-1, \tag{9}$$

where $\Psi_{n-1} = \Phi_{n-1}$, and

$$\Psi_s = \Phi_{n-1} \cdots \Phi_s$$

for $s = 1, ...., n-2$.

This system is a sequence of the Euler-type equations and therefore can be integrated. Keeping in mind that a solution of the single Euler-type equation

$$\frac{\partial f}{\partial x_s} = \Psi_s(f) \frac{\partial f}{\partial x_n}$$

is given by the implicit equation

$$f = u_0 \left( x_n + \Psi_s(f) x_s \right),$$

where $u_0(x_n)$ is an initial condition, when $x_s = 0$, and $\Psi_s$ is an arbitrary nonvanishing function, we get solutions $f$ of system (8) in the form:

$$f = u_0 \left( x_n + \Psi_{n-1}(f) x_{n-1} + \cdots + \Psi_1(f) x_1 \right),$$

where $u_0(x_n)$ is an initial condition, when $x_1 = \cdots = x_{n-1} = 0$, and $\Psi_s$ are arbitrary nonvanishing functions.

Thus, we have proved the following result.

**Theorem 8** *Web functions of hyperplanar webs have the form*

$$f = u_0 \left( x_n + \Psi_{n-1}(f) x_{n-1} + \cdots + \Psi_1(f) x_1 \right), \tag{10}$$

*where $u_0(x_n)$ are initial conditions, when $x_1 = \cdots = x_{n-1} = 0$, and $\Psi_s$ are arbitrary nonvanishing functions.*

**Example 9** *Assume that $n = 3$, $f_1(x_1, x_2, x_3) = x_1$, $f_2(x_1, x_2, x_3) = x_2$, $f_3(x_1, x_2, x_3) = x_3$, and take $u_0 = x_3$, $\Psi_1(f_4) = f_4^2$, $\Psi_2(f_4) = f_4$ in (10). Then we get the hyperplanar 4-web with the remaining web function*

$$f_4 = \frac{x_2 - 1 \pm \sqrt{(x_2 - 1)^2 - 4x_1x_3}}{2x_1}.$$

*It follows that the level surfaces $f_4 = C$ of this function are defined by the equation*

$$x_1(C^2x_1 - Cx_2 + x_3 + C) = 0,$$

*i.e., they form a one-parameter family of 2-planes*

$$C^2x_1 - Cx_2 + x_3 + C = 0.$$

*Differentiating the last equation with respect to $C$ and excluding $C$, we find that the envelope of this family is defined by the equation*

$$(x_2)^2 - 4x_1x_3 - 2x_2 + 1 = 0.$$

*Therefore, the envelope is the second-degree cone.*

**Example 10** *Assume that $n = 3$, $f_1(x_1, x_2, x_3) = x_1$, $f_2(x_1, x_2, x_3) = x_2$, $f_3(x_1, x_2, x_3) = x_3$, and take $u_0 = x_3, \Psi_1(f_4) = 1, \Psi_2(f_4) = f_4^2$ in (10). Then we get the linear 4-web with the remaining web function*

$$f_4 = \left(\frac{1 \pm \sqrt{1 - 4x_2(x_1 + x_3)}}{2x_2}\right)^2.$$

*The level surfaces $f_4 = C^2$ of this function are defined by the equation*

$$x_2(x_1 + C^2x_2 + x_3 - C) = 0,$$

*i.e., they form a one-parameter family of 2-planes*

$$x_1 + C^2x_2 + x_3 - C = 0.$$

*Differentiating the last equation with respect to $C$ and excluding $C$, we find that the envelope of this family is defined by the equation*

$$4x_1x_2 + 4x_2x_3 - 1 = 0.$$

*Therefore, the envelope is the hyperbolic cylinder.*

In the next example no one foliation of a web $W_3$ coincides with a foliation of coordinate lines, i.e., all three web functions are unknown.

**Example 11** *Assume that $n = 3$ and take*

**(i)** $u_{01} = x_3,\ \Psi_1(f_1) = f_1^2,\ \Psi_2(f_1) = f_1$;

**(ii)** $u_{02} = x_3, \Psi_1(f_2) = 1, \Psi_2(f_2) = f_2^2$;

**(iii)** $u_{03} = x_3^2,\ \Psi_1(f_3) = f_3,\ \Psi_2(f_3) = 1$;

**(iv)** $u_{04} = x_3,\ \Psi_1(f_4) = \Psi_2(f_4) = f_4$

*in (10). Then we get the linear 4-web with the web functions*

$$f_1 = \frac{x_2 - 1 \pm \sqrt{(x_2 - 1)^2 - 4x_1x_3}}{2x_1},$$

$$f_2 = \left(\frac{1 \pm \sqrt{1 - 4x_2(x_1 + x_3)}}{2x_2}\right)^2$$

(*see Examples 9 and 10*) *and*

$$f_3 = (\frac{1 \pm \sqrt{1 - 4x_1(x_2 + x_3)}}{2x_1})^2,$$
$$f_4 = \frac{x_3}{1 - x_1 - x_2}.$$

*It follows that the leaves of the foliation* $X_1$ *are tangent* 2*-planes to the second-degree cone*

$$(x_2)^2 - 4x_1x_3 - 2x_2 + 1 = 0$$

(*cf. Example 9 and 10*), *the leaves of the foliation* $X_2$ *and* $X_3$ *are tangent* 2*-planes to the hyperbolic cylinders*

$$4x_1x_2 + 4x_2x_3 - 1 = 0 \text{ and } 4x_1x_2 + 4x_1x_3 - 1 = 0$$

(*cf. Example 10*), *and the leaves of the foliation* $X_4$ *are* 2*-planes of the one-parameter family of parallel* 2*-planes*

$$Cx_1 + Cx_2 + x_3 = 1,$$

*where* $C$ *is an arbitrary constant.*

# References


[1] Goldberg, V. V. and V. V. Lychagin, *Geodesic webs on a two-dimensional manifold and Euler equations*, Acta Math. Appl., 2009 (to appear).



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