On Rank Problems for Planar Webs and Projective Structures

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Summary. We present some old and recent results on rank problems and linearizability of geodesic planar webs.

1 Introduction

In this paper we continue our studies of geodesic planar webs [13].

We give a modification of the Abel's elimination method. This method allows one to find all abelian relations admitted by a planar web and therefore to determine the rank of the web. It requires to solve step-by-step a series of ordinary differential equations. In [25] (see also [24]) the same modification is given by a little bit different approach.

On the other hand, we present the method of finding the web rank by means of differential invariants of the web, i.e., the determination of the web rank without solving the differential equations. Pantazi [22] found some necessary and sufficient conditions for a planar web to be of maximum rank. The paper [22] was followed by the papers [23] and [20]. Pirio in [24] presented a more detailed exposition of results of Pantazi in [22] and [23] and Mihăileanu in [20]. The characterization of webs of maximal rank in [22] and [20] is not given in terms of the web invariants.

We give also an alternative construction (the previous one was given in [13]) of the unique projective structure associated with a planar 4-web. Note that Theorem 7 was first proved in [6] (see §29, p. 246) and that the result in [6] was recently generalized in [26] for any dimension. Our method exploits differential forms and gives an explicit formula for the projective connection. Remark that this method, as well as one in ([13]), can be used in any dimension. Presence of the projective structure allows us to connect a differential invariant (which we call the Liouville tensor) with any planar 4-web. This tensor gives a criterion for linearizability of geodesic planar webs (cf. [4]).

2 Planar Webs

All constructions in the paper are local, and we do not specify domains in which they are valid. Functions, differential forms, etc. are real and of class C^∞.

A *planar d-web* is given by d one-dimensional foliations in the plane which are in general position, i.e., the directions corresponding to different foliations are distinct. The local diffeomorphisms of the plane act in the natural way on d-webs, and they say that two d-webs are (locally) *equivalent* if there exists a local diffeomorphism which sends one d-web to another.

Because all 2-webs are locally equivalent, we begin with 3-webs.

A 3-web can be defined either by three differential 1-forms, say, $\omega_1, \omega_2, \omega_3$, where

$$\omega_1 \wedge \omega_2 \neq 0, \ \ \omega_2 \wedge \omega_3 \neq 0, \ \ \omega_1 \wedge \omega_3 \neq 0,$$

or by the first integrals of the foliations, say, f_1, f_2, f_3, where

$$df_1 \wedge df_2 \neq 0, \ \ df_2 \wedge df_3 \neq 0, \ \ df_1 \wedge df_3 \neq 0.$$

The above functions f_1, f_2, f_3 are called *web functions*.

Remark that the web functions are defined up to *gauge transformations*

$$f_i \mapsto \Phi_i(f_i),$$

where $i = 1, 2, 3$, and $\Phi_i : \mathbb{R} \to \mathbb{R}$ are local diffeomorphisms of the line.

The implicit function theorem states that there is a relation

$$W(f_1, f_2, f_3) = 0$$

for these functions.

The above relation is called (see, for example, [5]) the *web equation*.

Any pair of functions in this equation is locally indistinguishable and can be viewed as local coordinates on the plane.

Keeping in mind this observation, we consider a space $\mathbb{R}^3$ with coordinates u_1, u_2, u_3 and two-dimensional surface

$$\Sigma \subset \mathbb{R}^3$$

given by the equation

$$W(u_1, u_2, u_3) = 0.$$

We say that $\Sigma \subset \mathbb{R}^3$ is a *web surface* if any two functions u_i, u_j are local coordinates on Σ.

In the case of d-webs one can choose d local first integrals of the corresponding foliations, say, $f_1, f_2, f_3..., f_d$, which are also called *web functions*. They define a map

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^d,$$

where

$$\sigma : (x, y) \in \mathbb{R}^2 \mapsto (u_1 = f_1(x, y), ..., u_d = f_d(x, y)) \in \mathbb{R}^d.$$

The image Σ of this map is a two-dimensional surface in $\mathbb{R}^d$. Remark that any pair of functions u_i, u_j are local coordinates on Σ.

From this point of view, the local theory of planar d-webs is just a geometry of web surfaces in $\mathbb{R}^d$ considered with respect to the gauge transformations.

3 Basic Constructions

Let us begin with 3-webs, and let differential 1-forms $\omega_1, \omega_2, \omega_3$ define such a web. These forms are determined up to multipliers $\omega_i \leftrightarrow \lambda_i \omega_i$, where λ_i are smooth nonvanishing functions. Hence, these forms can be normalized in such a way that

$$\omega_1 + \omega_2 + \omega_3 = 0 \tag{1}$$

with only possible scaling $\omega_i \leftrightarrow \lambda \omega_i$.

One can prove that in this case there is a unique differential 1-form γ such that the so-called *structure equations*

$$d\omega_i = \omega_i \wedge \gamma \tag{2}$$

hold for all $i = 1, 2, 3$ (see [4]).

The form γ determines the *Chern connection* Γ in the cotangent bundle T^*M with the following covariant differential:

$$d_\Gamma : \omega_i \longmapsto -\omega_i \otimes \gamma.$$

The curvature of this connection is equal to

$$R_\Gamma : \omega_i \longmapsto -\omega_i \otimes d\gamma.$$

If we write

$$d\gamma = K\omega_1 \wedge \omega_2,$$

then the function K is called the *curvature function* of the 3-web.

Note that the curvature form $d\gamma$ is an invariant of the 3-web while the curvature function K is a relative invariant of the web of weight two.

Let $\langle \partial_1, \partial_2 \rangle$ be the basis dual to $\langle \omega_1, \omega_2 \rangle$. We put $\partial_3 = \partial_2 - \partial_1$. Then leaves of the 3-web are trajectories of the vector fields ∂_2, ∂_1, and ∂_3.

The form γ can be decomposed as follows:

$$\gamma = g_1\omega_1 + g_2\omega_2,$$

where g_1 and g_2 are smooth functions.

Moreover, in this case one has (see [11])

$$[\partial_1, \partial_2] = -g_2\partial_1 + g_1\partial_2 \tag{3}$$

and
$$K = \partial_1 \left(g_2 \right) - \partial_2 \left(g_1 \right). \tag{4}$$

Remark also that the covariant derivatives with respect to the Chern connection have the form
$$\nabla_X(\omega_i) = -\gamma(X)\omega_i$$

and
$$\nabla_X(\partial_i) = -\gamma(X)\partial_i.$$

It shows that the leaves of all three foliations are geodesic with respect to the Chern connection.

Let $d_\nabla : \Omega^1(M) \to \Omega^1(M) \otimes \Omega^1(M)$ be the covariant differential with respect to the Chern connection.

The induced connection in the tangent bundle gives the differential
$$d_\nabla^* : \mathcal{D}(M) \to \mathcal{D}(M) \otimes \Omega^1(M),$$

where
$$d_\nabla : \partial_i \to \partial_i \otimes \gamma.$$

In a similar way the Chern connection induces the covariant differential in the tensor bundles.

Let us denote by $\Theta^{p,q}(M) = (\mathcal{D}(M))^{\otimes p} \otimes \left(\Omega^1(M) \right)^{\otimes q}$ the module of tensors of type (p, q).

Then the covariant differential
$$d_\nabla : \Theta^{p,q}(M) \to \Theta^{p+1,q}(M)$$

acts as follows:
$$d_\nabla : u\partial_{j_1} \otimes \cdots \otimes \partial_{j_p} \otimes \omega_{i_1} \otimes \cdots \otimes \omega_{i_q} \longmapsto \partial_{j_1} \otimes \cdots \otimes \partial_{j_p} \otimes \omega_{i_1} \otimes \cdots \otimes \omega_{i_q} \otimes (du + (p - q)\gamma u)$$

where $u \in C^\infty(M)$.

We say that u is of weight $k = q - p$ and call the form
$$\delta^{(k)}(u) \stackrel{\text{def}}{=} du - ku\gamma \tag{5}$$

the *covariant differential* of u.

Decomposing the form $\delta^{(k)}(u)$ in the basis $\{\omega_1, \omega_2\}$, we obtain
$$\delta^{(k)}(u) = \delta_1^{(k)}(u) \ \omega_1 + \delta_2^{(k)}(u) \ \omega_2,$$

where
$$\delta_i^{(k)}(u) = \partial_i(u) - (k) g_i u$$

are the covariant derivatives of u with respect to the Chern connection, $i = 1, 2$.

Note that $\delta_1^{(k)}(u)$ and $\delta_2^{(k)}(u)$ are of weight $k + 1$.

One can check that the covariant derivatives satisfy the classical Leibnitz rule

$$\delta_i^{(k+l)}(uv) = \delta_i^{(k)}(u)\ v + u\ \delta_i^{(l)}(v)$$

if u is of weight k and v is of weight l, and the following commutation relation:

$$\delta_2^{(s+1)} \circ \delta_1^{(s)} - \delta_1^{(s+1)} \circ \delta_2^{(s)} = sK. \tag{6}$$

Note that the curvature K is of weight two.

In what follows, we shall omit the superscript indicating the weight in the cases when the weight is known.

For the general d-web defined by differential 1-forms $\omega_1, \omega_2, \omega_3, \ldots, \omega_d$ we normalize $\omega_1, \omega_2, \omega_3$ as above and choose ω_i for $i \geq 4$ in such a way that the normalizations

$$a_i \omega_1 + \omega_2 + \omega_{i+2} = 0 \tag{7}$$

hold for $i = 1, ..., d - 2$, with $a_1 = 1$.

Note that $a_i \neq 0, 1$ for $i \geq 2$.

Moreover, for any fixed i, the value $a_i(x)$, of the function a_i at the point x is the cross-ratio of the four straight lines in the cotangent space T_x^* generated by the covectors $\omega_{1,x}, \omega_{2,x}, \omega_{3,x}$, and $\omega_{i+2,x}$, and therefore it is a web invariant. The functions a_i are called the *basic invariants* (cf. [10] or [9], pp. 302–303) of the web.

Because of locality of our consideration, one can choose a function f in such a way that $\omega_3 = df$ and find coordinates x, y such that $\omega_1 \wedge dx = 0$ and $\omega_2 \wedge dy = 0$.

Let also $\omega_{i+3} \wedge dg_i = 0$, for some functions $g_i(x, y)$, $i = 1, ..., d - 3$.

Then $\omega_1 = -f_x dx$ and $\omega_2 = -f_y dy$.

The dual basis $\{\partial_1, \partial_2\}$ has the form

$$\partial_1 = -f_x^{-1}\partial_x, \quad \partial_2 = -f_y^{-1}\partial_y,$$

and the connection form is

$$\gamma = -H\omega_3,$$

where

$$H = \frac{f_{xy}}{f_x\ f_y}$$

(see [11]).

The curvature function has the following explicit expression:

$$K = -\frac{1}{f_x f_y}\left(\log \frac{f_x}{f_y}\right)_{xy}, \tag{8}$$

and the basic invariants have the form

$$a_i = \frac{f_y g_{i,x}}{f_x g_{i,y}}$$

for $i = 1, ..., d - 3$.

Note that if a three-web W_3 is given by a web equation $W(u_1, u_2, u_3) = 0$, then the curvature K is expressed as follows (see [5], §9):

$$K = A_{12} + A_{23} + A_{31}, \qquad (9)$$

where

$$A_{rs} = \frac{1}{W_r W_s} \frac{\partial^2}{\partial u_r \partial u_s} \log \frac{W_r}{W_s},$$

and subscripts r and s mean the partial derivatives of the function with respect to the variables u_r and u_s, where $r, s = 1, 2, 3$.

Recall that a planar d-web is said to be (locally) *parallelizable* if it is (locally) equivalent to a d-web of parallel straight lines in the affine plane.

It is known (see, for example, [5], §8) that *a planar 3-web is locally parallelizable if and only if $K = 0$*.

For planar d-webs, $d \geq 4$, the following statement holds (cf. [10] or [9], Section 7.2.1 for $d = 4$): *a planar d-web $\langle \omega_1, \omega_2, \omega_3, \omega_4, ..., \omega_d \rangle$ is locally parallelizable if and only if its 3-subweb $\langle \omega_1, \omega_2, \omega_3 \rangle$ is locally parallelizable (i.e., $K = 0$), and all basic invariants a_i are constants.*

4 Rank

We begin with an interpretation of the classical Abel addition theorem ([2]) in terms of planar webs (cf. [5]).

Let us consider linear webs on the affine plane, i.e., such planar webs leaves of which are straight lines. There is an elegant method to construct such webs. Take a straight line $rx + sy = 1$ on the affine plane and assume that the coefficients $(r, s) \in \mathbb{R}^2$ satisfy an algebraic equation $P_d(r, s) = 0$ of degree d.

Given (x, y), then the system

$$\begin{cases} rx + sy = 1, \\ P_d(r, s) = 0 \end{cases}$$

has at most d roots.

Assume that in a domain on the plane (x, y) the above system has exactly d roots. Then in this domain we have a linear d-web.

Take now a cubic polynomial

$$P_3(s, t) = s^2 - 4r^3 - g_2 r - g_3,$$

where g_2 and g_3 are constants.

Then the system

$$\begin{cases} rx + sy = 1, \\ s^2 - 4r^3 - g_2 r - g_3 = 0 \end{cases}$$

in the domain

$$x^4 - 24xy^2 - 12g_2 y^4 > 0, \ y \neq 0,$$

has three distinct real roots and consequently three pairwise distinct straight lines $(r_i(x,y), s_i(x,y))$, passing through the point (x,y). In other words, we have a linear 3-web.

Assume that

$$g_2^3 - 27g_3^2 \neq 0.$$

Then the solutions of the equation $s^2 - 4r^3 - g_2 r - g_3 = 0$ can be parameterized by the Weierstrass' elliptic function with the invariants g_2 and g_3:

$$r = \wp(t), s = \wp'(t).$$

Hence, the roots $(r_i(x,y), s_i(x,y))$ correspond to three solutions $(t_i(x,y))$ of the equation

$$\wp(t)x + \wp'(t)y - 1 = 0.$$

Let us put

$$f(t) = \wp(t)x + \wp'(t)y - 1$$

and compute the integral

$$\int t \frac{f'(t)}{f(t)} dt$$

along the boundary of the period parallelogram of the Weierstrass function. We get

$$t_1(x,y) + t_2(x,y) + t_3(x,y) = \text{const}.$$

This is the *abelian relation.*

This relation can be understood geometrically if we note that, by the construction, the functions $t_1(x,y)$, $t_2(x,y)$, and $t_3(x,y)$ are first integrals of the corresponding 3-web.

In more general case, let us consider an arbitrary planar d-web defined by d web functions

$$f_1, \ldots, f_d.$$

Then by *abelian relation* we mean a relation

$$F_1(f_1) + \cdots + F_d(f_d) = \text{const}.$$

given by d functions $(F_1, \ldots, F_d)$ of one variable.

We say that two abelian relations $(F_1, \ldots, F_d)$ and $(G_1, \ldots, G_d)$ are *equivalent* if

$$F_i = G_i + \mathop{\text{const}}_i,$$

for all $i = 1, \ldots, d$.

The set of equivalence classes of abelian relations admits the natural vector space structure with respect to addition

$$(F_1, \ldots, F_d) + (G_1, \ldots, G_d) = (F_1 + G_1, \ldots, F_d + G_d)$$

and multiplication by numbers

$$\alpha (F_1, \ldots, F_d) = (\alpha F_1, \ldots, \alpha F_d).$$

The dimension of this vector space is called the *rank* of the web.

In the case when d-web is defined by differential 1-forms

$$\omega_1, \ldots, \omega_d,$$

the differentiation of the abelian relation leads us to the *abelian equation*

$$\lambda_1 \omega_1 + \cdots + \lambda_d \omega_d = 0,$$

for functions $\lambda_1, \ldots, \lambda_d$ under the condition that all differential 1-forms $\lambda_i \omega_i$ are closed:

$$d(\lambda_i \omega_i) = 0.$$

The abelian equation is a system of the first-order linear PDEs for the functions $(\lambda_1, \ldots, \lambda_d)$, and the rank of the web is the dimension of the solution space.

Example 1. The following example illustrates the above constructions for 3-webs.

Consider the 3-web W_3 given by web functions:

$$x, y, f(x, y).$$

Then

$$\omega_1 = -f_x dx, \quad \omega_2 = -f_y dy, \quad \omega_3 = df,$$

and the condition

$$\lambda_1 \omega_1 + \lambda_2 \omega_2 + \lambda_3 \omega_3 = 0$$

implies

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda.$$

The abelian relations take now the form

$$\begin{cases} (\lambda f_x)_y & = 0, \\ (\lambda f_y)_x & = 0, \\ \lambda_x f_y - \lambda_y f_x = 0, \end{cases}$$

or

$$\begin{cases} (\ln \lambda)_x = -(\ln f_x)_y, \\ (\ln \lambda)_y = -(\ln f_y)_x. \end{cases}$$

The compatibility condition for this system has the form

$$(\ln f_x)_{xy} = (\ln f_y)_{xy}$$

or

$$K = 0.$$

So, we can conclude this consideration by the following statement: *rank of a 3-web does not exceed one, and the rank equals to one if and only if the 3-web is parallelizable.*

5 Abel's Method

In this section we discuss the rank problem in the classical setting. A method of finding the rank, or in other terms, a method of solving abelian relations was proposed by Abel himself (see [1]). This method is just a consistent elimination of the functions from the abelian relation by using only differentiation.

Let us consider a planar d-web defined by web functions $f_1, \ldots, f_d$ and the corresponding abelian relation

$$F_1(f_1) + \cdots + F_d(f_d) = \text{const}. \tag{10}$$

Modifying Abel's method and adjusting it to equation (10), we can explain it as follows:

a) Taking the differential of equation (10), we get

$$F_1' \, df_1 + F_2' \, df_2 + \cdots + F_d' \, df_d = 0. \tag{11}$$

b) Taking the wedge product of (11) with df_1, we eliminate F_1 and get the following equation:

$$F_2' + J_{21}^{31} F_3' + \cdots + J_{21}^{d1} F_d' = 0, \tag{12}$$

where

$$J_{kl}^{ij} = \frac{\partial(f_i, f_j)}{\partial(f_k, f_l)}$$

is the Jacobian of the functions f_i, f_j with respect to functions f_k, f_l.

c) Taking the wedge product of the differential of (12) with df_2, we eliminate F_2 and get the following equation:

$$J_{21}^{31} F_3'' + a_3 F_3' + \cdots = 0, \tag{13}$$

where a_3 is a certain function.

d) Divide equation (13) by the first coefficient and take the differential of the obtained equation; if F_3'' appears in differentiation, take its value from equation (13). This gives the following equation:

$$F_3''' df_3 + b_3 F_3'(f_3) + \cdots = 0, \tag{14}$$

where b_3 is a certain function.

e) Taking the exterior product of (14) with df_3, we eliminate F_3''' and get the following equation:

$$c_3 F_3'(f_3) + \cdots = 0, \tag{15}$$

where c_3 is a certain function.

f) Dividing equation (15) by c_3 and taking the wedge product of the differential of the obtained equation and df_3, we eliminate the function F_3.

g) Use the procedure outlined above to eliminate the functions $F_4, \ldots, F_{d-1}$. Finally, we obtain a linear differential equation with respect to the function $F_d(f_d)$. This equation can be viewed as family of homogeneous ordinary linear differential equations.

h) Substitute the solution $F_d(f_d)$ into (10) and apply the outlined procedure to find another function, say, $F_{d-1}(f_{d-1})$.

On Abel's elimination method as well as on less general method of monodromy see [25].

Below we give few examples of application of the Abel method.

5.1 3-Webs

Here we apply the Abel elimination method for 3-webs to show once more that a planar 3-web admits a nontrivial abelian relation if and only if the 3-web is parallelizable.

Suppose that a 3-web is given by the web functions $f(x,y), x, y$ and let

$$F(f) + G(x) + H(y) = 0. \tag{16}$$

be an abelian relation.

Take the differential of (16):

$$F'(f)\, df + G'(x)\, dx + H'(y)\, dy = 0, \tag{17}$$

and the wedge product of (17) with df:

$$-f_y\, G'(x) + f_x\, H'(y) = 0. \tag{18}$$

Then

$$G'(x) - \frac{f_x}{f_y}\, H'(y) = 0. \tag{19}$$

Taking the wedge product of the differential of (19) with dx, we get

$$H''(y) + \left(\log \frac{f_x}{f_y} \right)_y H'(y) = 0. \tag{20}$$

In order equation (20) has a nontrivial solution, it is necessary and sufficient that the function $\left(\log \dfrac{f_x}{f_y} \right)_y$ does not depend on x, i.e.,

$$\left(\log \frac{f_x}{f_y} \right)_{xy} = 0. \tag{21}$$

This means that $K = 0$, i.e., the 3-web is parallelizable.

5.2 4-Webs of Rank Three

Assume that a 4-web is given by the following web functions:

$$f = x + y, \ g = xy, \ x, \ y.$$

We will apply the Abel elimination method to find all abelian relations admitted by this web.

Let

$$F(f) + G(g) + H(x) + K(y) = 0 \tag{22}$$

be an abelian relation.

Taking the differential of (22):

$$F'(f) \, df + G'(g) \, dg + H'(x) dx + K'(y) \, dy = 0, \tag{23}$$

and the wedge product of (23) with dy, we eliminate $K'(y)$:

$$F'(f) + y \, G'(g) + H'(x) = 0. \tag{24}$$

Once more, taking the differential of (24):

$$F''(f) \, df + y \, G''(g) \, dg + G'(g) \, dy + H''(x) \, dx = 0, \tag{25}$$

and the wedge product of (25) with dg, we eliminate $G''(g)$:

$$(x - y) \, F''(f) - y \, G'(g) + x/, H''(x) = 0. \tag{26}$$

Using equation (24), we eliminate $G'(g)$ in (26):

$$(x - y) \, F''(f) + F'(f) + H'(x) + x H''(x) = 0. \tag{27}$$

Dividing equation (27) by x, taking the differential of the result and taking the wedge product of the differential with dx, we eliminate $H(x)$ and arrive at the equation

$$F'''(f) = 0. \tag{28}$$

Up to an arbitrary constant, the solution of equation (28) is

$$F(f) = a\,f^2 + b\,f, \tag{29}$$

where a and b are arbitrary constants. By (29), equation (27) gives

$$x\,H''(x) + H'(x) + 2ax = 0. \tag{30}$$

Up to an arbitrary constant, the solution of equation (30) is

$$H(x) = -a\,x^2 - b\,x + k\log x, \tag{31}$$

where k is an arbitrary constant.

By (29) and (31), equation (26) gives

$$G'(g) = -a - \frac{k}{g}. \tag{32}$$

Up to an arbitrary constant, the solution of equation (32) is

$$G(g) = -2a\,g - k\log g, \tag{33}$$

Now by (29), (31) and (33), we find from equation (22) that

$$K(y) = -a\,y^2 - b\,y + k\,\log y. \tag{34}$$

Thus, the rank is equal to three, and we have the following three independent abelian relations:

$(a = 0, b = 0, k = -1)$

$$x + y - f = 0;$$

$(a = -1, b = 0, k = 0)$

$$x^2 + y^2 + (-f^2) + (2g) = 0;$$

$(a = 0, b = -1, k = 0)$

$$\log x + \log y + (-\log g) = 0.$$

The fact that the rank of this web is three was also proved in [12] by use of differential invariants of webs.

5.3 4-Webs of Rank Two

Consider a 4-web given by the following web functions:

$$f = x^2 + y^2, \ g = x + y, \ x, \ y.$$

We will apply the Abel elimination method and find all abelian relations admitted by this web.

Let

$$F(f) + G(g) + H(x) + K(y) = 0 \tag{35}$$

be an abelian relation.

Taking the differential:

$$F'(f)\, df + G'(g)\, dg + H'(x)\, dx + K'(y)\, dy = 0, \tag{36}$$

and the wedge product of (36) with dy, we eliminate $K(y)$:

$$2x\, F'(f) + G'(g) + H'(x) = 0. \tag{37}$$

Once more, taking the differential of (37):

$$2x\, F''(f)\, df + 2F'(f)\, dx + G''(g)\, dg + H''(x)\, dx = 0, \tag{38}$$

and the wedge product of (38) with dx, we eliminate $H(x)$:

$$2(g^2 - f)\, F''(f) + G''(g) = 0. \tag{39}$$

Taking the wedge product of the differential of (39) with dg, we eliminate $G(g)$:

$$2(g^2 - f)\, F'''(f) - 2F''(f) = 0. \tag{40}$$

Equation (40) is equivalent to the system

$$\begin{cases} -2f\, F'''(f) - 2F''(f) = 0, \\ F'''(f) = 0. \end{cases} \tag{41}$$

Therefore, up to an additive constant,

$$F(f) = kf,$$

where k is a constant.

Now it follows from (39) that, up to an additive constant,

$$G(g) = bg,$$

where b is a constant.

Equation (37) implies that, up to an additive constant,

$$H(x) = -kx^2 - bx,$$

and equation (35) gives that

$$K(y) = -ky^2 - by.$$

.

Thus, the rank of the web is equal to two, and we have the following two basic abelian relations:

$(k = 0, b = 1)$

$$(x + y) + (-x) + (-y) = 0,$$

$(k = 1, b = 0)$

$$(x^2 + y^2) + (-x^2) + (-y^2) = 0.$$

In [12] by use of differential invariants of webs, it was shown that this 4-web is of rank two.

5.4 4-Webs of Rank One

Assume that a 4-web is given by the following web functions:

$$f = \frac{(x - y)^2}{x}, \ g = \frac{(x - y)^2}{y}, \ x, \ y.$$

We will apply the Abel elimination method to find all abelian relations admitted by this web.

Let

$$F(f) + G(g) + H(x) + K(y) = 0 \tag{42}$$

be an abelian relation.

Taking the differential of (42):

$$F'(f) \, df + G'(g) \, dg + H'(x) dx + K'(y) \, dy = 0, \tag{43}$$

and the wedge product of (43) with df, we eliminate $F(f)$:

$$G'(g) - \frac{2xy^2}{(x - y)^3} H'(x) - \frac{(x + y)y^2}{(x - y)^3} K'(y) = 0. \tag{44}$$

Once more, taking the differential of (44):

$$G''(g) \, dg - \frac{2xy^2}{(x - y)^3} H''(x) - \frac{2y(2x + y)(-ydx + xdy)}{(x - y)^3} H'(x)$$
$$- \frac{(x + y)y^2}{(x - y)^3} K''(y) - \frac{2y(x + 2y)(-ydx + xdy)}{(x - y)^3} K'(y) = 0, \tag{45}$$

and the wedge product of (45) with dg, we eliminate $G(g)$:

$$H''(x) + \frac{2x+y}{x(x+y)} H'(x) + \frac{y}{x} K''(y) + \frac{x+2y}{x(x+y)} K'(y) = 0. \qquad (46)$$

Taking the wedge product of the differential of (46) with dx, we eliminate $H''(x)$:

$$H'(x) - \frac{(x+y)^2 y}{x} K'''(y) - \frac{(2x+3y)(x+y)}{x} K''(y) - K'(y) = 0. \qquad (47)$$

Taking the wedge product of the differential of (47) with dx, we eliminate $H'(x)$:

$$(x+y)y K^{iv}(y) + 3(x+2y) K'''(y) + 6K''(y) = 0. \qquad (48)$$

Equation (48) is equivalent to the system

$$\begin{cases} y\, K^{iv}(y) + 3x K'''(x) = 0, \\ y^2\, K^{iv}(y) + 6y\, K'''(y) + 6K''(y) = 0. \end{cases} \qquad (49)$$

It follows from (49) that

$$y\, K'''(y) + 2K''(y) = 0. \qquad (50)$$

Up to an additive constant, the solution of (50) is

$$K(y) = -k \log y + by, \qquad (51)$$

where k and b are arbitrary constants.

It follows from (47) and (51) that

$$H'(x) = \frac{k}{x} + b. \qquad (52)$$

Up to an additive constant, the solution of (52) is

$$H(x) = k \log x + bx. \qquad (53)$$

It follows from (44), (51) and (53) that

$$G'(g) = -\frac{k}{g} + \frac{by(3x+y)}{g(x-y)}. \qquad (54)$$

Equation (54) implies that

$$b = 0 \qquad (55)$$

and that, up to an additive constant,

$$G(g) = -k \log g, \quad H(x) = k \log x, \quad K(y) = -k \log y. \qquad (56)$$

Finally, equations (42) and (56) give

$$F(f) = k \log f, \qquad (57)$$

and the 4-web admits only one independent abelian relation

$$\log f - \log g + \log x - \log y = 0. \qquad (58)$$

By use of differential invariants of webs introduced in [12], one can show that this 4-web is of rank one.

5.5 4-Webs of Rank Zero

Assume that a 3-web is given by the following web functions:

$$f = (x + y)e^x, \; g = xy, \; x, \; y.$$

We will apply the Abel elimination method to show that this web admits no abelian relations.

Let

$$F(f) + G(g) + H(x) + K(y) = 0 \tag{59}$$

be an abelian relation.

Taking the differential of (59):

$$F'(f)\,df + G'(g)\,dg + H'(x)\,dx + K'(y)\,dy = 0, \tag{60}$$

and the wedge product of (60) with df, we eliminate $F(f)$:

$$[(x + y)x + x - y]\,G'(g) - H'(x) + (1 + x + y)\,K'(y) = 0. \tag{61}$$

Once more, taking the differential of (61):

$$[(x + y)x + x - y]\,G''(g)\,dg - H''(x)\,dx + (1 + x + y)\,K''(y)\,dy \\ + [(2x + y + 1)\,dx + (x - 1)\,dy]\,G'(g) + (dx + dy)\,K'(y) = 0, \tag{62}$$

and the wedge product of (62) with dx, we eliminate $H(x)$:

$$x\,[(x + y)x + x - y]\,G''(g) + (x - 1)\,G'(g) + K''(y) + K'(y) = 0. \tag{63}$$

Taking the differential of (63):

$$x\,[(x + y)x + x - y]\,G'''(g)\,dg + (x - 1)\,G''(g)\,dg \\ + K'''(y)\,dy + K''(y)\,dy + G''(g)\,dx = 0, \tag{64}$$

and the wedge product of (64) with dy, we eliminate $K(y)$:

$$g\,(x^2 + g + x - \frac{g}{x}]\,G'''(g) + (3g - \frac{2g}{x} + 3x^2 + 2x + 1)\,G''(g) = 0. \tag{65}$$

Equation (65) is equivalent to the system

$$\begin{cases} g^2 G'''(g) + (3g + 1)\,G''(g) = 0, \\ G''(g) = 0, \\ G'''(g) = 0, \end{cases}$$

i.e., to the equation $G''(g) = 0$. Up to an arbitrary constant, the solution of the latter equation is $G = ag$, where a is an arbitrary constant.

If $G = ag$, then equation (63) becomes

$$K''(y) + K'(y) + a(x - 1) = 0.$$

It follows that $a = 0$ and $G(g) = 0$. The equation for $K(y)$ becomes $K''(y) + K'(y) = 0$. Up to an arbitrary constant, its solution is $K(y) = -be^{-y}$, where b is an arbitrary constant.

Now equation (61) becomes

$$H'(x) = b(1 + y)e^{-y} + b\,xe^{-y}.$$

It follows that $b = 0$ and $H'(x) = 0$. Hence, up to an arbitrary constant, $H(x) = 0$ and $K(y) = 0$.

Finally equation (59) implies that $F(f) = 0$.

Thus, the web under consideration admits no abelian relations.

6 Abelian Differential Equations

In this section we discuss properties of abelian equations.

Recall that the abelian equation for a planar d-web given by differential 1-forms

$$\omega_1, ..., \omega_d$$

is a first-order PDE system for functions $\lambda_1, ..., \lambda_d$ of the form

$$\lambda_1\omega_1 + \cdots + \lambda_d\omega_d = 0,$$
$$d(\lambda_1\omega_1) = \cdots = d(\lambda_d\omega_d) = 0.$$

Let us write down the abelian equation in more explicit form. To this end, we choose a 3-subweb, say, the 3-web given by

$$\omega_1, \omega_2, \omega_3,$$

and normalize the d-web as it was done earlier:

$$a_1\omega_1 + \omega_2 + \omega_3 = 0, \quad a_2\omega_1 + \omega_2 + \omega_4 = 0, ..., \ a_{d-2}\omega_1 + \omega_2 + \omega_d = 0,$$

with $a_1 = 1$ and $d\omega_3 = 0$.

It is easy to see that, if $i \leq 3$, then, due to the structure equations, we get

$$d(\lambda\omega_i) = d\lambda \wedge \omega_i + \lambda d\omega_i = (d\lambda - \lambda\gamma) \wedge \omega_i$$

or

$$d(\lambda\omega_i) = \delta(\lambda) \wedge \omega_i,$$

if we consider λ as a function of weight one.

Assuming that all λ_i are functions of weight one and the functions a_i are of weight 0, we get

$$d\left(\lambda_1\omega_1\right) = -\delta_2\left(\lambda_1\right)\omega_1 \wedge \omega_2,$$
$$d\left(\lambda_2\omega_2\right) = \delta_1\left(\lambda_1\right)\omega_1 \wedge \omega_2,$$
$$d\left(\lambda_3\omega_3\right) = \left(\delta_2\left(\lambda_3\right) - \delta_1\left(\lambda_3\right)\right)\omega_1 \wedge \omega_2,$$
$$d\left(\lambda_i\omega_i\right) = \left(\delta_2\left(a_{i-2}\lambda_i\right) - \delta_1\left(\lambda_i\right)\right)\omega_1 \wedge \omega_2$$

for $i = 4, ..., d$.

The normalization condition $\sum_1^d \lambda_i\omega_i = 0$ implies that

$$\lambda_1 = a_1 u_1 + \cdots + a_{d-2}u_{d-2},$$
$$\lambda_2 = u_1 + \cdots + u_{d-2},$$

where

$$u_1 = \lambda_3, \ldots, u_{d-2} = \lambda_d.$$

Therefore the abelian equation can be written in the explicit form as the following PDE system:

$$\Delta_1\left(u_1\right) = \cdots = \Delta_{d-2}\left(u_{d-2}\right) = 0,$$
$$\delta_1\left(u_1\right) + \cdots + \delta_1\left(u_{d-2}\right) = 0,$$

where $\Delta_i = \delta_1 - \delta_2 \circ a_i$.

Let

$$\pi : \mathbb{R}^{d-2} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

be the trivial vector bundle, where $\pi : \left(u_1, ..., u_{d-2}, x, y\right) \mapsto \left(x, y\right)$.

Denote by $\mathfrak{A}_1 \subset \mathbf{J}^1\left(\pi\right)$ the subbundle of the 1-jet bundle corresponding to the abelian equation, and by $\mathfrak{A}_k \subset \mathbf{J}^k\left(\pi\right)$ the $(k-1)$-prolongation of $\mathfrak{A}_1$.

Let

$$\pi_{k,k-1} : \mathfrak{A}_k \longrightarrow \mathfrak{A}_{k-1}$$

be the restrictions of the natural jet projections

$$\pi_{k,k-1} : \mathbf{J}^k\left(\pi\right) \longrightarrow \mathbf{J}^{k-1}\left(\pi\right).$$

Then, if $k \leq d - 2$, one can easily check that $\mathfrak{A}_k$ are vector bundles, the maps $\pi_{k,k-1}$ are projections and

$$\dim \ker \pi_{k,k-1} = d - k - 2.$$

In other words, we have the following tower of vector bundles:

$$\mathbb{R}^2 \xleftarrow{\pi} \mathbb{R}^{d+2} \xleftarrow{\pi_{1,0}} \mathfrak{A}_1 \xleftarrow{\pi_{2,1}} \mathfrak{A}_2 \xleftarrow{\pi_{3,1}} \cdots \xleftarrow{\pi_{d-3,d-4}} \mathfrak{A}_{d-3} \xleftarrow{\pi_{d-2,d-3}} \mathfrak{A}_{d-2}.$$

The last projection

$$\pi_{k,k-1} : \mathfrak{A}_{d-2} \longrightarrow \mathfrak{A}_{d-3}$$

is an isomorphism, and geometrically it can be viewed as a linear Cartan connection (see [19]) in the vector bundle

$$\pi_{d-3} : \mathfrak{A}_{d-3} \to \mathbb{R}^2.$$

This proves that *the abelian equation is formally integrable if and only if this linear connection is flat.*

It is easy to see that the dimension of this bundle is equal to $(d-2)(d-1)/2$.

The dimension of the solution space is the rank of the corresponding d-web. The above computation shows that the rank of a d-web is finite-dimensional and does not exceed

$$\frac{(d-1)(d-2)}{2}.$$

This result was first established by Bol [7] (see also [5]).

The compatibility conditions for the abelian equation can be found (see [12]) by use of multi-brackets (see [15]).

These conditions have the form

$$\varkappa = \Box_1 u_1 + \cdots + \Box_{d-2} u_{d-2} = 0,$$

where

$$\Box_i = \Delta_1 \cdots \Delta_{d-2} \cdot \delta_1 - \Delta_1 \cdots \Delta_{i-1} \cdot \delta_1 \cdot \Delta_{i+1} \cdots \Delta_{d-2} \cdot \Delta_i$$

are linear differential operators of order not exceeding $d - 2$.

Summarizing, we get the following

Theorem 1. *A d-web is of maximum rank if and only if $\varkappa = 0$ on $\mathfrak{A}_{d-2}$.*

Remark that $\varkappa$ can be viewed as a linear function on the vector bundle $\mathfrak{A}_{d-2}$, and therefore the above theorem imposes $(d-1)(d-2)/2$ conditions on the d-web (or on $d-2$ web functions) in order the web has the maximum rank. A calculation of these conditions is pure algebraic, and we shall illustrate this calculation below for planar 3-, 4- and 5-webs. Note also that expressions for $\varkappa$ in the case of general d-webs are extremely cumbersome while for concrete d-webs it is not the case.

7 Rank of 4-Webs

7.1 The Obstruction

In order to simplify notations, we put $a_2 = a$ in the normalization for 4-webs :

$$\omega_1 + \omega_2 + \omega_3 = 0,$$
$$a\omega_1 + \omega_2 + \omega_4 = 0,$$

and reserve the subscripts for the covariant derivatives of a. Thus, $a_2 = \delta_2(a)$, etc.

For abelian equations we shall use the functions u, v, where $u = u_1, v = u_2$. Then the abelian equations have the form

$$(u + av)\,\omega_1 + (u + v)\,\omega_2 + u\omega_3 + v\omega_4 = 0,$$

where

$$\lambda_1 = u + av, \lambda_2 = u + v, \lambda_3 = u, \lambda_4 = v,$$

and the functions u and v satisfy the equations

$$\delta_1\,(u) - \delta_2\,(u) = 0,$$
$$\delta_1\,(v) - \delta_2\,(av) = 0,$$
$$\delta_1\,(u) + \delta_1\,(v) = 0.$$

In the case of 4-webs, the tower of prolongations has the form

$$\mathbb{R}^2 \xleftarrow{\pi} \mathbb{R}^4 \xleftarrow{\pi_{1,0}} \mathfrak{A}_1 \xleftarrow{\pi_{2,1}} \mathfrak{A}_2,$$

where the isomorphism $\pi_{2,1} : \mathfrak{A}_2 \to \mathfrak{A}_1$ defines a linear Cartan connection on the three-dimensional vector bundle

$$\pi_1 : \mathfrak{A}_1 \to \mathbb{R}^2.$$

In what follows, we use coordinates in the jet spaces adjusted to the Chern connection and weight. Thus, for example, $u_{k,l}$ stands for the operator $\delta_1^k \delta_2^l$.

In these coordinates, the abelian equation takes the following form:

$$u_1 - u_2 = 0,$$
$$v_1 - av_2 - a_2 v = 0,$$
$$u_1 + v_1 = 0,$$

and the obstruction

$$\varkappa = (\varDelta_1 \varDelta_2 \delta_1 - \delta_1 \varDelta_1 \varDelta_2)u + (\varDelta_1 \varDelta_2 \delta_1 - \varDelta_1 \delta_1 \varDelta_2)v$$

equals

$$\varkappa = c_0 v_2 + c_1 v + c_2 u,$$

where c_0, c_1, and c_2 are certain functions of the curvature function K, the basic invariant a and their covariant derivatives K_i and a_i, a_{ij} (see formula (1) in [12]).

The coefficient c_0 in the expression of $\varkappa$ has an intrinsic geometric meaning.

Namely, by the *curvature function* of a 4-web we mean the arithmetic mean of the curvatures of its 3-subwebs $[1, 2, 3]$, $[1, 2, 4]$, $[1, 3, 4]$ and $[2, 3, 4]$.

Then (see [12]) the coefficient c_0 equals the curvature function of the 4-web.

7.2 4-Webs of Maximum Rank

A planar 4-web has the maximum rank three if and only if the obstruction $\varkappa$ identically equals zero, i.e., if and only if $c_0 = c_1 = c_2 = 0$. Computing these coefficients leads us to the following result (see [12]).

Theorem 2. *A planar* 4-*web is of maximum rank if and only if the following relations hold:*

$$c_0 = K + \frac{a_{11} - aa_{22} - 2\left(1 - a\right)a_{12}}{4a(1 - a)} + \frac{\left(-1 + 2a\right)a_1^2 - a^2 a_2^2 + 2\left(1 - a\right)^2 a_1 a_2}{4\left(1 - a\right)^2 a^2},$$

$$c_1 = \frac{K_2 - K_1}{4(1 - a)} + \frac{\left(a - 4\right)a_1 + \left(11 - 20a + 12a^2\right)a_2}{12\left(1 - a\right)^2 a} K + \frac{a_{112} - a_{122}}{4a(1 - a)}$$

$$+ \frac{a_1 - aa_2}{4a^2(1 - a)} a_{22} + \frac{\left(2a - 1\right)\left(a_1 - aa_2\right)}{4\left(1 - a\right)^2 a^2} a_{12} - \frac{a_2^2\left(\left(1 - 2a\right)a_1 + aa_2\right)}{4\left(1 - a\right)^2 a^2},$$

$$c_2 = \frac{aK_2 - K_1}{4a(1 - a)} + \frac{\left(1 - 2a\right)a_1 - \left(a - 2\right)aa_2}{4\left(1 - a\right)^2 a^2} K.$$

Vanishing of the coefficients c_1 and c_2 for 4-webs with $c_0 = 0$ is equivalent to linearizability of the web (see [4]). Therefore the above theorem can be formulated in pure geometric terms:

Theorem 3. *A* 4-*web is of maximum rank three if and only if it is linearizable and its curvature vanishes.*

Theorem 3 leads to interesting results in web geometry.

1. A linearizable planar 4-web is of maximum rank if and only if its curvature vanishes.
2. A planar 4-web of maximum rank is linearizable (algebraizable) (Poincaré).
3. If a planar 4-web with a constant basic invariant a has maximum rank, then it is parallelizable.
4. Parallelizable planar 4-webs have maximum rank.
5. The Mayrhofer 4-webs are of maximum rank.

Recall that a 4-web is called the *Mayrhofer* web if all 3-subwebs of this web are parallelizable.

7.3 4-Webs of Maximum Rank and Surfaces of Double Translation

A surface $S \subset \mathbb{R}^3$ is a *surface of translation* in if it admits a vector parametric representation $r = R(u, v)$, where $R(u, v)$ is a solution of the wave equation

$$R_{uv} = 0.$$

Then

$$r = f(u) + g(v), \tag{66}$$

or, in components of vectors

$$\begin{cases} x = f^1(u) + g^1(v), \\ y = f^2(u) + g^2(v), \\ z = f^3(u) + g^3(v). \end{cases}$$

A surface S is a *surface of double translation* if in addition to representation (66) it also admits a representation

$$r = h(s) + k(t), \tag{67}$$

such that the coordinate functions u, v, s, t on the surface are pairwise independent.

In other words, they define a 4-web on the surface S. If S is a surface of double translation, then it follows from (66) and (67) that

$$f^i(u) + g^i(v) - h^i(s) - k^i(t) = 0 \tag{68}$$

for $i = 1, 2, 3$. These relations can be viewed as abelian relations for the 4-web mentioned above.

If the surface S does not belong to a plane, then (68) gives three independent abelian relations for the web. Therefore, this web has the maximal rank, and as we have seen earlier, it is linearizable (algebraizable). This result was first proved by Sophus Lie in the form.

Theorem 4. ([16]) *If S is a surface of double translation not belonging to a plane, then the curves $f'(u), g'(v), h'(s)$ and $k'(t)$ belong to an algebraic curve of degree four.*

More on the subject, its further developments and references one can find in [8] and [3].

7.4 4-Webs of Rank Two

As we have seen, a 4-web admits an abelian equation (has a positive rank) if and only if the equation

$$c_0 v_2 + c_1 v + c_2 u = 0 \tag{69}$$

has a nonzero solution.

Suppose that $c_0 = 0$. Then if two other coefficients $c_1 = c_2 = 0$, then a 4-web is of maximum rank three. If $c_0 = 0$ but one of the coefficients c_1 or c_2 is not 0, then $c_1 v + c_2 u = 0$ and then, say u, satisfies a first-order PDE system of two equations. Therefore, the 4-web admits not more than one abelian equation (i.e., it is of rank one or zero).

Assume that $c_0 \neq 0$. Then we can find all first derivatives u_i, v_j from the abelian equation and (69):

$$u_1 = \frac{ac_1 - a_2c_0}{c_0}v + \frac{ac_2}{c_0}u,$$

$$u_2 = \frac{ac_1 - a_2c_0}{c_0}v + \frac{ac_2}{c_0}u,$$

$$v_1 = \frac{a_2c_0 - ac_1}{c_0}v - \frac{ac_2}{c_0}u,$$

$$v_2 = -\frac{c_1}{c_0}v - \frac{c_2}{c_0}u.$$

Therefore, a 4-web has rank two if and only if the above system is compatible.

Theorem 5. *A planar* 4-*web is of rank two if and only if* $c_0 \neq 0$, *and*

$$G_{ij} = 0, \ i, j = 1, 2, \tag{70}$$

where

$$
\begin{aligned}
G_{11} &= ac_0(c_{2,2} - c_{2,1}) + ac_2(c_{0,1} - c_{0,2}) - a\,(1-a)\,c_1c_2 \\
&\quad + (2a_2 - a_1 - aa_2)\,c_0c_2 - Kc_0^2, \\
G_{12} &= ac_0(c_{1,2} - c_{1,1}) + ac_1(c_{0,1} - c_{0,2}) - a\,(1-a)\,c_1^2 \\
&\quad + (2a_2 - a_1 - 2aa_2)\,c_0c_1 + \left(a_2^2 + a_{12} - a_{22}\right)c_0^2, \\
G_{21} &= c_0(c_{2,1} - ac_{2,2}) + c_2(ac_{0,2} - c_{0,1}) - 2a_2c_0c_2 + a\,(1-a)\,c_2^2, \\
G_{22} &= c_0(c_{1,1} - ac_{1,2}) + c_1(ac_{0,2} - c_{0,1}) + a\,(1-a)\,c_1c_2 - a_2c_0c_1 \\
&\quad - a_2(1-a)c_0c_2 + (a_{22} - K)\,c_0^2.
\end{aligned}
$$

Example 2. Consider the planar 4-web with the following web functions

$$x, \ y, \ \frac{x}{y}, \ xy(x+y).$$

The linearizability conditions (see [4]) for this web are not satisfied, and therefore, this 4-web is not linearizable, but in this case $G_{11} = G_{12} = G_{21} = G_{22} = 0$. Hence, the 4-web is of rank two.

This example gives us the following important property:
General 4-*webs of rank two are not linearizable.*

7.5 4-Webs of Rank One

As we have seen earlier, a 4-web can be of rank one if $c_0 = 0$ but one of the coefficients c_1 and c_2 of (69) is not 0 or if $c_0 \neq 0$. The following theorem outlines the four cases when a 4-web can be of rank one.

Theorem 6. *A planar* 4-*web is of rank one if and only if one of the following conditions holds:*

1. $c_0 = 0$, $J_1 = J_2 = 0$, *where*

$$\begin{aligned}
J_1 &= a_2 c_1 c_2 (c_1 - c_2) + a c_2^2 (c_{1,2} - c_{1,1}) \\
&\quad + c_1 c_2 (c_{1,1} + a(c_{2,1} - c_{1,2} - c_{2,2})) + c_1^2 (a c_{2,2} - c_{2,1}), \\
J_2 &= c_1^2 (c_1 - c_2)^2 K + (c_{1,11} - c_{1,12}) c_1 c_2 (c_2 - c_1) \\
&\quad + c_1^2 (c_1 - c_2)(c_{2,11} - c_{2,12}) - c_2 (2c_1 - c_2) c_{1,1}(c_{1,2} - c_{1,1}) \\
&\quad + c_1^2 c_{2,1}(c_{1,2} - c_{2,2} + c_{2,1}) + c_1^2 c_{1,1}(c_{2,2} - 2c_{2,1})
\end{aligned}$$

and $c_1 \neq c_2$, $c_1 \neq 0$.

2. $c_0 = 0, c_1 = c_2 \neq 0$, *and* $J_3 = 0$, *where*

$$J_3 = (a_{22} - a_{12})(1 - a) + a_2(a_2 - a_1) - (1 - a)^2 K.$$

3. $c_0 = 0, c_1 = 0, c_2 \neq 0$, *and* $J_4 = 0$, *where*

$$J_4 = a_{12} a - a_1 a_2 - K a^2.$$

4. $c_0 \neq 0$, *and* $J_{10} = J_{11} = J_{12} = 0$, *where*

$$J_{10} = G_{11} G_{22} - G_{21} G_{12},$$

$$\begin{aligned}
J_{11} &= c_0 (G_{21,1} G_{22} - G_{22,1} G_{21}) + (a_2 c_0 - a c_1) G_{21}^2 \\
&\quad + (a c_2 - a_2 c_0 + a c_1) G_{21} G_{22} - a c_2 G_{22}^2,
\end{aligned}$$

$$\begin{aligned}
J_{12} &= c_0 (G_{21,2} G_{22} - G_{22,2} G_{21}) + (a_2 c_0 - a c_1) G_{21}^2 \\
&\quad + a(c_2 - c_1) G_{21} G_{22} - c_2 G_{22}^2.
\end{aligned}$$

Proof. See [12].

Example 3. Consider the planar 4-web with the following web functions

$$x, \ y, \ \frac{xy^2}{(x-y)^2}, \ \frac{x^2 y}{(x-y)^2}.$$

For this web, we have $c_0 = 0$ and $J_1 = J_2 = 0$. Thus, we have the web of type 1 as indicated in Theorem 6, and this 4-web is of rank one.

In this example the 4-web is not linearizable. Therefore,
General 4-webs of rank one are not linearizable.

8 Planar 5-Webs of Maximum Rank

Let us consider a planar 5-web in the standard normalization

$$\omega_1 + \omega_2 + \omega_3 = 0, \quad a\omega_1 + \omega_2 + \omega_4 = 0, \quad b\omega_1 + \omega_2 + \omega_5 = 0,$$

where a and b are the basic invariants of the web.

The abelian equation for such a web has the form

$$(w + au + bv)\omega_1 + (w + u + v)\omega_2 + w\omega_3 + u\omega_4 + v\omega_5 = 0,$$

where we have

$$\lambda_1 = w + au + bv, \quad \lambda_2 = w + u + v,$$

and

$$\lambda_3 = w, \quad \lambda_4 = u, \quad \lambda_5 = v.$$

The functions w, u, and v satisfy the abelian equation

$$\delta_1(w) - \delta_2(w) = 0, \quad \delta_1(u) - \delta_2(au) = 0,$$
$$\delta_1(v) - \delta_2(bv) = 0, \quad \delta_1(w) + \delta_1(u) + \delta_1(v) = 0,$$

and their compatibility condition takes the form

$$\varkappa = (\Delta_1\Delta_2\Delta_3\delta_1 - \delta_1\Delta_2\Delta_3\Delta_1)(w) + (\Delta_1\Delta_2\Delta_3\delta_1 - \Delta_1\delta_1\Delta_3\Delta_2)(u)$$
$$+ (\Delta_1\Delta_2\Delta_3\delta_1 - \Delta_1\Delta_2\delta_1\Delta_3)(v) = 0.$$

In the canonical coordinates in the jet bundles, the abelian equation has the form

$$u_1 + v_1 + w_1 = 0, \qquad v_1 - bv_2 - b_2v = 0,$$
$$u_1 - au_2 - a_2u = 0, \quad w_1 - w_2 = 0,$$

and the obstruction $\varkappa$ equals

$$c_0w_{22} + c_1w_2 + c_2v_2 + c_3w + c_4u + c_5v = 0,$$

where the explicit form of expressions for the coefficients c_9, c_1, c_2, c_3, c_4 and c_5 can be found in [12].

This gives the following result [12]:

A planar 5-web is of maximum rank if and only if the invariants c_0, c_1, c_2, c_3, c_4 and c_5 vanish.

Similar to the case of 4-webs, the coefficient c_0 in the expression of $\varkappa$ for 5-webs has an intrinsic geometric meaning.

Namely, we call by the *curvature function* of a 5-web the arithmetic mean of the curvature functions of its ten 3-subwebs.

The straightforward calculation shows that the curvature function equals to c_0.

In other words [12], the curvature of a planar 5-web of maximum rank equals zero.

For the case of planar 5-webs with constant basic invariants a and b the invariants c_i, for $i = 0, 1, 2, 3, 4, 5$, vanish (and the web is of maximum rank) if and only if this web is parallelizable [12].

Example 4. We consider the Bol 5-web with the web functions

$$x, y, \frac{x}{y}, \frac{1-y}{1-x}, \frac{x-xy}{y-xy}.$$

For this web we have

$$K = 0,\ a = \frac{xy-x}{xy-y},\ b = \frac{y-1}{x-1},$$

and $c_i = 0$, for $i = 0, 1, ..., 5$.

Thus, the 3-web is of maximum rank.

Using the linearizability conditions for planar 5-webs [4], we see that the Bol 5-web is not linearizable.

The above example leads us to the following important observation:

General planar 5-webs of maximum rank are not linearizable.

9 Projective Structures and Planar 4-Webs

In this section we give more direct construction of the projective structure associated with 4-webs (see [13]).

Remind that an affine connection ∇ on the plane determines a covariant differential

$$d_\nabla : \Omega^1\left(\mathbb{R}^2\right) \to \Omega^1\left(\mathbb{R}^2\right) \otimes \Omega^1\left(\mathbb{R}^2\right).$$

This differential splits into the sum

$$d_\nabla = d_\nabla^a \oplus d_\nabla^s,$$

where

$$d_\nabla^a : \Omega^1\left(\mathbb{R}^2\right) \to \Omega^2\left(\mathbb{R}^2\right)$$

is the skew-symmetric part, and

$$d_\nabla^s : \Omega^1\left(\mathbb{R}^2\right) \to S^2\left(\Omega^1\right)\left(\mathbb{R}^2\right)$$

is the symmetric one.

The connection is torsion-free if and only the skew-symmetric part coincides with the de Rham differential:

$$d_\nabla^a = d.$$

A foliation given by a differential 1-form ω is geodesic (i.e., all leaves of the foliation are geodesics) with respect to connection ∇ if and only if (see [13]):

$$d_\nabla^s(\omega) = \theta \cdot \omega$$

for some differential 1-form θ.

Remark that it follows from the above formula that two affine connections, say, ∇ and ∇', are projectively equivalent (i.e., have the same geodesics) if and only if there exists a differential 1-form ρ such that

$$d_\nabla^s(\omega) - d_{\nabla'}^s(\omega) = \rho \cdot \omega$$

for all differential 1-forms ω.

Assume that a 4-web is given by differential 1-forms $\omega_i, i = 1, 2, 3, 4$, which are normalized

$$\omega_1 + \omega_2 + \omega_3 = 0,$$
$$a\omega_1 + \omega_2 + \omega_4 = 0,$$

and

$$d\omega_3 = 0.$$

Let ∇ be a torsion-free connection for which all foliations $\omega_i = 0$, $i = 1, 2, 3, 4$, are geodesics.

We call such 4-webs *geodesic*.

Then

$$d_\nabla^s(\omega_i) = \theta_i \cdot \omega_i$$

for all $i = 1, 2, 3, 4$ and some differential 1-forms θ_i.

Differentiating the normalization conditions, we get

$$\theta_1 \cdot \omega_1 + \theta_2 \cdot \omega_2 + \theta_3 \cdot \omega_3 = 0,$$
$$da \cdot \omega_1 + a\theta_1 \cdot \omega_1 + \theta_2 \cdot \omega_2 + \theta_4 \cdot \omega_4 = 0.$$

If

$$\theta_i = A_i\omega_1 + B_i\omega_2$$

for all $i = 1, 2, 3, 4$, then the above system is just a system of linear equations for coefficients A_i and B_i.

Solving this system, we find that

$$A_2 = A_1 + z, \quad B_2 = B_1 + z,$$
$$A_3 = A_1, \qquad B_3 = B_1 + z,$$
$$A_4 = A_1 + \frac{a_1}{a}, \; B_4 = B_1 + z,$$

where

$$z = \frac{a_1 - aa_2}{a(1 - a)}.$$

In other words, the affine connection is completely determined by the differential 1-form θ_1, and

$$\theta_2 = \theta_1 - z\omega_3,$$

$$\theta_3 = \theta_1 + z\omega_2,$$

$$\theta_4 = \theta_1 + \frac{a_1}{a}\omega_1 + z\omega_2.$$

This shows that all such affine connections are projectively equivalent. Taking the representative with

$$\theta_1 = \frac{z}{2}\omega_3,$$

we get the following result (this result was first obtained in [6], §29, p. 246):

Theorem 7. *There is a unique projective structure associated with a planar 4-web in such a way that the 4-web is geodesic with respect to the structure.*

The projective structure is an equivalence class of the torsion-free affine connection ∇ with the following symmetric differential:

$$d_\nabla^s(\omega_1) = \frac{z}{2}\,\omega_3 \cdot \omega_1,$$

$$d_\nabla^s(\omega_2) = -\frac{z}{2}\,\omega_3 \cdot \omega_2.$$

We say that a planar d-web is *geodesic* with respect to an affine connection if all leaves of all foliations are geodesic.

The above theorem gives a criterion for a d-web to be geodesic. For simplicity we take the case of 5-webs. Let a 5-web be given by differential 1-forms $\omega_i, i = 1, 2, 3, 4, 5$, which are normalized as follows:

$$\omega_1 + \omega_2 + \omega_3 = 0,$$

$$a\omega_1 + \omega_2 + \omega_4 = 0,$$

$$b\omega_1 + \omega_2 + \omega_5 = 0,$$

and

$$d\omega_3 = 0.$$

This web is geodesic if and only if the fifth foliation $\omega_5 = 0$ is geodesic with respect to the canonical projective structure determined by the 4-web ($\omega_i, i = 1, 2, 3, 4$).

We have

$$d_\nabla^s(\omega_5) = \theta_5 \cdot \omega_5 + (zb(1 - b) - (b_1 - bb_2))\omega_1^2.$$

Therefore, in order to have a geodesic 5-web, the last term should vanish.

Theorem 8. *A 5-web is geodesic if and only if the basic invariants a and b satisfy the following condition:*

$$\frac{a_1 - aa_2}{a(1 - a)} = \frac{b_1 - bb_2}{b(1 - b)}. \tag{71}$$

The linearizability problem (see [4]) for planar webs can be reformulated now as follows: *a planar d-web is linearizable if and only if the web is geodesic and the canonical projective structure of one of its 4-subwebs is flat.*

The flatness of a projective structure can be checked by the Liouville tensor (see [18], [17], [14]). This tensor can be constructed as follows (see, for example, [21]). Let ∇ be a representative of the canonical projective structure, and Ric be the Ricci tensor of the connection ∇. Define a new tensor $\mathfrak{P}$ as

$$\mathfrak{P}(X,Y) = \frac{2}{3}Ric(X,Y) + \frac{1}{3}Ric(Y,X)$$

for all vector fields X, Y.

The Liouville tensor $\mathfrak{L}$ is defined as follows:

$$\mathfrak{L}(X,Y,Z) = \nabla_X(\mathfrak{P})(Y,Z) - \nabla_Y(\mathfrak{P})(X,Z)$$

for all vector fields X, Y, Z.

The tensor is skew-symmetric in X and Y, and therefore it belongs to

$$\mathfrak{L} \in \Omega^1(\mathbb{R}^2) \otimes \Omega^2(\mathbb{R}^2).$$

It is known (see [18], [21], [17], [14]) that *the Liouville tensor depends on the projective structure defined by ∇ and vanishes if and only if the projective structure is flat.*

For the case of the projective structure associated with a planar 4-web we shall call this tensor the *Liouville tensor* of the 4-web

Consider three invariants:

$$w = \frac{f_y}{f_x}, \quad \alpha = \frac{aa_y - wa_x}{wa(1-a)}, \quad k = (\log w)_{xy}. \tag{72}$$

Then the Liouville tensor has the form

$$\mathfrak{L} = (L_1\omega_1 + \frac{L_2}{w}\omega_2) \otimes \omega_1 \wedge \omega_2,$$

where L_1 and L_2 are relative differential invariants of order three. The explicit formulas for these invariants are

$$3L_1 = w(-(kw)_x + \alpha_{xx} + \alpha\alpha_x) + (\alpha w_{xx} + (\alpha^2 + 3\alpha_x)w_x - 2\alpha_{xy} - 2\alpha\alpha_y)$$
$$+ w^{-1}(-\alpha w_{xy} - 2\alpha_y w_x + \alpha w_x^2) + w^{-2}\alpha w_x w_y,$$
$$3L_2 = w^2(-(kw^{-1})_y + 2\alpha\alpha_x) + w(2\alpha^2 w_x - 2\alpha_{xy} - \alpha\alpha_y)$$
$$+ (-\alpha w_{xy} - 2\alpha_y w_x + \alpha_{yy}) + w^{-1}(\alpha w_x w_y - \alpha_y w_y).$$

$$\tag{73}$$

Summarizing, we get the following result.

Theorem 9. *A planar d-web is linearizable if and only if the web is geodesic and the Liouville tensor of one of its 4-subwebs vanishes.*

Corollary 1. *If the basic invariants of all 4-subwebs of a d-web are constants, then the d-web is linearizable if and only if it is parallelizable.*

Proof. First of all, the web is geodesic because of conditions (71).

Moreover, for a 4-subweb , condition $a = \mathrm{const}$. implies $\alpha = 0$, and by Theorem 9 and (73), the 4-web is linearizable if and only if

$$(kw)_x = 0,$$

and

$$(kw^{-1})_y = 0.$$

Then $w = A(x)B(y)$ and by (8), $K = 0$. Therefore, due to Section 3, the 4-web is parallelizable.

The d-web is parallelizable too, because it geodesic and has constant basic invariants.

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