On a class of linearizable planar geodesic webs

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Abstract

We present a complete description of a class of linearizable planar geodesic webs which contain a parallelizable 3-subweb.

1 Introduction

The paper is a continuation of [3]. In the paper [3] we considered some classical problems of the theory of planar webs. In particular, at the end of the paper we proved that *a planar d-web is linearizable if and only if the web is geodesic and the Liouville tensor of one of its* 4*-subwebs vanishes.* In the current paper we describe all linearizable planar geodesic webs satisfying the following additional condition: the curvature K of one of its 3-subwebs vanishes.

2 The Problem

Below we give some (not all) definitions and notions which will be used in the paper. For additional information a reader is advised to look into [3].

We consider the plane M endowed with a torsion-free connection ∇ and a geodesic d-web in M, i.e., a d-web all leaves of all foliations of which are geodesic with respect to the connection ∇. We have proved in [3] that *there is a unique projective structure associated with a planar* 4*-web in such a way that the* 4*-web is geodesic with respect to the structure.*

The flatness of the projective structure can be checked by the Liouville tensor (see [6], [5], [4]). This tensor can be constructed as follows (see, for example, [7]).

Let ∇ be a representative of the canonical projective structure, and Ric be the Ricci tensor of the connection ∇. Define a new tensor $\mathfrak{P}$ as

$$\mathfrak{P}(X,Y) = \frac{2}{3}Ric(X,Y) + \frac{1}{3}Ric(Y,X),$$

where X and Y are arbitrary vector fields.

The Liouville tensor $\mathfrak{L}$ is defined as follows:

$$\mathfrak{L}(X,Y,Z) = \nabla_X(\mathfrak{P})(Y,Z) - \nabla_Y(\mathfrak{P})(X,Z)$$

where X, Y and Z are arbitrary vector fields.

The tensor is skew-symmetric in X and Y, and therefore it belongs to

$$\mathfrak{L} \in \Omega^1(\mathbb{R}^2) \otimes \Omega^2(\mathbb{R}^2).$$

It is known (see [6], [7], [5], [4]) that *the Liouville tensor depends on the projective structure defined by* ∇ *and vanishes if and only if the projective structure is flat.*

For the case of the projective structure associated with a planar 4-web we shall call this tensor the *Liouville tensor* of the 4-web.

Let us consider a 4-web with a 3-subweb given by a web function $f(x,y)$ and a basic invariant a (see [3] for more details) and introduce the following three invariants:

$$w = \frac{f_y}{f_x}, \quad \alpha = \frac{aa_y - wa_x}{wa(1-a)}, \quad k = (\log w)_{xy}. \tag{1}$$

Then the Liouville tensor has the form [3]:

$$\mathfrak{L} = (L_1\omega_1 + \frac{L_2}{w}\omega_2) \otimes \omega_1 \wedge \omega_2,$$

where L_1 and L_2 are relative differential invariants of order three.

The explicit formulas for these invariants are

$$\begin{aligned} 3L_1 &= w(-(kw)_x + \alpha_{xx} + \alpha\alpha_x) + (\alpha w_{xx} + (\alpha^2 + 3\alpha_x)w_x - 2\alpha_{xy} - 2\alpha\alpha_y) \\ &\quad + w^{-1}(-\alpha w_{xy} - 2\alpha_y w_x + \alpha w_x^2) + w^{-2}\alpha w_x w_y, \\ 3L_2 &= w^2(-(kw^{-1})_y + 2\alpha\alpha_x) + w(2\alpha^2 w_x - 2\alpha_{xy} - \alpha\alpha_y) \\ &\quad + (-\alpha w_{xy} - 2\alpha_y w_x + \alpha_{yy}) + w^{-1}(\alpha w_x w_y - \alpha_y w_y). \end{aligned} \tag{2}$$

As we said in Introduction, at the end of the paper [3] we proved that a planar d-web is linearizable if and only if the web is geodesic and the Liouville tensor of one of its 4-subwebs vanishes.

In the current paper we consider *a class of planar d-webs for which the curvature K of one of its 3-subwebs vanishes.*

In order to prove the main theorem, we need the following lemma.

Lemma 1 *If $K = 0$, we can reduce w (see (1)) to one: $w = 1$.*

Proof. In fact, because

$$K = -\frac{1}{f_x f_y}\left(\log \frac{f_x}{f_y}\right)_{xy},$$

it follows from $K = 0$ that $(\log w)_{xy} = 0$. Hence $\log w = u(x) + v(y)$, where $u(x)$ and $v(y)$ are arbitrary functions. It follows that $w = a(x)b(y)$, where $a(x) = e^{u(x)}$ and $b(y) = e^{v(y)}$. Taking the gauge transformation $x \to X(x), y \to Y(y)$, with $X'(x) = e^{u(x)}$ and $Y'(y) = e^{-v(y)}$, we get that $w = 1$. ∎

We shall prove now the main theorem.

Theorem 2 *A planar d-web, for which the curvature K of one of its 3-subwebs vanishes, is linearizable if and only if the web is geodesic, and the invariants α defined by its 4-subwebs have one of the following forms:*

(i)

$$\alpha = \frac{\wp'(2x+y+\lambda_1, g_2, g_3) - \wp'(x+2y+\lambda_2, g_2, -g_3)}{\wp(2x+y+\lambda_1, g_2, g_3) - \wp(x+2y+\lambda_2, g_2, -g_3)}, \tag{3}$$

where $\wp$ is the Weierstrass function, g_2 and g_3 are invariants, and λ_1 and λ_2 are arbitrary constants.

(ii)

$$\alpha = k\frac{e^{k(x-y+C)}+1}{e^{k(x-y+C)}-1}, \tag{4}$$

where k and C are arbitrary constants.

(iii)

$$\alpha = -k\tan\frac{x-y+C}{2}, \tag{5}$$

where k and C are arbitrary constants.

(iv)

$$\alpha = \frac{2}{x-y+C}, \tag{6}$$

where C is an arbitrary constant.

Here x, y are such coordinates that the 3-subweb is defined by the web functions x, y and $x+y$.

Proof. By Theorem 9 of [3], the conditions of linearizability are $L_1 = 0, L_2 = 0$. By (1) and Lemma 1, the condition $K = 0$ implies $k = 0, w = 1$.

It follows that the conditions $L_1 = 0, L_2 = 0$ become

$$\begin{cases} \alpha_{xx} - 2\alpha_{xy} + \alpha\alpha_x - 2\alpha\alpha_y = 0, \\ \alpha_{yy} - 2\alpha_{xy} + 2\alpha\alpha_x - \alpha\alpha_y = 0. \end{cases} \tag{7}$$

Conditions (7) can be written in the form

$$\begin{cases} (\partial_x - 2\partial_y)(\alpha_x + \frac{1}{2}\alpha^2) = 0, \\ (\partial_y - 2\partial_x)(\alpha_y - \frac{1}{2}\alpha^2) = 0. \end{cases} \tag{8}$$

Therefore, relations (8) imply

$$\begin{cases} \alpha_x + \frac{1}{2}\alpha^2 = A(2x+y), \\ \alpha_y - \frac{1}{2}\alpha^2 = B(x+2y) \end{cases} \tag{9}$$

for some functions A and B.

Differentiating the first equation of (9) with respect to y and the second one with respect to x, we get the following compatibility conditions for (9):

$$\alpha\alpha_y + \alpha\alpha_x = A' - B',$$

which by (9) is equivalent to

$$(A+B)\alpha = A' - B'. \tag{10}$$

We assume that $A + B \neq 0$. (The case $A + B = 0$ will be considered separately.) Then equation (10) implies

$$\alpha = \frac{A' - B'}{A + B}. \tag{11}$$

Next, we substitute α from (11) into equations (7). As a result, we obtain that

$$\begin{cases} (2A'' - B'')(A+B) - (A' - B')(2A' + B') + \frac{1}{2}(A' - B'^2 \\ = A(A+B)^2, \\ (A'' - 2B'')(A+B) - (A' - B')(A' + 2B') - \frac{1}{2}(A' - B'^2 \\ = B(A+B)^2. \end{cases} \tag{12}$$

Adding and subtracting equations (12), we find that

$$(A'' - B'')(A+B) - (A'^2 - B'^2) = \frac{(A+B)^3}{3} \tag{13}$$

and

$$A'' + B'' = A^2 - B^2. \tag{14}$$

Therefore,

$$\begin{cases} A'' - A^2 = c, \\ (B'' + B^2 = -c, \end{cases} \tag{15}$$

for a constant $c \in \mathbb{R}$.

Multiplying equations (15) by A' and B', respectively, we get

$$A'A'' - A'^2 = cA';$$
$$B'B'' + B'^2 = -cB',$$

and

$$\Big(\frac{1}{2}A'^2 - \frac{1}{3}A'^3 - cA\Big)' = 0;$$
$$\Big(\frac{1}{2}B'^2 + \frac{1}{3}B'^3 + cB\Big)' = 0,$$

respectively.

This means that
$$\frac{1}{2}A'^2 - \frac{1}{3}A'^3 - cA = a(s) \tag{16}$$
and
$$\frac{1}{2}B'^2 + \frac{1}{3}B'^3 + cB = b(t), \tag{17}$$
where $s = x + 2y$ and $t = 2x + y$.

Now equations (16), (17) and (13) give
$$b = a = \text{const}\,. \in \mathbb{R} \tag{18}$$

Remind that solutions of the equation
$$y'^2 = 4y^3 - g_2 y - g_3 \tag{19}$$
have the form
$$y = \wp(x + \lambda, g_2, g_3), \tag{20}$$
where $\wp$ is the Weierstrass function, g_2 and g_3 are invariants, and λ is an arbitrary constant.

By (18), equations (16) and (17) can be written as
$$\begin{aligned} A'^2 &= \frac{2}{3}A^3 + 2cA + 2a, \\ B'^2 &= -\frac{2}{3}B^3 - 2cB - 2a. \end{aligned} \tag{21}$$
Taking $A = \beta\wp$ and $B = \gamma\wp$, substituting them into (21) and comparing the result with (19), we find that
$$\beta = 6, \gamma = -6; g_2 = -\frac{c}{3}, g_3 = -\frac{a}{18},$$
i.e., g_2 and g_3 are the same for both equations (21).

By (20), the solutions of (21) are
$$\begin{cases} A = 6\wp(t + \lambda_1, g_2, g_3), \\ B = -6\wp(t + \lambda_2, g_2, -g_3), \end{cases} \tag{22}$$
where g_2 and g_3 are arbitrary constants.

Equations (22) can be now written as
$$\begin{cases} A = 6\wp(2x + y + \lambda_1, g_2, g_3), \\ B = -6\wp(x + 2y + \lambda_2, g_2, -g_3). \end{cases} \tag{23}$$

Finally, equations (11) and (23) give the following expression (3) for the invariant α:
$$\alpha = \frac{\wp'(2x + y + \lambda_1, g_2, g_3) - \wp'(x + 2y + \lambda_2, g_2, -g_3)}{\wp(2x + y + \lambda_1, g_2, g_3) - \wp(x + 2y + \lambda_2, g_2, -g_3)}.$$

Consider now the cases for which $A+B=0$, i.e., the cases

$$A=v, \quad B=-v, \quad v\in\mathbb{R}.$$

Then system (9) has the form

$$\begin{cases} \alpha_x+\frac{1}{2}\alpha^2=v, \\ \alpha_y-\frac{1}{2}\alpha^2=-v \end{cases} \tag{24}$$

and is consistent.

It follows from (24) that $\alpha_x+\alpha_y=0$. The solution of this equation is $\alpha=\alpha(x-y)$. As a result, we can write two equations (24) as one equation

$$\alpha'+\frac{1}{2}\alpha^2=v. \tag{25}$$

Three cases are possible:

(ii) $v=\frac{1}{2}k^2,\ k\neq 0$. Then the solution of (25) has the form (4).

(iii) $v=-\frac{1}{2}k^2,\ k\neq 0$. Then the solution of (25) has the form (5).

(ii) $v=0$. Then the solution of (25) has the form (6).

■

Corollary 3 *If for a geodesic d web the basic invariants are solutions of the Euler equation and one of its* 3*-subwebs is parallelizable, then this web is linearizable.*

References

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[1]In the bibliography we will use the following abbreviations for the review journals: JFM for Jahrbuch für die Fortschritte der Mathematik, MR for Mathematical Reviews, and Zbl for Zentralblatt für Mathematik.

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