# A piece-wise affine contracting map with positive entropy

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**Abstract**

We construct the simplest chaotic system with a two-point attractor.[1]

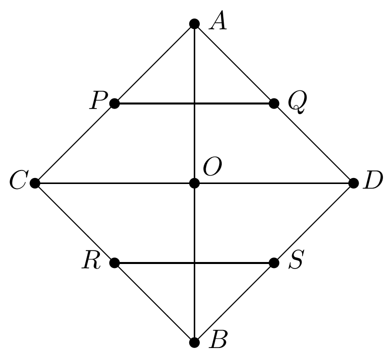
If $f : X \to X$ is an isometry of the metric space $(X, d)$, then the topological entropy vanishes: $h_{\text{top}}(f) = 0$ (for definitions and notations consult e.g. [4]).

This follows from the fact, that the iterated distance $d_n^f = \max\limits_{0\le i<n}(f^i)^*(d)$ equals $d$. If $f$ is distance non-increasing, the same equality holds and again $h_{\text{top}}(f) = 0$.

Whenever $f$ can have discontinuities of some tame nature, so that $f$ is piece-wise continuous, even the isometry result becomes difficult. In dimension 2 for invertible maps it was proven by Gutkin and Haydn [3]. In arbitrary dimension Buzzi proved that piece-wise affine isometries have zero topological entropy [2].

In the same paper after the theorem (remark 4) it is claimed that the result holds for arbitrary piece-wise (non-strictly) contracting maps. This latter is however wrong and the goal of this note is to present a counter-example.

**Example:** Let $X$ be a rhombus $ADBC$ with vertices $(\pm 1, 0), (0, \pm 1)$, see the figure below. Let $O$ be its center and $P, Q, R, S$ be on the sides as is shown.

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Let $f$ be partially defined on the interior of four big triangles forming the rhombus. These triangles are bijectively mapped by $f$ as follows:

$$ACO \longrightarrow APQ, \quad ADO \longrightarrow BRS, \quad BCO \longrightarrow AQP, \quad BDO \longrightarrow BSR.$$

Thus the piece-wise affine map is defined.

If $P, Q, R, S$ are middle-points of the intervals $AC, AD$ and $BC, BD$, then the map is not strictly contracting. But if they are closer to the vertices $A$ and $B$ respectively than to $C, D$, then $f$ is strictly contracting. In any case, the attractor of the system is the 2-point set $\{A, B\}$. Notice that the points belong to the singularity set, where the map $f$ is not (uniquely) defined.

Taking $\varepsilon = \frac{1}{2}$ we observe that the cardinality of minimal $(n, \varepsilon)$-spanning set satisfies: $2^{n+2} \leq N(f, n, \varepsilon) \leq 2^{n+3}$. In fact, if we partition $CD$ into $2^n$ equal intervals $Z_i Z_{i+1}$, then every $d_n^f$ $\varepsilon$-ball is contained in some triangle $AZ_iZ_{i+1}$ or $BZ_iZ_{i+1}$ and every such a triangle is covered by two $d_n^f$ $\varepsilon$-balls.

Therefore the topological entropy $h_{\text{top}}(f) = \log 2$ is positive. In addition, the Lyapunov spectrum is strictly negative at each point (for strict contractions), no invariant measure exists and so the variational principle breaks.

The result of Buzzi [2] generalizes however in the following fashion:

**Theorem.** *Let $f$ be a piece-wise affine map with restriction to each continuity component being conformal (non-strict) contraction. Then $h_{\text{top}}(f) = 0$.*

Now we can repeat Buzzi's remark 4 [2]: The proof of his theorem 3 applies almost literally to the above case of piece-wise affine conformal contracting maps. Therefore we omit the proof.

**Remark.** *It is obvious that if the attractor consists of one point only, then $h_{\text{top}}(f) = 0$. If the phase space $X \subset \mathbb{R}^1$ is one-dimensional and the map is (non-strictly) contracting, then again $h_{\text{top}}(f) = 0$. We don't even need to require piece-wise affine property. This follows from the Buzzi proposition 4 [1], yielding $h_{\text{top}}(f) \leq h_{\text{mult}}(f)$, where $h_{\text{mult}}(f)$ is the multiplicity entropy, because the latter always vanishes in dimension one.*

Thus our example with 2 points attractor and 2-dimensional phase-space $X$ is the simplest possible example with positive topological entropy.

# References


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