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General continuity principles for the Bergman kernel

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INTRODUCTION

In [4] the authors studied continuity principles for the Bergman kernel $K_D(w) := K_D(w,w)$ and the Bergman metric ds^2_D in dependence of the domain D. More precisely, the following result was shown:

Theorem
Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with C^∞-smooth boundary and $\{D_t\}_{-1\leq t\leq 1}$ a two-sided bumping family of D at some point $p \in \partial D$, where ∂D is strictly pseudoconvex. Then there is for any $\epsilon > 0$ and any neighborhood U of p a number $t_0 \in (0,1)$ such that

$$\left|K_D(w)K^{-1}_{D_t}(w) - 1\right| < \epsilon$$

and

$$(1-\epsilon)ds^2_D \leq ds^2_{D_t} \leq (1+\epsilon)ds^2_D$$

on $D\backslash U$ for all $t \in (-t_0, t_0)$.

For the precise definition of what was meant by a two-sided bumping family the reader is refered to [4]. It implies, in particular, that the domains D_t are only local perturbations of D near p in the sense, that for any neighborhood U of p there is a $t_0 > 0$, such that $D\backslash U = D_t\backslash U$ for all $t \in (-t_0, t_0)$. Furthermore, a certain monotonicity property of the D_t in dependence of t was required.

In this article, we want to extend such continuity principles to much more general situations. For instance, for local bumping near a point $p \in \partial D$ we want to eliminate the hypothesis of strict pseudoconvexity at p, the monotonicity in t and the restriction of the Bergman kernel to the diagonal. Furthermore, we want to prove another continuity principle, which allows much more global perturbations of D. Continuity principles for strictly pseudoconvex domains were proved by R.E.Greene and St. Krantz in [5].

Definition: Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain. By a local perturbation family for D near a point $p \in \partial D$, we mean a family of pseudoconvex domains $\{D_t\}_{0\leq t\leq 1}$, such that $D = D_0$ and for each neighborhood U of p there is a $t_0 > 0$, such that $D_t\backslash U = D\backslash U$ for all $t \in [0, t_0]$

Remark: Notice, that we do not make any regularity assumption for the boundaries ∂D_t in this definition. Also, there is no monotonicity assumption made for the family $\{D_t\}$.

Furthermore, in the case of smooth boundaries, such non-trivial local perturbation families can exist near p even if ∂D is of infinite type at p. For the case of finite 1-type of D at p, Cho [2] has constructed nice non-trivial perturbation families.

Our main result for local perturbations is:

Theorem 1

Let $\{D_t\}_{0\leq t\leq 1}$ be a local perturbation family for D near $p \in \partial D$. Then there is for any $\epsilon > 0$ and any neighborhood U of p a $t_0 > 0$, such that one has for all $t \in [0, t_0]$ and all $z, w \in D \backslash U$

$$\left| \frac{K_{D_t}(w)}{K_D(w)} - 1 \right| < \epsilon \tag{1}$$

$$\frac{|K_{D_t}(z,w) - K_D(z,w)|}{K_D^{1/2}(z) K_D^{1/2}(w)} < \epsilon \tag{2}$$

and

$$(1-\epsilon) ds_D^2(w) \leq ds_{D_t}^2(w) \leq (1+\epsilon) ds_D^2(w) \tag{3}$$

Next, we want to state a theorem about uniform continuity of the Bergman kernel and metric for more global perturbations. For this, we assume, that $D \subset \mathbb{C}^n$ is a bounded pseudoconvex domain with C^∞-smooth boundary and define:

Definition: Let $q \in \partial D$ be an arbitrary point and V an open neighborhood of q. By a regular perturbation family for D outside V we mean a family $\{D_t\}_{0\leq t\leq 1}$ of pseudoconvex domains with C^∞-smooth boundaries with the following properties:
a) $D = D_0$,
b) for any neighborhood U of $\partial D \backslash V$ there is a $t_0 > 0$, such that $D_t \backslash U = D \backslash U$ for all $t \in [0, t_0]$,
c) there is a function $r = r(t, z) : [0,1] \times \mathbb{C}^n \to \mathbb{R}$ with the following regularity properties: r is C^∞ in z and for all $(\alpha, \beta) \in (\mathbb{N}_o^n)^2$ with $|\alpha| + |\beta| \leq 2$ the derivative $\frac{\partial^{|\alpha|+|\beta|} r}{\partial z^\alpha \partial \bar{z}^\beta}$ is continuous on $[0,1] \times \mathbb{C}^n$. For all $t \in [0,1]$, $D_t = \{z \in \mathbb{C}^n : r(t,z) < 0\}$ and $d_z r(t,z) \neq 0$ for all $z \in \partial D_t$.

We will show in this article

Theorem 2

Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain with C^∞-smooth boundary and $q \in \partial D$ an arbitrary point. Furthermore, let $\{D_t\}_{0\leq t\leq 1}$ be a regular perturbation family for D outside a certain open neighborhood V of q. Suppose, that ∂D is of finite 1-type at all points of $\partial V \cap \partial D$ and let $V' \subset\subset V$ be another open neighborhood of q. Then there is for any $\epsilon > 0$ a $t_0 > 0$, such that the inequalities (1), (2) and (3) from theorem 1 hold for all $z, w \in V' \cap D$ and for all $t \in [0, t_0]$.

Remarks: a) As we will see in section 5, the assumption in theorem 2, that ∂D is of finite 1-type at $\partial V \cap \partial D$ cannot be (totally) dropped.
b) We want to point out, that inequality (2) is much too weak in case the domain D has finite 1-type everywhere, because then it is known from the work of N.Kerzman [6] and D. Catlin [1], that the Bergman kernel function $K_D(z,w)$ is C^∞ on $(\overline{D} \times \overline{D})\backslash\Delta_{\overline{D}}$, where $\Delta_{\overline{D}}$ is the diagonal. In this case, the division by $K_D^{1/2}(z)K_D^{1/2}(w)$ in (2) should not be necessary as long as the pair (z,w) stays away from the diagonal. In general, however, it is needed.

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1. A NEW VARIANT OF SOLVING THE $\overline{\partial}$-EQUATION IN L^2-SPACES

It will be very convenient for the purpose of this article to have a new variant of solving the $\overline{\partial}$-equation in certain L^2-spaces given by Kähler metrics and weight functions. Since this also might be useful for other purposes, we explain it first.

We will use the following notations: X is a connected, paracompact complex manifold of dimension n. For a Hermitian metric ds^2 on X and a continuous function $\varphi : X \to \mathbb{R}$ and any measurable (p,q)-form on X we set

$$\|u\|^2_{ds^2,\varphi} := \int_X e^{-\varphi}|u|^2\, dV$$

where $|u| = |u|_{ds^2}$ denotes the pointwise norm of u and dV means the volume form with respect to ds^2. Furthermore, for any C^∞ strictly plurisubharmonic function $\psi : X \to \mathbb{R}$ we will denote by $\partial\overline{\partial}\psi$ also the Kähler metric induced by ψ on X (the usual abuse of notation).

One has

Theorem 3
Suppose, that the manifold X admits a complete Kähler metric and let $\psi : X \to \mathbb{R}$ be a C^∞ strictly plurisubharmonic function satisfying the estimate $\partial\overline{\partial}\psi \geq \partial\psi\overline{\partial}\psi$. Furthermore, let $\varphi : X \to \mathbb{R}$ be an arbitrary plurisubharmonic C^∞-function. Then, for any $\overline{\partial}$-closed $(n,1)$-form v on X satisfying $\|v\|_{\partial\overline{\partial}\psi,\varphi} < \infty$, there is a measurable $(n,0)$-form u satisfying $\overline{\partial}u = v$ and $\|u\|_\varphi \leq C\|v\|_{\partial\overline{\partial}\psi,\varphi}$ with a numerical constant $C > 0$ (independent of X, ψ, φ, v).

Proof: Fix any complete Kähler metric ds^2 on X and set $ds^2_\epsilon := \partial\overline{\partial}\psi + \epsilon ds^2$ for any $\epsilon \geq 0$. Then we have $\|v\|_{\epsilon,\varphi} := \|v\|_{ds^2_{\epsilon,\varphi}} \leq \|v\|_{\partial\overline{\partial}\psi,\varphi}$, since v is of type $(n,1)$. Since $\partial\overline{\partial}\psi \geq \partial\psi\overline{\partial}\psi$ we have the following estimate as a special case of what was proved in Proposition 1.6 of [8]:

$$\left(i\partial\overline{\partial}(\varphi+\psi)\wedge\Lambda_\epsilon f, f\right)_{\epsilon,\varphi} \leq C_0\left(\left\|\overline{\partial}f\right\|^2_{\epsilon,\varphi} + \left\|\overline{\partial}^*_{\epsilon,\varphi}f\right\|^2_{\epsilon,\varphi}\right) \tag{4}$$

for any $\epsilon \geq 0$ and any C^∞ compactly supported $(n,1)$-form f on X.
(Here Λ_ϵ denotes the adjoint of the exterior multiplication by the fundamental form of ds^2_ϵ, $(\cdot,\cdot)_{\epsilon,\varphi}$ is the inner product associated to $\|\cdot\|_{\epsilon,\varphi}$ and $\overline{\partial}^*_{\epsilon,\varphi}$ is the adjoint of $\overline{\partial}$ with respect to $\|\cdot\|_{\epsilon,\varphi}$.)
Now, by a straightforward computation as in the proof of Proposition 1.3 in [7] one obtains from (4)

$$\left|(f,v)_{\epsilon,\varphi}\right|^2 \leq C_0\|v\|^2_{0,\varphi}\left(\left\|\overline{\partial}f\right\|^2_{\epsilon,\varphi} + \left\|\overline{\partial}^*_{\epsilon,\varphi}f\right\|^2_{\epsilon,\varphi}\right) \tag{5}$$

for any f as above. Since for $\epsilon > 0$ the metric ds^2_ϵ is complete, the estimate (5) implies, that there is a u_ϵ satisfying $\|u_\epsilon\|^2_{\epsilon,\varphi} \leq C_0\|v\|^2_{0,\varphi}$ and $\overline{\partial}u_\epsilon = v$, whenever $\epsilon > 0$. Letting $\epsilon \searrow 0$, we obtain the desired solution as weak limit of a subsequence of $(u_\epsilon)_{\epsilon>0}$. □

2. The maximizing functions

As already done in [4], we consider the so-called maximizing functions defined by

Definition: Let D be a bounded domain in $\mathbb{C}^n$. We denote by $\|\cdot\| = \|\cdot\|_D$ the L^2-norm on D with respect to the Lebesgue-measure and put

$$H^2(D) := \{f \in \mathcal{O}(D) : \|f\| < \infty\}$$

For a point $w \in D$ and for any $\alpha \in \mathbb{N}_0^n$ we define $D^\alpha := \frac{\partial^{|\alpha|}}{\partial z^\alpha}$ and

$$H^2_{(\alpha)}(D;w) := \left\{g \in H^2(D) : D^\beta g(w) = 0 \ \forall\, \beta \in \mathbb{N}_0^n \ \textit{with}\ |\beta| \leq |\alpha|, \beta \neq \alpha\right\}$$

By $B^{(\alpha)}_D(z,w) \in H^2_{(\alpha)}(D,w)$ we denote the (unique) function satisfying

$$D^\alpha B^{(\alpha)}_D(w,w) = \max\left\{|D^\alpha g(w)| \ \textit{for}\ g \in H^2_{(\alpha)}(D;w) \ \textit{with}\ \|g\|_D \leq 1\right\}$$

and we write $D^\alpha B^{(\alpha)}_D(w) := D^\alpha B^{(\alpha)}_D(w,w)$.

The main part of the proofs of theorems 1 and 2 consists, in fact, in showing the following statement

Theorem 4
Suppose, that the domain D and the perturbation family $\{D_t\}_{0\leq t\leq 1}$ for D satisfy either the conditions of theorem 1 or the conditions of theorem 2. Let U be an arbitrary open neighborhood of p (in the situation of theorem 1) and let $V' \subset\subset V$ be an arbitrary neighborhood of q (in the situation of theorem 2). Furthermore, fix an $\alpha \in \mathbb{N}_0^n$. Then there is for any $\epsilon > 0$ a $t' > 0$, such that

$$\left| \frac{D^\alpha B_{D_t}^{(\alpha)}(w)}{D^\alpha B_D^{(\alpha)}(w)} - 1 \right| < \epsilon \tag{6}$$

for all $t \in [0, t']$ and for all $w \in D \backslash U$ (in the situation of theorem 1) resp. $w \in V' \cap D$ (in the situation of theorem 2). In fact, one also has for $\widehat{D}_{t'} := \bigcap_{t\in[0,t']} D_t$ the inequality

$$\left| \frac{D^\alpha B_{D_t}^{(\alpha)}(w)}{D^\alpha B_{\widehat{D}_{t'}}^{(\alpha)}(w)} - 1 \right| < \epsilon \tag{6'}$$

for all t and w as above.

Remark: We take this opportunity to point out, that definition 1 and the estimate in Theorem 2 in [4] should have been written exactly as the above definition resp. the inequality (6).

Notice, that the statements about the continuity of the Bergman kernel on the diagonal and the continuity of the Bergman metric in theorems 1 and 2 are immediate consequences of theorem 4 (see also [4]). But also the continuity inequality (2) follows from theorem 4 in both situations. In order to show this, we show at first the following lemma about maximizing functions:

Lemma 1
Let $w \in D$ and $\alpha \in \mathbb{N}_0^n$ be given and suppose that $f \in H^2_{(\alpha)}(D)$ is a function with $\|f\| \leq 1$ and such that

$$\left| \frac{D^\alpha f(w)}{D^\alpha B_D^{(\alpha)}(w)} - 1 \right| < \epsilon$$

Then one has

$$\left\| f - B_D^{(\alpha)}(\cdot, w) \right\| < C\epsilon$$

with a numerical constant $C > 0$.

Proof: Let $(g_j)_{j\in\mathbb{N}_0}$ be a complete orthonormal basis for $H^2_{(\alpha)}(D)$ with $g_0 = B_D^{(\alpha)}(\cdot, w)$. Then one has for all $j = 1, 2, 3, \ldots$ necessarily $D^\alpha g_j(w) = 0$, since otherwise a function g of the form $g = ag_0 + bg_j$ could be constructed with $\|g\| = 1$, but $|D^\alpha g(w)| > D^\alpha g_0(w)$.

We represent $f = \sum_{j=0}^{\infty} c_j g_j$. Then $D^\alpha f(w) = c_0 D^\alpha g_0(w) = c_0 D^\alpha B_D^{(\alpha)}(w)$, such that we obtain from the hypothesis of the theorem the estimate

$$|c_0 - 1| < \epsilon$$

From this, we get $\|f\|^2 - |c_0|^2 = \sum_{j=1}^{\infty} |c_j|^2 \leq 1 - |c_0|^2 \leq 1 - (1-\epsilon)^2 \leq 2\epsilon$, such that we now can estimate

$$\left\| f(\cdot) - B_D^{(\alpha)}(\cdot, w) \right\|^2 = |1 - c_0|^2 + \sum_{j=1}^{\infty} |c_j|^2 \leq \epsilon^2 + 2\epsilon \leq 3\epsilon$$

(if $\epsilon \leq 1$). □

With this lemma, it is now easy to show, that inequality (2) (from theorem 1 and 2) also is a consequence of theorem 4. Namely:

Proof of inequality (2): Suppose, we are in a situation, where (6') is valid for $\alpha = 0$. Then we can apply lemma 1 to the function $f := B_{D_t}(\cdot, w)|\widehat{D}_{t'}$ and obtain $\|(B_{D_t}(\cdot, w)|\widehat{D}_{t'}) - B_{\widehat{D}_{t'}}(\cdot, w)\| < C\epsilon$. This implies according to the definition of the maximizing function and the fact, that always $B_{D'}(z, w) = K_{D'}^{-1/2}(w) K_{D'}(z, w)$ for any domain $D' \subset\subset \mathbb{C}^n$ and any point $z \in D'$, the estimate

$$\left| B_{D_t}(z, w) - B_{\widehat{D}_{t'}}(z, w) \right| \leq C\epsilon K_{\widehat{D}_{t'}}^{1/2}(z)$$

From this, (2) follows easily. Namely, writing $K_t := K_{D_t}$ and we get

$$\left| \frac{K_t(z, w)}{K_{\widehat{D}_{t'}}^{1/2}(z) K_t^{1/2}(w)} - \frac{K_{\widehat{D}_{t'}}(z, w)}{K_{\widehat{D}_{t'}}^{1/2}(z) K_{\widehat{D}_{t'}}^{1/2}(w)} \right| < C\epsilon$$

Next, we estimate

$$G := \frac{|K_t(z, w)|}{K_{\widehat{D}_{t'}}^{1/2}(z)} \left| \frac{1}{K_t^{1/2}(w)} - \frac{1}{K_{\widehat{D}_{t'}}^{1/2}(w)} \right|$$

Together with $\left|\frac{K_t(w)}{K_{\widehat{D}_{t'}}(w)} - 1\right| < \epsilon$, which is a consequence of (6'), and the fact, that $|K_t(z, w)| \leq K_t^{1/2}(z) K_t^{1/2}(w)|$, one obtains $G < C'\epsilon$. This gives

$$\frac{\left| K_t(z, w) - K_{\widehat{D}_{t'}}(z, w) \right|}{K_{\widehat{D}_{t'}}^{1/2}(z) K_{\widehat{D}_{t'}}^{1/2}(w)} < C''\epsilon$$

for all $t \in [0, t']$. By using this twice, namely for $t = 0$ and for arbitrary t, the inequality (2) follows easily.

Next we will show an important general estimate on the behavior of the mass of maximizing functions near the boundary of pseudoconvex domains. It will for us be a useful tool in the proof of theorem 1. Namely, by being able to use it in the case of local perturbation families, we can avoid in theorem 1 any monotonicity hypothesis with respect to the perturbations.

Theorem 5

Let D be any bounded pseudoconvex domain in $\mathbb{C}^n$ and $p \in \partial D$ an arbitrary point. Then there exists a positive number C, depending only on the dimension n and the diameter of D, such that for any $\alpha \in \mathbb{N}_o^n$, $\tau > 0$, $\eta > 0$ and $w \in D$ with $|w-p| \geq \eta$ the estimate

$$\left\| B_D^{(\alpha)}(\cdot, w) \right\|_{D \cap B(p, E(\eta))} < C\eta^\tau$$

holds. (Here $B(p,r)$ denotes the open ball of radius r centered at p and $E(\eta) = E_{\tau,\alpha}(\eta) := \exp(-\exp(\eta^{-(n+|\alpha|+\tau)}))$.)

Proof: In a first part of the proof we assume, that $D \subset B := B(0, e^{-1})$ and $p = 0$. The result then has to be proved only for $\eta < e^{-1}$. We put

$F(z) := \log(\log(-\log|z|))$ and $W_k := \{z \in B : k-1 < F(z) < k\}$

for $k \in \mathbb{R}$ and write $\psi(z) := -\log(-\log|z|)$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be any C^∞-function satisfying $\chi|(-\infty, 1/2) = 1$ and $\chi|(1,\infty) = 0$ and put $\rho_k(z) := \chi(F(z) - k)$. Given any point $w \in D \backslash B(p,\eta)$ we define $\varphi_w(z) := 2(n+|\alpha|)\log|z-w|$. Since $d\rho_k = \chi'(F(z)-k)dF$, we have

$$|d\rho_k|_{\partial\overline{\partial}\psi} \leq \sqrt{2}e^{-k}\sup|\chi'|$$

Hence we obtain

$$|d\rho_k|_{\partial\overline{\partial}\psi} \leq \sqrt{2}\eta^N \sup|\chi'| \tag{7}$$

if $k \geq -N\log\eta+1$. We also observe that (8)

$$\sup\left\{ |z-w|^{-2(n+|\alpha|)} : z \in B(p, E(\eta))\right\} = (\eta - E(\eta))^{-2(n+|\alpha|)} < 2^{2(n+|\alpha|)}\eta^{-2(n+|\alpha|)}$$

We now put $v_k := \overline{\partial}(\rho_k(z) B_D^{(\alpha)}(z,w)) \wedge dz_1 \wedge \cdots \wedge dz_n$. Then v_k is a C^∞ $(n,1)$-form on D satisfying $\overline{\partial} v_k = 0$ and $\operatorname{supp} v_k \subset \overline{W}_k$. Moreover, from (7) and (8) we obtain

$$\|v_k\|^2_{\partial\overline{\partial}\psi,\varphi_w} < C_0\eta^{2\tau} \; if \; k \geq -(n+|\alpha|+\tau)\log\eta + 1$$

Here $C_0 = 2^{n+|\alpha|+1}\sup|\chi'|^2$. We fix $k := -(n+|\alpha|+\tau)\log\eta+1$. Since D is pseudoconvex, also $D\backslash\{w\}$ admits a complete Kähler metric. Therefore, theorem 3 gives us an $(n,0)$-form u_k on $D\backslash\{w\}$ satisfying $\overline{\partial} u_k = v_k$ and

$$\left| \int_{D\backslash\{w\}} \frac{u_k \wedge \overline{u}_k}{|z-w|^{2(n+|\alpha|)}} \right| \leq C\|v_k\|^2_{\partial\overline{\partial}\psi,\varphi_w} < CC_0\eta^{2\tau} \tag{9}$$

Let $u_k = f_k(z)\,dz_1\wedge\cdots\wedge dz_n$. Then the integrability property (9) implies that the function $\rho_k(z)B_D^{(\alpha)}(z,w) - f_k(z)$ extends holomorphically to D and lies in $H^2(D)$. We call it $g_w(z)$. It enjoys the following properties: $g_w \in H^2_{(\alpha)}(D)$ and

$$D^\alpha g_w(w) = D^\alpha B_D^{(\alpha)}(w,w) \tag{10}$$

$$\|g_w\|_D \le 1 + C_1\eta^\tau \tag{11}$$

(Here one can take $C_1 = \sqrt{CC_0}$.)
From (10) and (11) we obtain

$$\left|(1+C_1\eta^\tau)^{-1}D^\alpha g_w(w) - D^\alpha B_D^{(\alpha)}(w,w)\right| = C_1\eta^\tau(1+C_1\eta^\tau)^{-1}D^\alpha B_D^{(\alpha)}(w) \tag{12}$$

By applying lemma 1 to this, we get

$$\left\|(1+C_1\eta^\tau)^{-1}g_w - B_D^{(\alpha)}(\cdot,w)\right\|_D \le C_2\eta^\tau \tag{13}$$

for some numerical constant C_2. This shows, that, in the case $D \subset B(0,e^{-1})$, it suffices for the proof of the desired inequality of Theorem 5 to show, that

$$\left\|(1+C_1\eta^\tau)^{-1}g_w\right\|_{D\cap B(p,E(\eta))} < C_3\eta^\tau$$

This, however, follows immediately from (9), if one uses, that $g_w(z) = f_k(z)$ on $D\cap B(p,E(\eta))$. The general case can very easily be deduced from this. □

3. Proof of Theorem 1

As we have already observed, theorem 1 is a consequence of theorem 4 in the situation of theorem 1. It is, therefore, this, what we have to show here. In doing this, theorem 5 is an extremely useful tool.

We are given a local perturbation family $\{D_t\}_{0\le t\le 1}$ for D at a point $p\in\partial D$. We fix an $\epsilon > 0$ and a neighborhood U of p and define $\delta_0 := \inf_{z\in\partial U}|z-p|$. Of course, we may assume, that $\delta_0 < 1$. Given any $\delta\in(0,\delta_0)$, we can choose a $t_0 > 0$ so that $D_t\backslash B(p,E_{1,\alpha}(\delta)) = D\backslash B(p,E_{1,\alpha}(\delta))$ for all $t\in[0,t_0]$.
Let now $w\in D\backslash U$ be any point. Then, for $\widehat{D} = \widehat{D}_{t_0} := \bigcap_{t\in[0,t_0]} D_t$ there is according to theorem 5 a constant $C > 0$, such that

$$\left\|B_{\widehat{D}}^{(\alpha)}(\cdot,w)\right\|_{\widehat{D}\cap B(p,E_{1,\alpha}(\delta))} < C\delta^{|\alpha|+1} \tag{14}$$

Hence, by using the same cut-off functions ρ_k for suitable k as in the proof of theorem 5 and by applying the $\bar\partial$-machinery with the Kähler metric $\partial\bar\partial(-\log(-\log(A|z-p|)))$ with a suitable constant $A>0$ and weight $2(n+|\alpha|)\log|z-w|$ to the form $v_k := \bar\partial(\rho_k B^{(\alpha)}_{\widehat{D}}(z,w))\wedge dz_1\wedge\cdots dz_n$ we can produce holomorphic functions $g_t \in H^2_{(\alpha)}(D_t;w)$ satisfying

$$D^\alpha g_t(w) = D^\alpha B^{(\alpha)}_{\widehat{D}}(w,w) \tag{15}$$

and

$$\|g_t\|_{D_t} \le 1 + C'\delta^{|\alpha|+1} \tag{16}$$

for all $t\in[0,t_0]$. Here the constant C' depends only on n and the diameter of D. By using the monotonicity of $D^\alpha B^{(\alpha)}_{\widehat{D}}$ with respect to the domain $\widehat{D}$ (observe, that $\widehat{D}\subset D\cap D_t$), we obtain (6) from (15) and (16) after assuming, that δ was chosen sufficiently small. □

Remark: Notice, that it also follows from this proof, that

$$\left|\frac{D^\alpha B^{(\alpha)}_{D_t}(w)}{D^\alpha B^{(\alpha)}_{\widehat{D}}(w)} - 1\right| < \epsilon \tag{17}$$

for all $t\in[0,t_0]$.

4. Proof of Theorem 2

As we know already, for proving theorem 2 it suffices to show theorem 4 in the situation of theorem 2. In order to do this, we show at first:

Lemma 2

Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^∞-smooth boundary, and let $W\subset\partial D$ be a (relatively) open subset. Suppose, that there exists a neighborhood U of $\partial_{\partial D}W$ in $\mathbb{C}^n$ and a C^∞ function $r:\mathbb{C}^n\to\mathbb{R}$ satisfying

a) $D=\{z: r(z)<0\}$,

b) dr vanishes nowhere on ∂D,

c) $\partial\bar\partial(-\log(-r))\ge\partial r\bar\partial r/2r^2$ on D and there exists a $\delta>0$, such that

$$\partial\bar\partial(-\log(-r)) \ge (-r)^{-\delta}\partial\bar\partial|z|^2 + \frac{\partial r\bar\partial r}{2r^2} \text{ on } D\cap\overline{U}$$

Then there exists a C^∞ function $\rho: D\to\mathbb{R}$ satisfying the following properties:

$$W\backslash U \subset \partial\{z:\rho(z)<c\}\cap\partial D\subset W \text{ for any } c\in\mathbb{R} \tag{18}$$

$$\lim_{c\to\infty}\left(\sup\left\{|d\rho(z)|_{\partial\bar\partial(-\log(-r))} : z\in D, \rho(z)>c\right\}\right) = 0 \tag{19}$$

Proof: Let τ be any real-valued C^∞ function on ∂D satisfying $\tau|W\backslash U = 0$ and $\tau|\partial D\backslash(U \cup W) = 1$ and let $\tilde{\tau} : \overline{D} \to \mathbb{R}$ be any C^∞ extension of τ from ∂D to D. Put $\rho(z) := \tau(z) \cdot \log(-\log(-r(z)))$. Then (18) is clearly satisfied by ρ. As for (19), it follows immediately from c), since

$$d\rho = \log(-\log(-r))\, d\tau + \frac{\tau}{r \log(-r)}\, dr$$

□

Proof of Theorem 4 in the situation of Theorem 2: Let the situation of Theorem 2 be given. Since ∂D is of finite 1-type at all points of $\partial V \cap \partial D$, there exists, because of condition c) of the definition of a regular perturbation family, a $t_0 > 0$, such that the functions $r_t := r(t, \cdot) : \mathbb{C}^n \to \mathbb{R}$ satisfy c) with a fixed $\delta > 0$ on $D_t \cap U_0$ for all $t \in [0, t_0]$ and a fixed neighborhood U_0 of $\partial V \cap \partial D$ (notice, that finite 1-type is an open condition with respect to such perturbations and that the 1-type is uniformly bounded, see [3]). Let now $W := V \cap \partial D$ and choose a C^∞ function ρ on D satisfying (18) and (19) of Lemma 2. Since $\partial\{z : \rho(z) < c\}$ approaches ∂D as $c \to \infty$, one can find for any $\epsilon > 0$ a $t' \in (0, t_0)$, such that $\overline{\{z : \rho(z) < c+1\}}\backslash W \subset D_t$ and $|\, d\rho|_{\partial\overline{\partial}(-\log(-r_t))} < \epsilon$ on $\{z \in D_t : \rho(z) > c\}$ for any $t \in [0, t']$ (here we use again condition c) of the definition of a regular perturbation family). Let $\chi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function as in the proof of Theorem 5, and put $\omega_\epsilon(z) := \chi(\rho(z) - c)$. Then we have

$$|\, d\omega_\epsilon|_{\partial\overline{\partial}(-\log(-r_t))} < \epsilon \sup_{\mathbb{R}} |\chi'|$$

for all $t \in [0, t']$. Again we put $\widehat{D}_{t'} := \bigcap_{t \in [0,t']} D_t$. The ω_ϵ are then used as cut-off functions, when for any $w \in V' \cap D$ good approximate extensions of $B^{(\alpha)}_{\widehat{D}_{t'}}(\cdot, w)$ from $\widehat{D}_{t'}$ to D_t are constructed for all $t \in [0, t']$ by using Theorem 3 on D_t with the metric $-\partial\overline{\partial}(\log(-r_t))$ and weight $2(n + |\alpha|)\log|z - w|$. We leave the details to the reader. We obtain from this the desired estimate

$$\left| \frac{D^\alpha B^{(\alpha)}_{D_t}(w)}{D^\alpha B^{(\alpha)}_{D_{t'}}(w)} - 1 \right| < \epsilon$$

for all $t \in [0, t']$. □

5. A Counterexample

In this section we briefly give an example of a C^∞ smooth pseudoconvex domain and a regular perturbation family for it, for which inequality (1) does not hold, the reason being the lack of a finite type condition required in Theorem 2.

Let $\Delta^2 := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1 \text{ and } |z_2| < 1\}$. Furthermore, let $\{R_t\}_{0 \le t \le 1}$ be any (continuous) family of convex domains in $\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$ satisfying

$$\partial R_0 \cap \{xy = 0\} = \{x \le -1 \text{ or } y \le -1\} \cap \{xy = 0\}$$

and

$$\partial R_t \cap \{xy = 0\} = \{x \le -2 \text{ or } y \le -2\} \cap \{xy = 0\}$$

if $t \in (0, 1]$. Let $\widehat{R}_t$ be the closure of R_t in $(\mathbb{R} \cup \{-\infty\})^2$. We set

$$D_t := \left\{ (z_1, z_2) \in \Delta^2 : (\log|z_1|, \log|z_2|) \in \widehat{R}_t \right\}$$

Then, with a suitable regularity assumption on $\{R_t\}_{0 \le t \le 1}$, $\{D_t\}_{0 \le t \le 1}$ is a regular perturbation family of pseudoconvex domains with C^∞ boundaries outside the set $V := \{(z_1, z_2) \in \partial D : |z_1| < e^{-2} \text{ or } |z_2| < e^{-2}\}$. But obviously inequality (1) fails to hold for it, as can be seen in the following way: We define for $\mu \in \mathbb{N}$ the point $w_\mu := 1 - 1/\mu$ and consider the maximizing function for the bidisc Δ^2:

$$f_\mu((z, w)) := B^{(0)}_{\Delta^2}((z, w), (0, w_\mu)) = \frac{1}{\pi^2} \frac{1 - |w_\mu|^2}{(1 - w\overline{w}_\mu)^2}$$

Furthermore, we put $\Delta_\mu := \{w \in \Delta : |w - 1| < 1/\mu^{1/3}\}$ and fix any arbitrarily small $t > 0$. Then one has $\lim_{\mu \longrightarrow \infty}(\sup_{D_t \setminus (\Delta_\mu \times \Delta)} |f_\mu|) = 0$. Therefore,

$$\lim_{\mu \longrightarrow \infty} \|f_\mu\|^2_{D_t} = \lim_{\mu \longrightarrow \infty} \int_{(\Delta_\mu \times \Delta) \cap D_t} |f_\mu|^2 \, dV = \frac{1}{e^4}$$

This shows, that, in fact,

$$\lim_{\mu \longrightarrow \infty} \frac{B_{D_t}((0, w_\mu))}{B_{\Delta^2}((0, w_\mu))} \ge e^2$$

On the other hand, it is immediate, that one has

$$\lim_{\mu \longrightarrow \infty} \frac{B_{D_0}((0, w_\mu))}{B_{\Delta^2}((0, w_\mu))} \le e$$

Together, it follows, that (1) cannot be satisfied for the family D_t.

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