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Monte Carlo Methods With Variance Reductions For System Reliability Evaluation

京都大学工学部精密工学科　熊本博光

Hiromitsu Kumamoto
Dept. of Precision Mechanics, Faculty of Engrg, Kyoto Univ.

1 INTRODUCTION

This paper briefly describes several Monte Carlo methods the author has developed for calculating the reliability (or equivalently unreliability, availability or unavailability) of a large complex system represented by a logic model such as a Boolean function, a reliability block diagram or a fault tree. More detailed descriptions can be found in references [2]-[6]. When a direct or a crude Monte Carlo method is used, a large number of trials are required to obtain reasonably precise estimates of the reliability. The Monte Carlo methods presented in this paper are based on variance-reduction techniques, and resultant variance reductions are proved theoretically. A usual term-wise calculation and a decomposition method become impractical for large systems since the reliability involves a large number of terms. Although several approximations have been proposed [1], they yield only lower and upper bounds of the reliability.

2 PROBLEM STATEMENT

2.1 Assumptions

1) The system has n components, numbered $1,\ldots,n$.
2) Each components is either functioning or failed.
3) States of components are statistically independent.
4) The system is either functioning or failed. The system is coherent.
5) Every state vector has a non-zero probability.

2.2 Notation

1) x_i: component state (random variable).

$$x_i \equiv \begin{cases} 1, & \text{if component } i \text{ is functioning} \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

2) $\overline{x}_i$: $1 - x_i$ is a complement of x_i.

$$\overline{x}_i \equiv \begin{cases} 1, & \text{if component } i \text{ is failed} \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

3) x: $(x_1,\ldots,x_n)$ is a component state vector.

4) $\overline{x}$: $(\overline{x}_1, \ldots, \overline{x}_n) = (1 - x_1, \ldots, 1 - x_n)$ is a complement vector of x.
5) b: $(b_1, \ldots, b_n)$ is a sample vector of x or $\overline{x}$.
6) $\overline{b}$: $(\overline{b}_1, \ldots, \overline{b}_n) = (1 - b_1, \ldots, 1 - b_n)$ is a complement vector of b.
7) i: is a suffix denoting a component.
8) $\phi(x)$: coherent structure function of x [1].

$$\phi(x) \equiv \begin{cases} 1, & \text{if system } i \text{ is functioning} \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

The coherent function can be regarded as a monotonically increasing function with some trivial exceptions such as exclusions of irrelevant variables, zero function, etc.
9) $\psi(\overline{x})$: $1 - \phi(x)$, complement of ϕ.

$$\psi(\overline{x}) \equiv \begin{cases} 1, & \text{if system } i \text{ is failed} \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

This function is also coherent and monotonically increasing with respect to $\overline{x}$.
10) ν, ρ: suffix denoting Monte Carlo trial.
11) N: sample size, i.e., total number of Monte Carlo trials.
12) R: system reliability (or availability).
13) Q: system unreliability (or unavailability).
14) $R_0 = 1 - Q_0$: direct Monte Carlo estimator.
15) $R_1 = 1 - Q_1$: bound Monte Carlo estimator.
16) $R_2 = 1 - Q_2$: dagger-sampling Monte Carlo estimator.
17) $R_3 = 1 - Q_3$: coverage Monte Carlo estimator.
17) $R_4 = 1 - Q_4$: less-information Monte Carlo estimator.

The problem is to calculate the system reliability,

$$R \equiv Pr\{\phi(x) = 1\} \tag{5}$$

$$= \sum_b \phi(b) Pr\{x = b\} \tag{6}$$

$$= E_x\{\phi(x)\} \tag{7}$$

or equivalently, the problem is a calculation of system unreliability

$$Q \equiv Pr\{\psi(\overline{x}) = 1\} = \sum_{\overline{b}} \psi(\overline{b}) Pr\{\overline{x} = \overline{b}\} = E_{\overline{x}}\{\psi(\overline{x})\} \tag{8}$$

3 DIRECT MONTE CARLO

Generate N independent samples $c_1, \ldots, c_N$ of x. Evaluate R or Q by the unbiased binomial estimator

$$R_0 \equiv N^{-1} \sum_{\nu=1}^{N} \phi(c_\nu), \; Q_0 \equiv N^{-1} \sum_{\nu=1}^{N} \psi(\overline{c}_\nu) \tag{9}$$

This estimator has variance

$$Var\{R_0\} = Var\{Q_0\} = N^{-1} R(1 - R) \tag{10}$$

4 BOUND MONTE CARLO [2]

Let $\phi_L(x)$ and $\phi_U(x)$ be two binary functions satisfying (11) and (12).

$$\phi_L(b) \leq \phi(b) \leq \phi_U(b), \text{ for all } b. \tag{11}$$

$$\begin{aligned} \phi_L(b) &= 1 \text{ for some } b \\ \phi_U(b) &= 0 \text{ for some } b \end{aligned} \tag{12}$$

Lower and upper bound functions $\phi_L(x)$ and $\phi_U(x)$ can be constructed by using partial path sets and cut sets of $\phi(x)$ [1], respectively.

Assume that system reliability can be calculated easily for functions ϕ_L and ϕ_U.

$$R_L \equiv \sum_b \phi_L(b) Pr\{x = b\} \tag{13}$$

$$R_U \equiv \sum_b \phi_U(b) Pr\{x = b\} \tag{14}$$

The following inequalities hold:

$$0 < R_L \leq R \leq R_U < 1 \tag{15}$$

If the equality $R_U = R_L$ holds, then $R = R_L = R_U$ and the problem is trivial; R can be obtained without the use of the Monte Carlo methods. In the discussion that follows we assume the inequality

$$R_U - R_L > 0 \tag{16}$$

Apply the straight-forward control variate method to (6); we have

$$R = \sum_b [\phi(b) - \phi_L] Pr\{x = b\} + \sum_b \phi_L(b) Pr\{x = b\} \tag{17}$$

$$= \sum_b [\phi(b) - \phi_L(b)] Pr\{x = b\} + R_L \tag{18}$$

We consider now generating random samples with probability different from $Pr\{x = b\}$ according to the importance sampling method. Define two sets.

$$X \equiv \{b | \phi(b) - \phi_L(b) = 1\} \tag{19}$$

$$Y \equiv \{b | \phi_U(b) - \phi_L(b) = 1\} \tag{20}$$

Since $X \subset Y$, we rewrite (18) as follows:

$$R = \sum_{b \in X} [\phi(b) - \phi_L(b)] Pr\{x = b\} + R_L \tag{21}$$

$$= [R_U - R_L] \sum_{b \in Y} [\phi(b) - \phi_L] Pr\{y = b\} + R_L, \tag{22}$$

where $y \equiv (y_1, \ldots, y_n) \in Y$ is a natural containment of x within set Y.

$$Pr\{y = b\} \equiv Pr\{x = b\}/[R_U - R_L] \tag{23}$$

Since $\phi_L(b) = 0$ for all $b \in Y$, we rewrite (22) as follows:

$$R = [R_U - R_L] \sum_{b \in Y} \phi(b) Pr\{y = b\} + R_L \tag{24}$$

$$= [R_U - R_L] \cdot E_y\{\phi(y)\} + R_L \tag{25}$$

A new Monte Carlo method is obtained from (25): Generate N independent samples $s_1, \ldots, s_N$ of y. Evaluate R by the unbiased binomial estimator R_1

$$R_1 = 1 - Q_1 \equiv N^{-1}[R_U - R_L] \sum_{\nu=1}^{N} \phi(s_\nu) + R_L \tag{26}$$

It can be shown that this estimator has a smaller variance than the direct Monte Carlo estimator.

$$Var\{R_1\} = N^{-1}(R_U - R)(R - R_L) \tag{27}$$

$$< Var\{R_0\} = N^{-1}R(1 - R) \tag{28}$$

A sampling method is given in reference [2].

5 DAGGER-SAMPLING MONTE CARLO [3]

5.1 DAGGER-SAMPLING

In dagger-sampling, a small number of uniform random numbers generate a large number of sample vectors which are negatively correlated. This can be best illustrated by simple examples.

Consider first the case where each component fails with probability 0.01, i.e., $Pr\{x_i = 0\} = Pr\{\overline{x}_i = 1\} = 0.01$ for component i. Dagger-sampling generates 100 samples for component 1 in the manner shown in **Figure 1.**

A group of 100 intervals between 0 and 1 is introduced for component 1. Interval ν is used for generating samples for component 1 in trial ν, and has subinterval $[(\nu - 1) \times 0.01, \nu \times 0.01)$. The length of the subinterval is equal to 0.01, the probability of the occurrence of component 1 failure. The first interval has subinterval $[0, 0.01)$, and the last interval $[0.99, 1)$.

Only one uniform random number is generated for the group of 100 unit-intervals. Assume that the random number falls in subinterval of interval ν. Then, component 1 is assumed to be failed in trial ν and not to be failed in the other 99 trials. For example, random number 0.4256 determines that component failure occurs in trial 43 and does not occur in the other 99 trials (Fig. 1). The component failure occurs with

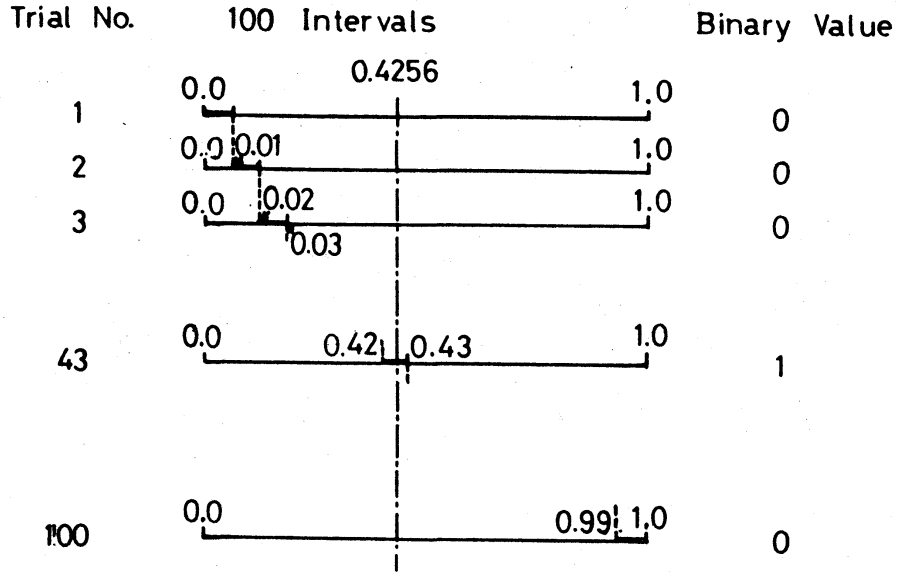

Figure 1: Generation of 100 samples for component 1. (Dagger-sampling; binary value is the one for $\overline{x}_1$)

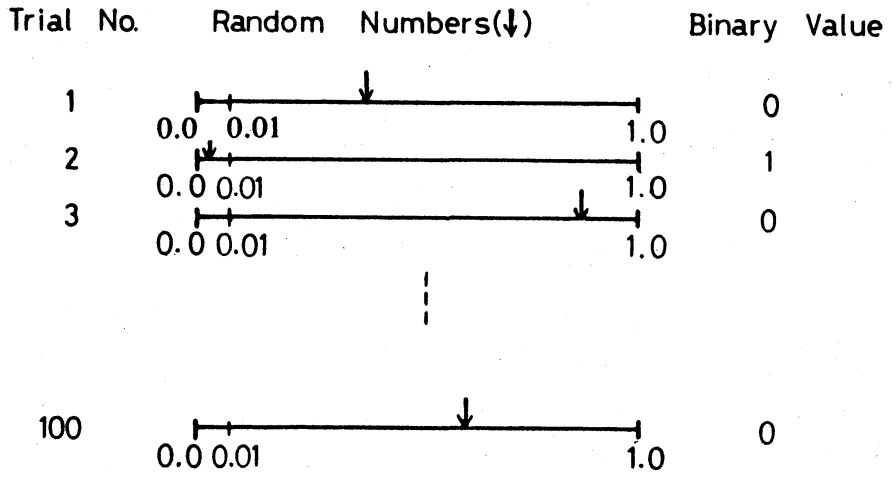

Figure 2: Generation of 100 samples for component 1. (Direct Monte Carlo)

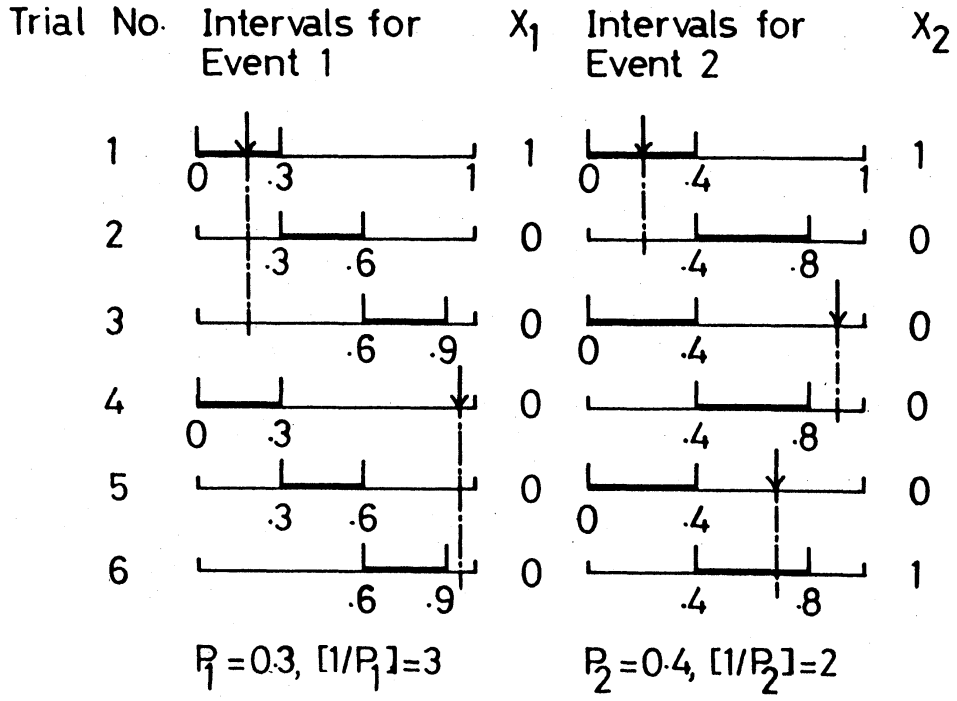

Figure 3: Dagger-sampling in general case.

probability 0.01 in each trial. One uniform random number pierces 100 intervals, and determines 100 trials for the component 1: hence the description "Dagger-sampling."

Samples for component 2 are generated similarly by using another uniform random number for the group of 100 intervals. Any number of component state vectors can be sampled by independent repetitions of each 100 generations.

The direct Monte Carlo method in this example generates 100 samples for component 1, using 100 uniform random numbers (**Figure 2**). Thus, dagger-sampling needs only 1/100 the random numbers of the direct Monte Carlo method.

We now consider the general cases where component i fails with probability P_i. Let $[1/P_i]$ be the largest integer not larger than $1/P_i$. Dagger-sampling generates $[1/P_i]$ samples for event i, using one random number in the manner shown in **Figure 3**: we introduce $[1/P_i]$ subintervals; the length of each subinterval is P_i; if the random number is less than $[1/P_i]P_i$, then one out of $[1/P_i]$ samples is the occurrence of component i failure, similarly to Fig. 1; otherwise, all $[1/P_i]$ samples are the non-occurrence of component i failure (see trials 4-6 for component 1).

5.2 Dagger-Sampling Estimator

Let $z_1, \ldots, z_N$ be N component state vectors generated by the dagger-sampling. The system unreliability or unavailability $Q = 1 - R$ can be estimated by the unbiased

binomial estimator $Q_2 = 1 - R_2$.

$$Q_2 \equiv N^{-1} \sum_{\nu=1}^{N} \psi(z_\nu) \tag{29}$$

The variance of Q_2 is given by:

$$Var\{Q_2\} = N^{-2} \left(\sum_{\nu=1}^{N} Var\{\psi(z_\nu)\} + \sum_{\nu \neq \rho} Cov\{\psi(z_\nu), \psi(z_\rho)\} \right) \tag{30}$$

The first sum of the r.h.s. of (30) is the variance of direct Monte Carlo estimator $Q_0 = 1 - R_0$, since $Var\{\psi(\overline{c}_\nu)\} = Var\{\psi(\overline{c}_\rho)\}$ and $Cov\{\psi(\overline{c}_\nu), \psi(\overline{c}_\rho)\} = 0$ for component state vectors $\overline{c}_\nu$ and $\overline{c}_\rho$ generated by the direct Monte Carlo.

In dagger-sampling, two sample vectors z_ν and z_ρ are not independent, but are correlated because a smaller number of uniform random numbers generates more sample vectors. Fig. 1 and 3 show that: if a component failure occurs during a trial, then the failure does not occur in other trials for the same group. Thus, the correlation of two sample vectors z_ν and z_ρ is negative as long as the vectors have some elements in the same group: if elements $z_{\nu,i}$ and $z_{\rho,i}$ in trials ν and ρ are generated by a common random number, then

$$E\{z_{\nu,i} z_{\rho,i}\} - E\{z_{\nu,i}\} E\{z_{\rho,i}\} = -P_i^2 < 0 \tag{31}$$

Since the structure function $\psi(\overline{x})$ is coherent, it is monotonically increasing with respect to $\overline{x}$. Thus, the negative correlation between z_ν and z_ρ also applies to $\psi(z_\nu)$ and $\psi(z_\rho)$, and we have, from (30) that

$$Var\{Q_2\} < Var\{Q_0\} \tag{32}$$

6 COVERAGE MONTE CARLO [4]

6.1 More Nomenclature and Notation

1) IEF: inclusion-exclusion formula.
2) SOP: sum of products.
3) CV: coefficient of variation.
4) C_j: a minimal cut set of function $\psi(\overline{x})$ [1].
5) j, k: suffix denoting a minimal cut set.
6) m: total number of minimal cut sets.
7) $g_j(\overline{b})$: structure function of C_j, i.e.,

$$g_j(\overline{b}) = \prod_{i \in C_j} \overline{b}_i = \prod_{i \in C_j} (1 - b_i) \tag{33}$$

$$= \begin{cases} 1, & \text{if minimal cut set } j \text{ is occurring} \\ 0, & \text{otherwise} \end{cases} \tag{34}$$

8) $\alpha(\overline{b})$: coverage of vector $\overline{b}$, nonnegative integer,

$$\alpha(\overline{b}) = \sum_{j=1}^{m} g_j(\overline{b}) \tag{35}$$

9) B_k: set of vectors $\overline{b}$ whose coverage is no less than k,

$$B_k = \{\overline{b} | \alpha(\overline{b}) \geq k\} \tag{36}$$

10) P_i: $Pr\{\overline{x}_i = 1\}$, failure probability of component i.
11) D_1: value of the first SOP of IEF (assumed available).

$$D_1 = \sum_{j=1}^{m} Pr\{C_j\} = \sum_{j=1}^{m} \left[\prod_{i \in C_j} P_i \right] \tag{37}$$

12) D_2: value of the second SOP of IEF (assumed available).

$$D_2 = \sum_{j=1}^{m} \sum_{k>j}^{m} Pr\{C_j, C_k\} = \sum_{j=1}^{m} \sum_{k>j}^{m} \left[\prod_{i \in C_j \cup C_k} P_i \right] \tag{38}$$

13) D: $D_1 - Q$ is a difference between the first IEF and the true unreliability.

6.2 Coverage Monte Carlo Estimator

The system unavailability Q is

$$Q = \sum_{\overline{b}} \psi(\overline{b}) Pr\{\overline{x} = \overline{b}\} \tag{39}$$

This can be rewritten as

$$Q = \sum_{\overline{b} \in B_1} Pr\{\overline{x} = \overline{b}\} \tag{40}$$

This is also expressed by:

$$Q = \sum_{\overline{b} \in B_1} \sum_{j=1}^{m} [1/\alpha(\overline{b})] g_j(\overline{b}) Pr\{\overline{x} = \overline{b}\} \tag{41}$$

A final expression of $D = D_1 - Q$ can be derived as

$$D = \sum_{\overline{b} \in B_2} \sum_{j=1}^{m} \sum_{j>k}^{m} [2/\alpha(\overline{b})] \cdot g_j(\overline{b}) g_k(\overline{b}) Pr\{\overline{x} = \overline{b}\} \tag{42}$$

or equivalently

$$D = D_2 E_{\overline{b},j,k}\{2/\alpha(\overline{b})\} \tag{43}$$

$$(\overline{b}, j, k) \sim g_j(\overline{b}) g_k(\overline{b}) / D_2, \; j < k \tag{44}$$

Thus, the coverage estimator is:

$$Q_3 = D_1 - N^{-1} D_2 \sum_{\nu=1}^{N} 2/\alpha(\bar{b}_\nu) \tag{45}$$

The pair (j,k), $j < k$ in (44) can be sampled from a discrete distribution of $\{1,2,\ldots,m(m-1)/2\}$ by the alias method. Component $\bar{b}_i$ of $\bar{b}$ can be sampled as follows: If component i is in $C_j \cup C_k$, then $\bar{b}_i = 1$; otherwise, $\bar{b}_i = 1$ or 0 with P_i or $1 - P_i$, respectively.

It can be proved that the coverage estimator yields smaller variance than the direct Monte Carlo. Further, the CV of the estimator converges to zero as the component failure probabilities get smaller.

7 LESS-INFORMATION MONTE CARLO [5]

7.1 Further Notation

1) $\bar{x}_i(t)$: state of component i at time t

$$\bar{x}_i(t) \equiv \begin{cases} 1, & \text{if component } i \text{ is failed} \\ 0, & \text{otherwise} \end{cases} \tag{46}$$

2) $\bar{x}(t)$: $(\bar{x}_1,\ldots,\bar{x}_n)$ component state vector at time t.
3) $P_i(t)$: $Pr\{\bar{x}_i(t) = 1\}$ probability that component i is failing at time t.
4) $A_i(t)$: $1 - P_i(t)$
5) $Q(t)$: probability that the system is failed at time t.
6) τ_i: time to failure of component i.
7) τ: $(\tau_1,\ldots,\tau_n)$
8) T: time to failure of the system.

7.2 General Framework

7.2.1 Direct Monte Carlo

The system failure probability is:

$$Q(t) = Pr\{T \le t\} = E_\tau\{Pr\{T \le t|\tau\}\} \tag{47}$$

To compute $Q(t)$ by a direct Monte Carlo method, we first generate time to failure of each component. A direct Monte Carlo estimator is

$$Q_0(t) = N^{-1} \sum_{\nu=1}^{N} Pr\{T_\nu \le t|\tau_\nu\},\ \tau_\nu \sim Pr\{\tau\} \tag{48}$$

$Pr\{T \le t|\tau\}$ is either one or zero when t is fixed since T is a function of τ.

7.2.2 Less-Information Monte Carlo

We now derive a Monte Carlo method based on less prior information than time to failure of components. Suppose prior information is Y which provides less information than τ. In other words, knowledge of τ implies knowledge of Y, but the reverse is not true. $Q(t)$ can be expressed by

$$Q(t) = Pr\{T \leq t\} = E_Y\{Pr\{T \leq t|Y\}\} \tag{49}$$

Thus, a proposed Monte Carlo estimator based on prior information Y is

$$Q_4(t) = N^{-1} \sum_{\nu=1}^{N} Pr\{T_\nu \leq t|Y_\nu\},\ Y_\nu \sim Pr\{Y\} \tag{50}$$

It can be shown that the variance of the proposed estimator does not exceed that of the direct estimator. The proposed estimator based on less prior information requires exact computation of $Pr\{T \leq t|Y\}$ that is not immediately possible. If this is possible, then we can obtain some variance reduction.

7.3 Uniform Distribution of Time to Failure

Suppose that only the chronological order of component failures $(i_1, \ldots, i_u, \ldots, i_n) \equiv S_1$ is available and that u-th component failure causes the system failure: component i_1 fails first, i_n fails last, and system failure occurs when component i_u fails. Information S_1 clearly has less information than τ. In order to construct a Monte Carlo method based on S_1, we must be able to compute $Pr\{T \leq t|S_1\}$. It can be shown that this quantity can be calculated by

$$Pr\{T \leq t|S_1\} = \frac{\sum_{j=u}^{n} Pr\{j, S_1\}}{\sum_{j=0}^{n} Pr\{j, S_1\}} \tag{51}$$

where j is the number of components that failed in the period $(0, t]$.

Assume that time to failure of component i is distributed uniformly in a period $(0, t]$, and also uniformly in the other period $(t, t+r]$ given component i is functioning at time t (r is an arbitrary positive number). The two periods have probabilities $P_i(t)$ and $1 - P_i(t)$, respectively. Then,

$$Pr\{T \leq t|S_1\} = \frac{\sum_{j=u}^{n} \Psi_j}{\sum_{j=0}^{n} \Psi_j} \tag{52}$$

$$\Psi_j \equiv \frac{P_{i_1}(t) \cdots P_{i_j}(t)}{j!} \frac{A_{i_{j+1}}(t) \cdots A_{i_n}(t)}{(n-j)!} \tag{53}$$

This gives the sequential destruction method previously described in [7]. Note that the system unreliability at a particular time t is being evaluated by an artificial introduction of uniform time to failure distribution.

7.4 Exponential Distribution of Time to Failure

Suppose that time to failure of component i is exponential with failure rate parameter λ_i, i.e., $P_i = 1 - \exp(-\lambda_i t)$. Given $S_2 = (i_1, \ldots, i_u)$, i.e., the failure order of the first u components where failure u causes the system failure, we would like to compute $Pr\{T \leq t|S_2\}$.

We can compute it by solving numerically a set of linear differential equations. This method takes excessive computation time for large systems. Another method is to compute the inverse of the Laplace transformation of $Pr\{T \leq t|S_2\}$ as described in [6]. However, this method may have serious round-off errors.

We derive a formula for computing $Pr\{T \leq t|Y\}$ based on prior information $Y \equiv S_3 \equiv (\{i_1, \ldots, i_{u-1}\}, i_u)$ where failure u causes the system failure and $\{\cdot\}$ denotes a set, i.e., the order of the first $u-1$ failures does not matter; S_2 implies S_3.

It can be shown that $Pr\{T \leq t|S_3\}$ is

$$Pr\{T \leq t|S_3\} = \frac{\int_0^t \Psi_u(v)dv}{\int_0^\infty \Psi_u(v)dv} \tag{54}$$

$$\Psi_u(v) \equiv \exp(-\lambda_{i_u} v) \prod_{j=1}^{u-1} [1 - \exp(-\lambda_{i_j} v)][\prod_{j=u+1}^{n} \exp(-\lambda_{i_j} v)] \tag{55}$$

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