THE TEACHING OF MATHEMATICS

2019, Vol. XXII, 1, pp. 1–16

STRUCTURING SYSTEMS OF NATURAL, POSITIVE RATIONAL, AND RATIONAL NUMBERS

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Abstract. In this paper, we are concerned with the extensions of the system N of natural numbers with 0 to the system Q_+ of positive rational numbers with 0 and then, the extension of this latter system to the system Q of rational numbers. Structuring these systems, we start with the system of natural numbers forming the list of its basic operative properties (i.e. the properties of operations and the order relation) and using only these postulated properties, we deduce from them a series of properties of proportions and equalities that relate differences. While deducing, we suppose that all involved ratios and differences are defined in N, i.e. they have the value which is a natural number.

Keeping in mind these properties, we define equivalence of arbitrary ratios (without supposing their values in N) and then, we extend the meaning od addition, multiplication and order relation to the equivalence classes of these ratios. Thus, the system of positive rational numbers with 0 is constructed and the list of its basic properties is accomplished.

Comparing this new list and that with properties of N, two lists are identical except that the variables are denoted by different letters and the new list contains an item more: the existence of multiplicative inverse. Omitting this item, two systems become formally identical. Therefore, they have the same postulated properties, as well as those deduced from them. This is a precise formulation of the Peacock's principle of permanence, telling which properties are transferable and it is also the way of its logical justification.

Similarly, Q_+ is extended by equivalence classes of formal differences in Q_+ and addition, multiplication and the order relation are defined in this set of equivalence classes. Thus, the system Q of rational numbers is constructed and the list of its basic properties accomplished. Taking the operative properties of the systems N, Q_+ and Q as axioms, the N-structure, Q_+-structure and Q-structure (standardly called the ordered field) are defined, respectively. Since the axioms of N-structure are least restrictive, it follows that the systems of positive rational and rational numbers and of real numbers are examples of N-structure. Thereby, all operative properties of N and those deduced from them are also valid in these number systems, when they are transcribed writing corresponding variables. This is the necessary precision and logical justification for the Peacock's principle of permanence.

At the end, we prove that each system satisfying axioms of N-structure contains an isomorphic copy of the system N with 0, what characterizes the system of natural numbers with 0 as the smallest system satisfying the axioms of N-structure.

MathEduc Subject Classification: F43

MSC Subject Classification: 97F40

Key words and phrases: Natural numbers; positive rational numbers with zero; rational numbers; Peacock's principle of permanence; characterizing properties of the system of natural numbers with zero.

1. Introduction

Genesis of natural numbers has always been related to the experience of discrete realities. Such realities are collections of visible objects in the surrounding space, which we call *sets at the sensory level*. This dependence of the idea of natural number on the perception of sets at the sensory level, we express in the form of *Cantor principle of invariance of number*:

Starting with the perception of a set A of visible objects and abstracting (forgetting)

(i) *The nature of these objects*

and

(ii) *Any structure with which the set A is endowed* (ordering, grouping of its elements, etc.)

an abstract idea $\overline{\overline{A}}$ of number results.

Manipulations with sets lead to the ideas of set operations and abstracting further, the ideas of arithmetic operations are created. These latter operations are used when the number blocks are built (up to 10, 20, 100, 1000 and so on inductively), what results in the construction of positional number system. Then it is supposed that the number of elements of each finite set can be counted, in other words to each such set a unique positional notation is attached as the indicator of its cardinality and the cardinalities of two sets are compared, by comparing the positional notations that are attached to them.

In [4] and [5], several properties of the system of natural numbers have been established and used to sketch the extensions of this system to the systems of positive rational numbers and integers. In this paper we consider the extension of the system of natural numbers with 0 to the system of positive rational numbers with 0, and then, as the next step, the extension of this latter system to the system of all rational numbers. These extensions will be performed here in a much more systematic way, looking at these systems of numbers as being the structures with the prescribed sets of properties.

As the first step, we take the system $\mathbb{N}$ of natural numbers to be a structure which consists of the set $\mathbb{N}$ of all natural numbers (including 0) together with two operations: addition $(+)$ and multiplication $(\cdot)$, and the order relation $(<)$ with the assumed operative properties written in *List 1* (Section 2), which we call basic operative properties of $\mathbb{N}$. Then, these properties are used to deduce a series of properties of operations and the order relation in $\mathbb{N}$, and in particular, the stress is laid on the properties of proportions, i.e. equalities that relate rations as well as the equalities that relate differences. Let us also add that all quotients and all differences are supposed, of course, to be defined in $\mathbb{N}$ (i.e. their values are natural numbers). We are guided with these properties when we extend $\mathbb{N}$ by ratios $m : n$, $(n > 0)$ and when the system Q_+ of positive rational numbers with 0 is constructed. Then, the basic properties of Q_+ (*List 2*, Section 3) are verified and the existence of the multiplicative inverse proved. Comparing List 1 and List 2, we see that the latter list is enriched with the property stating the existence of the multiplicative inverse. Omitting this property, two lists become identical (up to the letters used to denote variables) and from the listed properties the same sets of deduced properties follow (with variables denoted by different letters denoting elements of different sets). Hence, in transition from $\mathbb{N}$ to Q_+, all basic and deduced properties of $\mathbb{N}$ are equally valid in Q_+, what provides a precise formulation to and serves as the logical justification of the *Peacock's principle of permanence* in this specific case of extension.

Let us note that Euclid in his Elements, ([2]), derives a long series of properties of proportions, relating ratios of magnitudes of the same kind (in particular, line segments). This is due to his following of the Eudoxus theory of real numbers. We relate the meaning of natural, positive rational and rational numbers to the discrete phenomena and we use algebra as early as the system of natural numbers. (Let us also note that Early Algebra is a branch of didactics of mathematics mostly having its objectives pedagogically oriented. The book [3] is an up-to-date presentation of this discipline.)

Extending the system Q_+ to the set of all formal differences $q - r$ and imbedding this system into this set of differences, taking $q - 0$ instead of q, we will be guided by the properties of $\mathbb{N}$ (considered as being transferred to Q_+ and enumerated by the same number to which the prime is added), when the equivalence relation, the operations and the ordering are defined. The set of equivalence classes will be denoted by Q and together with the corresponding operations and the order relation, this set will be called the system of rational numbers. After the verification based on properties of Q_+, *List 3* (Section 4) of the basic operative properties of Q is formed. Reasoning in the same way as in the case of the former extension of $\mathbb{N}$ to Q_+, we conclude that the properties in *List 1* as well as the properties deduced from them are formally carried over to the system Q, what also justifies logically the Peacock's principle of permanence in the case of this extension. Let us also notice that the properties in *List 3* are the system of axioms of the ordered field.

We will call provisionally a structure satisfying conditions in *List 1*, the N-structure and a structure satisfying condition in *List 3*, the Q-structure (or, as it is standardly called, an ordered field). Each ordered field is an example of N-structure, as the number systems $\mathbb{Q}_+$ and $\mathbb{N}$ are also such examples. As we prove it in the section *Concluding remarks*, each example of an N-structure contains an isomorphic copy of the system $\mathbb{N}$ of natural numbers with 0. Hence, *the system of natural numbers with* 0 *is the smallest system satisfying the conditions in List 1*. This characterizes natural numbers in the same way as the smallest ordered field is the characteristic of rational numbers. The system of real numbers is a continuous ordered field, what is again an example of N-structure and a case of the transfer of basic properties and those deduced from them.

Finally, let us add that whenever $k + l = m$, we write $m - k = l$ and when $k \cdot l = m$, $(k > 0)$, we write $m : k = l$, thus having subtraction and division defined partially as binary operations.

2. Structuring the system of natural numbers

It is a common practice to denote the set of natural numbers by the letter $\mathbb{N}$. But when the set $\mathbb{N}$ is viewed together with two arithmetic operations, addition $(+)$ and multiplication $(\cdot)$, and with the order relation $(<)$, then it is referred to as the *system of natural numbers*. Here, we view that system as a mathematical structure

$$(\mathbb{N}, +, \cdot, <),$$

assigning to the operations and the order relation only the properties listed below. The variables $k, l, m, \dots$ are arbitrary natural numbers and, for the sake of brevity, we usually write kl instead of $k \cdot l$.

(i) $(\forall k)(\forall l)\, k + l = l + k$

(ii) $(\forall k)(\forall l)(\forall m)\, (k + l) + m = k + (l + m)$

(iii) $(\exists 0)(\forall k)\, k + 0 = k$

(iv) $(\forall k)(\forall l)\, kl = lk$

(v) $(\forall k)(\forall l)(\forall m)\, (kl)m = k(lm)$

(vi) $(\exists 1)(0 < 1 \textit{ and } (\forall k)\, k \cdot 1 = k)$

(vii) $(\forall k)(\forall l)(\forall m)\, k(l + m) = kl + km$

(viii) $(\forall k)(\forall l)\, (k < l \iff (\exists m > 0)\, k + m = l)$

(ix) $(\forall k)(\forall l)\, (k < l \textit{ or } k = l \textit{ or } l < k)$

(x) $(\forall k)(\forall l)(\forall m)$ $(k < l \iff k + m < l + m)$

(xi) $(\forall k)(\forall l)(\forall m > 0)$ $(k < l \iff km < lm)$

List 1

We call these properties the *basic operative properties of* $\mathbb{N}$ and, as it will be shown latter, they suffice when $\mathbb{N}$ is extended to the systems of positive rational numbers and rational numbers.

Two other operational signs, "$-$" and "$:$", can be introduced, defining "partial" operations on $\mathbb{N}$, in the following way:

(1) If $k + l = m$, then l is called the *difference* of m and k, which is denoted as $l = m - k$;

(2) If $kl = m$ and $k \neq 0$, then l is called the *quotient* of m and k, which is denoted as $l = m : k$.

Now we use basic operative properties of $\mathbb{N}$ to deduce from them a series of other properties of operations and the order relation in $\mathbb{N}$. In what follows, the variables k, l, m, n are supposed to be arbitrary natural numbers, except for the properties where differences and quotients are involved, when their values are bound to the cases for which such expressions are defined.

(3) a) $(k + l) - l = k$, $(k - l) + l = k$;

b) $(k \cdot l) : l = k$, $(k : l) \cdot l = k$.

a) Applying definition (1) to $k + l = k + l$, and $k - l = k - l$, relations under a) follow.

b) Applying definition (2) to $k \cdot l = k \cdot l$, and $k : l = k : l$, relations under b) follow.

(4) a) $k = l \iff k + m = l + m$.

b) If $m \neq 0$, then $k = l \iff k \cdot m = l \cdot m$.

a) Applying (1) and (3)a), we have that $k = l \iff (k + m) - m = l \iff k + m = l + m$.

b) Applying (2) and (3)b), we have that, for $m \neq 0$, $k = l \iff (k \cdot m) : m = l \iff k \cdot m = l \cdot m$.

(5) $k(l - m) = kl - km$.

Set $l - m = x$; then by (1), $l = m + x$, and by (vii), $kl = km + kx$, hence, $kx = kl - km$, while $k(l - m) = kx$. Thus, $k(l - m) = kl - km$.

(6) *Additive and multiplicative identity elements (postulated in (iii) and (vi)) are uniquely determined.*

Indeed, if 0_1 and 0_2 were two additive identity elements, we would have $0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2$. The proof for multiplicative case is similar.

(7) $k \cdot 0 = 0$.

$k + (k \cdot 0) \overset{\text{(vi)}}{=} (k \cdot 1) + (k \cdot 0) \overset{\text{(vii)}}{=} k(1 + 0) \overset{\text{(iii)}}{=} k \cdot 1 \overset{\text{(vi)}}{=} k$, and it follows by (6) that $k \cdot 0 = 0$.

We note that, based on (7), the implication part $k = l \implies km = lm$ of property (4)b) holds without the assumption $m \neq 0$.

(8) $k : l < (=, >)\, m : n$ *if and only if* $kn < (=, >)\, lm$.

Set $k : l = x$, $m : n = y$; then $k = lx$, $m = ny$, $kn = lnx$, $lm = lny$. Now, since $l, n \neq 0$, we have that $x = y \overset{(4)}{\iff} lnx = lny \iff kn = lm$.

Since $l, n > 0$, we have that $x < y \overset{\text{(xi)}}{\iff} lnx < lny \iff kn < lm$.

(9) $k(l : m) = (kl) : m$.

Set $l : m = x$, $l = mx$; then $kl = k(mx) = m(kx)$, hence $(kl) : m = kx$. Since $k(l : m) = kx$, the given relation follows.

(10) $(k : l) : m = k : (lm)$.

Set $k : (lm) = x$; then $k = lmx$, $k : l = mx$, and $(k : l) : m = x$.

(11) $k : (l : m) = (km) : l$.

Set $k : (l : m) = x$; then $k = x(l : m) \overset{(9)}{=} (xl) : m$, $km = xl$, and $x = (km) : l$.

(12) $(k : l) : (m : n) = (kn) : (lm)$.

$(k : l) : (m : n) \overset{(10)}{=} k : (l(m : n)) \overset{(9)}{=} k : ((lm) : n) \overset{(11)}{=} (kn) : (lm)$.

(13) *For each* $m > 0$, $k : l = (mk) : (ml)$.

The following conditions are equivalent for $m > 0$: $k : l = x$, $k = lx$, $mk = (ml)x$, $(mk) : (ml) = x$.

(14) $(k : l) \cdot (m : n) = (km) : (ln)$.

Indeed, $(k:l)\cdot(m:n) \overset{(9)}{=} ((k:l)m):n \overset{(9)}{=} ((km):l):n \overset{(10)}{=} (km):(ln)$.

(15) $(m \pm n):k = m:k \pm n:k$.

Let $(m+n):k = x$; then $m+n = kx$. Set also $m:k = y$, $n:k = z$; then $m+n = ky+kz = k(y+z)$, and since $kx = k(y+z)$, we get by (4) (since $k \neq 0$) that $x = y+z$. The proof in the case of minus sign is similar.

(16) $(k:l) \pm (m:n) = (kn \pm lm):(ln)$.

Applying (13) and (15), we get $(k:l)+(m:n) = (kn):(ln)+(lm):(ln) = (kn+lm):(ln)$. The proof in the case of minus sign is similar.

The following are some properties of the difference defined by (1).

(17) $k-l < (=,>) m-n \iff k+n < (=,>) l+m$.

Set $k-l = x$, $m-n = y$; then $k = l+x$, $m = n+y$, $k+n = (l+x)+n = (l+n)+x$, $l+m = l+(n+y) = (l+n)+y$. Now, $x = y \overset{(3)}{\iff} (l+n)+x = (l+n)+y \iff k+n = l+m$.

Similarly, $x < y \overset{(x)}{\iff} (l+n)+x < (l+n)+y \iff k+n < l+m$.

(18) $(k-l)-m = k-(l+m)$.

Indeed, $(k-l)-m = x \overset{(1)}{\iff} k-l = m+x \overset{(1)}{\iff} k = (m+x)+l \overset{(i),(ii)}{\iff} k = x+(m+l) \overset{(1)}{\iff} x = k-(m+l)$.

(19) $(k+l)-m = k+(l-m)$.

Indeed, $k+(l-m) = x \overset{(4)}{\iff} (k+(l-m))+m = x+m \overset{(ii)}{\iff} k+((l-m)+m) = x+m \overset{(3)}{\iff} k+l = x+m \overset{(1)}{\iff} x = (k+l)-m$.

(20) $k-(l-m) = (k+m)-l$.

Indeed, $k-(l-m) = x \overset{(1)}{\iff} k = (l-m)+x \overset{(4)}{\iff} k+m = ((l-m)+x)+m \overset{(ii)}{=} ((l-m)+m)+x \overset{(3)}{=} l+x \overset{(1)}{\iff} x = (k+m)-l$.

(21) $(k-l)+m = (k+m)-l$.

Let $k-l = x$. Then $k = l+x$, $k+m = (l+x)+m = l+(x+m)$, $x+m = (k+m)-l$. On the other hand $x+m = (k-l)+m$, and the desired relation follows.

(22) $(k-l)+(m-n) = (k+m)-(l+n)$.

Indeed, let $k-l = x$, $m-n = y$, and $x+y = (k-l)+(m-n)$. Then $k = x+l$, $m = n+y$, $k+m = (x+l)+(y+n) = (x+y)+(l+n)$, $x+y = (k+m)-(l+n)$.

(23) $(k-l)-(m-n) = (k+n)-(l+m)$.

Let $k-l = x$, $m-n = y$, $x-y = (k-l)-(m-n)$. Then $k = l+x$, $m = n+y$, $k+n = (l+n)+x$, $l+m = (l+n)+y$, $x-y = (x+(l+n))-(y+(l+n)) = (k+n)-(l+m)$.

(24) $(k-l)(m-n) = (km+ln)-(kn+lm)$.

Indeed, $(k-l)(m-n) \overset{(5)}{=} (k-l)m-(k-l)n \overset{(5)}{=} (km-lm)-(kn-ln) \overset{(23)}{=} (km+ln)-(lm+kn)$.

3. Positive rational numbers with zero and their properties

Now we introduce the set of all ratios $k : l$, where k and $l > 0$ are arbitrary natural numbers. Mapping k onto $k : 1$, the set $\mathbb{N}$ of natural numbers is imbedded into the set of all ratios. Guided by the properties of $\mathbb{N}$, we define equivalent ratios and on the set of equivalence classes we define two operations and the order relation, constructing so the *system* $(\mathbb{Q}_+, +, \cdot, <)$ *of positive rational numbers with zero.*

Inspired by property (8) (Section 2), we take that two ratios $k : l$ and $m : n$ are equivalent if $kn = lm$. It is easy to verify that, in this way, an equivalence relation on the set of all ratios is defined. For the sake of simplicity, we will be denoting an equivalence class by one of its elements, i.e., instead of $[k : l]$ we write simply $k : l$, and we will be referring to that class as *a positive rational number (or zero).*

Inspired by property (15) (Section 2), the *sum of two rational numbers* $k : l$ and $m : n$ is to be

$$(k : l) + (m : n) = (kn + lm) : (ln).$$

To prove that this definition is correct, let us suppose that $k : l = k' : l'$ and $m : n = m' : n'$, i.e. $kl' = k'l$ and $mn' = m'n$. Then $(kn + lm)l'n' = (kl')nn' + (mn')ll' = (k'l)nn' + (m'n)ll' = (k'n' + l'm')ln$, and so $(kn + lm) : (ln)$ and $(k'n' + l'm') : (l'n')$ belong to the same equivalence class.

Inspired by property (14) (Section 2), the *product of two rational numbers* $k : l$ *and* $m : n$ is taken to be

$$(k : l)(m : n) = (km) : (ln).$$

It is easy to see that the product does not depend on the choice of representatives of the equivalence classes. Indeed, suppose that $k : l$ and $k' : l'$ belong to the same class, as well as $m : n$ and $m' : n'$. Then $kl' = k'l$ and $mn' = m'n$, hence $kl'mn' = k'lm'n$. By (8), it follows that $(km) : (ln) = (k'm') : (l'n')$.

Inspired by (8), the order relation in $\mathbb{Q}_+$ is defined taking

$$k : l < m : n \quad \text{if} \quad kn < lm.$$

To prove the independence of the choice of representatives, let again $kl' = k'l$, $mn' = m'n$ and suppose that $k : l < m : n$, i.e., $kn < lm$. Then $knk'lmn' < lmkl'm'n$, and it follows that $k'n' < l'm'$, i.e., $k' : l' < m' : n'$.

Difference and quotient of two elements in $\mathbb{Q}_+$ are defined similarly as in $\mathbb{N}$, by:

(1') If $(k : l) + (m : n) = i : j$, then $m : n$ is called the *difference* of $i : j$ and $k : l$ and denoted as $m : n = (i : j) - (k : l)$;

(2') If $(k : l)(m : n) = i : j$ and $k > 0$, then $m : n$ is called the *quotient* of $i : j$ and $k : l$ and denoted as $m : n = (i : j) : (k : l)$.

Now we prove that

$$(k : l) - (m : n) = (kn - lm) : (ln) \quad \text{and} \quad (k : l) : (m : n) = (kn) : (lm), \ \ m > 0.$$

Let us put $(k:l)-(m:n)=i:j$; then $k:l=(m:n)+(i:j)=(mj+ni):(nj)$, $knj=lmj+lni$, $(kn-lm)j=lni$, $(kn-lm):(ln)=i:j$.

Similarly, putting $(k:l):(m:n)=i:j$, we have that $k:l=(m:n)(i:j)=(mi):(nj)$. Thus, $k(nj)=l(mi)$ and then $i:j=(kn):(lm)$.

The set $\mathbb{N}$ of natural numbers is imbedded in $\mathbb{Q}_+$, identifying k with $k:1$. If $m:n$ is also a representative of $k:1$, then these two ratios are equivalent, i.e., $m=kn$ holds, showing that all other representatives are $(kn):n$.

Let us also notice that

$$\begin{aligned} &(k:1)\pm(l:1)=(k\pm l):1, && (k:1)\cdot(l:1)=(kl):1,\\ &(k:1):(l:1)=k:l, && k:1<l:1 \text{ if and only if } k<l. \end{aligned}$$

As we see, this imbedding preserves operations and the order relation.

For a positive rational number $k:l$, $(k>0)$, the number $l:k$ is also positive and $(k:l)\cdot(l:k)=(kl):(lk)=1:1$. Hence, each positive $k:l$ from $\mathbb{Q}_+$ has its *multiplicative inverse* $l:k$ that will be denoted as $(k:l)^{-1}$. Let us check that multiplicative inverse does not depend on the chosen representative. Let $k':l'$ be another representative, i.e., let $kl'=k'l$. Then $(k:l)(l':k')=(kl'):(k'l)=1:1$.

Using the letters $q,r,s,t,\dots$ to denote variables in $\mathbb{Q}_+$, we list the basic operative properties of this system in the form of *List 2* and then, we proceed further with verification of these properties.

(i) $(\forall q)(\forall r)\, q+r=r+q$

(ii) $(\forall q)(\forall r)(\forall s)\,(q+r)+s=q+(r+s)$

(iii) $(\exists 0)(\forall q)\, q+0=q$

(iv) $(\forall q)(\forall r)\, qr=rq$

(v) $(\forall q)(\forall r)(\forall s)\,(qr)s=q(rs)$

(vi) $(\exists 1)(0<1 \textit{ and } (\forall q)\, q\cdot 1=q)$

(vii) $(\forall q\neq 0)(\exists r)\, qr=1$

(viii) $(\forall q)(\forall r)(\forall s)\, q(r+s)=qr+qs$

(ix) $(\forall q)(\forall r)\,(q<r \iff (\exists s>0)\, q+s=r)$

(x) $(\forall q)(\forall r)\,(q<r \textit{ or } q=r \textit{ or } r<q)$

(xi) $(\forall q)(\forall r)(\forall s)$
$(q<r \iff q+s<r+s)$

(xii) $(\forall q)(\forall r)(\forall s>0)$
$(q<r \iff qs<rs)$

List 2

Verification of the properties:

(i) Let $q=k:l$ and $r=m:n$. Then $q+r=(kn+lm):(ln)$ and $r+q=(ml+nk):(nl)$, what was to be proved.

(ii) Let $q=k:l$, $r=m:n$ and $s=i:j$. Then

$$\begin{aligned} (q+r)+s&=((kn+lm):(ln))+(i:j)=((knj+lmj)+lni):(lnj),\\ q+(r+s)&=(k:l)+((mj+ni):(nj))=(knj+(lmj+lni)):(lnj), \end{aligned}$$

and the desired equality follows.

(iii) $(k:l)+(0:1)=(k\cdot 1+l\cdot 0):(l\cdot 1)\overset{(7)}{=}k:l$.

Verification of properties (iv)–(vi) are easy and omitted.

(vii) For $q\neq 0$, $qq^{-1}=1$ holds by the definition of the multiplicative inverse q^{-1}.

(viii) Let $q=k:l$, $r=m:n$ and $s=i:j$. Then

$$\begin{aligned}
q(r+s)&=(k:l)((m:n)+(i:j))=(k:l)((mj+ni):(nj))\\
&=(kmj+kni):(lnj),\\
qr+qs&=(k:l)(m:n)+(k:l)(i:j)=(km:ln)+(ki:lj)\\
&=(kmlj+klni):(llnj)\overset{(13)}{=}(kmj+kni):(lnj),
\end{aligned}$$

and the right-hand sides of the last two equalities are equivalent to the same ratio.

(ix) Let $q=k:l$, $r=m:n$ and $q<r$, i.e. $kn<lm$ and so $lm-kn>0$. Denoting $i=lm-kn$, $j=ln$ and $s=i:j$, we have that $(k:l)+(i:j)=m:n$, i.e. $q+s=r$.

Conversely, if $q+s=r$, where $q=k:l$, $r=m:n$ and $s=i:j$ (with $i>0$), then $(kj+li):(lj)=m:n$, and so $n(kj+li)=lmj$. It follows that $knj+lni=lmj$, with $lni>0$, meaning that $knj<lmj$ and $kn<lm$. Therefore, $k:l<m:n$, i.e. $q<r$.

(x) Trichotomy follows because $k:l<(=,>)m:n \iff kn<(=,>)lm$.

(xi) Let $k:l$, $m:n$, $i:j$ be elements of $\mathbb{Q}_+$ $(l,n,j>0)$. Then, using definitions of operations and order relation in $\mathbb{Q}_+$, as well as basic properties of the system $\mathbb{N}$, we obtain that

$$\begin{aligned}
k:l<m:n \iff kn<lm &\overset{(\text{xi})}{\iff} knjj<lmjj\\
&\overset{(\text{iv}),(\text{v}),(\text{x})}{\iff} (nj)(kj)+(nj)(li)<(lj)(mj)+(lj)(ni)\\
&\overset{(\text{vii})}{\iff} (nj)(kj+li)<(lj)(mj+ni)\\
&\iff (kj+li):(lj)<(mj+ni):(nj)\\
&\iff (k:l)+(i:j)<(m:n)+(i:j).
\end{aligned}$$

(xii) Let again $k:l$, $m:n$, $i:j$ be elements of $\mathbb{Q}_+$ $(l,n,j>0$ and, in addition, $i>0)$. Then we get that

$$\begin{aligned}
k:l<m:n &\iff kn<lm \overset{(\text{xi})}{\iff} knij<lmij\\
&\iff (ki):(lj)<(mi):(nj) \iff (k:l)(i:j)<(m:n)(i:j).
\end{aligned}$$

When property (vii) is taken off *List 2*, systems $\mathbb{N}$ and $\mathbb{Q}_+$ have identical sets of basic properties (up to the letters denoting variables). Thus, all properties deduced from these sets are also identical. In particular, all basic properties of $\mathbb{N}$ and all those deduced from them are equally valid in $\mathbb{Q}_+$. This confirms the Peacock's principle of permanence in the case of this extension.

We shall deduce some additional properties that follow from the existence of multiplicative inverse.

(25′) *For each $q \in \mathbb{Q}_+$, $q \neq 0$, the multiplicative inverse is unique.*

Let q_1 and q_2 be two multiplicative inverses of q. Then $q_1 = q_1 \cdot 1 = q_1(qq_2) = (q_1 q)q_2 = 1 \cdot q_2 = q_2$.

(26′) *If $q, r \in \mathbb{Q}_+$ and $q > 0$, then the equation $qx = r$ has a unique solution in $\mathbb{Q}_+$.*

Since $q(q^{-1}r) = (qq^{-1})r = 1 \cdot r = r$, then $q^{-1}r$ is a solution for $qx = r$. Conversely, if $qx = r$, then $(qx)q^{-1} = rq^{-1}$, implying that $x = q^{-1}r$.

(27′) *Whenever $r > 0$, the quotient of numbers q and r from $\mathbb{Q}_+$ can be expressed as $q : r = qr^{-1}$.*

(28′) *$qr = 0$ if and only if $q = 0$ or $r = 0$.*

The implication ($q = 0$ or $r = 0$) $\implies qr = 0$ was proved in (7) (Section 2) in the case of natural numbers, and is transferred to $\mathbb{Q}_+$. To prove the converse, assume, e.g., that $r \neq 0$. By (7′), 0 is a solution (for q) of the equation $qr = 0$. Since, by (26′), this equation has a unique solution, it follows that $q = 0$.

4. Rational numbers and their properties

In this section, we will be concerned with the extension of system $\mathbb{Q}_+$ to system $\mathbb{Q}$ of rational numbers. As we will be often referring to the operative properties of $\mathbb{Q}_+$, we write (1′), (2′), (3′), ... to denote properties (1), (2), (3), ... from Section 2, imagining them as being transferred to $\mathbb{Q}_+$, i.e., taking their form completely unchanged except that the variables q, r, s, t, ... are used.

For arbitrary $q, r \in \mathbb{Q}_+$, we consider the set of all formal differences $q - r$. Inspired by (17′), we define the relation "$\sim$" on this set, taking

$$q - r \sim s - t \quad \text{if} \quad q + t = r + s.$$

1. *The relation "$\sim$" is an equivalence relation.*

It is trivial to verify that "$\sim$" is a reflexive and symmetric relation. For the transitivity, let us suppose that $q-r \sim s-t$ and $s-t \sim u-v$ or else that $q+t = r+s$ and $s + v = t + u$. Then, adding v to both sides of the first equality, and r to both sides of the second, we get $q + t + v = r + s + v$ and $r + s + v = r + t + u$. Hence, $q + t + v = r + t + u$ and cancelling t we get $q + v = r + u$, i.e., $q - r \sim u - v$.

From the definition of the relation "$\sim$", it immediately follows

2. *For each q, r and s in $\mathbb{Q}_+$:*

$$q - r \sim (q + s) - (r + s).$$

The equivalence class of $q - r$ we will denote, writing simply $q - r$ (instead of $[q - r]$).

In view of **2.**, for $q \geqslant r$: $q - r \sim (q - r) - 0$ and let us put $\alpha_{qr} = q - r$ and $\beta_{qr} = 0$ and for $q < r$: $q - r \sim 0 - (r - q)$ and let us put $\alpha_{qr} = 0$ and $\beta_{qr} = r - q$. Then, $q - r$ in the former case and $r - q$ in the latter are called the *absolute value* of $q - r$. By **2.**, $q - r \sim \alpha_{qr} - \beta_{qr}$ and $\alpha_{qr} - \beta_{qr}$ is called the *standard representative* of the class $q - r$.

Permuting the sum $r + s$ in (18′), we have

3. $(q - r) - s = (q - s) - r$, $(q \geqslant r + s)$.

Now we prove that the standard representative is unique within its equivalence class. Namely,

4. *If* $q - r \sim s - t$, *then* $\alpha_{qr} = \alpha_{st}$ *and* $\beta_{qr} = \beta_{st}$.

When $q \geqslant r$, from $q + t = r + s$, it follows that $q = (r + s) - t$ and $q - r = ((r+s)-t)-r \overset{\mathbf{3.}}{=} ((r+s)-r)-t \overset{(3')}{=} s-t$. Hence, $\alpha_{qr} = \alpha_{st}$ and evidently $\beta_{qr} = \beta_{st}$. But when $q < r$, it follows that $r = (q + t) - s$ and $r - q = ((q + t) - s) - q = ((q + t) - q) - s = t - s$. Hence, $\beta_{qr} = \beta_{st}$ and evidently $\alpha_{qr} = \alpha_{st}$.

The set of all equivalence classes is denoted by $\mathbb{Q}$ and called *the set of rational numbers.*

Now we proceed further, defining in $\mathbb{Q}$ the order relation and the operations of addition and multiplication. These definitions will be given in terms of standard representatives, what ensures their independence of the chosen representatives.

Inspired by (17′), we take that

$$q - r < s - t \quad \text{if} \quad \alpha_{qr} + \beta_{st} < \beta_{qr} + \alpha_{st}.$$

It is easy to verify that "$<$" is an order relation in $\mathbb{Q}$.

Inspired by (22′), the *sum of two rational numbers* $q - r$ *and* $s - t$ is taken to be

$$(q - r) + (s - t) = (\alpha_{qr} + \alpha_{st}) - (\beta_{qr} + \beta_{st}).$$

The element $0-0$ is the canonical representative of its class. For each $q-r \in \mathbb{Q}$, $(q - r) + (0 - 0) = (\alpha_{qr} + 0) - (\beta_{qr} + 0) = \alpha_{qr} - \beta_{qr} = q - r$. Thus, we see that $0 - 0$ is *additive identity element.*

For $q - r$, $\alpha_{qr} = \beta_{rq}$ and $\alpha_{rq} = \beta_{qr}$. Thus,

$$\begin{aligned}(q - r) + (r - q) &= (\alpha_{qr} - \beta_{qr}) + (\alpha_{rq} - \beta_{rq}) \\ &= (\alpha_{qr} + \alpha_{rq}) - (\beta_{rq} + \beta_{qr}) = 0 - 0,\end{aligned}$$

and we see that $r - q$ is the *additive inverse* for $q - r$. We denote this inverse as $r - q = (q - r)^{-}$.

Inspired by (24′), we define the *product of two rational numbers* $q - r$ *and* $s - t$, taking

$$(q - r) \cdot (s - t) = (\alpha_{qr}\alpha_{st} + \beta_{qr}\beta_{st}) - (\alpha_{qr}\beta_{st} + \beta_{qr}\alpha_{st}).$$

The element $(1 - 0) \in \mathbb{Q}$ has the form of standard representative and the role of *multiplicative identity element.* Indeed,

$$(q - r) \cdot (1 - 0) = (\alpha_{qr} \cdot 1 + \beta_{qr} \cdot 0) - (\alpha_{qr} \cdot 0 + \beta_{qr} \cdot 1) = \alpha_{qr} - \beta_{qr} = q - r.$$

Let us notice that the mapping $q \mapsto q - 0$ imbeds $\mathbb{Q}_+$ into $\mathbb{Q}$, preserving the operations and the order relation.

Using the letters $a, b, c, d, \dots$ to denote variables in $\mathbb{Q}$, we list now the basic operative properties of this system in the form of *List 3* and then we proceed further with verification of these properties.

(i) $(\forall a)(\forall b)\, a+b=b+a$

(ii) $(\forall a)(\forall b)(\forall c)\,(a+b)+c=a+(b+c)$

(iii) $(\exists 0)(\forall a)\, a+0=a$

(iv) $(\forall a)(\exists b)\, a+b=0$

(v) $(\forall a)(\forall b)\, ab=ba$

(vi) $(\forall a)(\forall b)(\forall c)\,(ab)c=a(bc)$

(vii) $(\exists 1)(0<1$ *and* $(\forall a)\, a\cdot 1=a)$

(viii) $(\forall a\neq 0)(\exists b)\, ab=1$

(ix) $(\forall a)(\forall b)(\forall c)\, a(b+c)=ab+ac$

(x) $(\forall a)(\forall b)\,(a<b \iff (\exists c>0)\, a+c=b)$

(xi) $(\forall a)(\forall b)\,(a<b$ *or* $a=b$ *or* $b<a)$

(xii) $(\forall a)(\forall b)(\forall c)$
$(a<b \iff a+c<b+c)$

(xiii) $(\forall a)(\forall b)(\forall c>0)$
$(a<b \iff ac<bc)$

List 3

Verification of the properties:

(i) and (ii) follow easily from the definition of addition, using properties (i) and (ii) of system $\mathbb{Q}_+$ (*List 2*), and (iii) has been already explained.

(iv) $a+a^-=0$ holds by the definition of the additive inverse a^-.

(v) Let $q-r$ and $s-t$ be elements of $\mathbb{Q}$. Then

$$\begin{aligned}(q-r)(s-t)&=(\alpha_{qr}\alpha_{st}+\beta_{qr}\beta_{st})-(\alpha_{qr}\beta_{st}+\beta_{qr}\alpha_{st}),\\(s-t)(q-r)&=(\alpha_{st}\alpha_{qr}+\beta_{st}\beta_{qr})-(\alpha_{st}\beta_{qr}+\beta_{st}\alpha_{qr}),\end{aligned}$$

and the conclusion follows from properties (i) and (iv) of $\mathbb{Q}_+$ (*List 2*).

(vi) Let $q-r$, $s-t$, $u-v$ be elements of $\mathbb{Q}$. Then

$$\begin{aligned}((q-r)(s-t))(u-v)&=((\alpha_{qr}\alpha_{st}+\beta_{qr}\beta_{st})-(\alpha_{qr}\beta_{st}+\beta_{qr}\alpha_{st}))(\alpha_{uv}-\beta_{uv})\\&=(\alpha_{qr}\alpha_{st}\alpha_{uv}+\beta_{qr}\beta_{st}\alpha_{uv}+\alpha_{qr}\beta_{st}\beta_{uv}+\beta_{qr}\alpha_{st}\beta_{uv})\\&\quad-(\alpha_{qr}\alpha_{st}\beta_{uv}+\beta_{qr}\beta_{st}\beta_{uv}+\alpha_{qr}\beta_{st}\alpha_{uv}+\beta_{qr}\alpha_{st}\alpha_{uv}),\\(q-r)((s-t)(u-v))&=(\alpha_{qr}-\beta_{qr})((\alpha_{st}\alpha_{uv}+\beta_{st}\beta_{uv})-(\alpha_{st}\beta_{uv}+\beta_{st}\alpha_{uv}))\\&=(\alpha_{qr}\alpha_{st}\alpha_{uv}+\alpha_{qr}\beta_{st}\beta_{uv}+\beta_{qr}\alpha_{st}\beta_{uv}+\beta_{qr}\beta_{st}\alpha_{uv})\\&\quad-(\beta_{qr}\alpha_{st}\alpha_{uv}+\beta_{qr}\beta_{st}\beta_{uv}+\alpha_{qr}\alpha_{st}\beta_{uv}+\alpha_{qr}\beta_{st}\alpha_{uv}),\end{aligned}$$

and the conclusion follows from the properties of system $\mathbb{Q}_+$.

(vii) As has been already shown, $1-0$ is a multiplicative identity element in $\mathbb{Q}$. The inequality $0-0<1-0$ follows from the definition of the order relation.

(viii) Let $(q-r)\in\mathbb{Q}$, $q-r\neq 0-0$. Since $\alpha_{qr}-\beta_{qr}\neq 0$, two cases are possible: (a) $\alpha_{qr}>0$ and $\beta_{qr}=0$, or (b) $\alpha_{qr}=0$ and $\beta_{qr}>0$. In the first case, take $s-t\in\mathbb{Q}$

with $\alpha_{st} = 1 : \alpha_{pq}$, $\beta_{st} = 0$. Then $(q - r)(s - t) = (\alpha_{qr} - 0)((1 : \alpha_{qr}) - 0) = 1 - 0$. In the second case take $\alpha_{st} = 0$, $\beta_{st} = 1 : \beta_{qr}$ and then $(q - r)(s - t) = (0 - \beta_{qr})(0 - (1 : \beta_{qr})) = 1 - 0$. Hence, in both cases, $s - t$ is the multiplicative inverse of $q - r$.

(ix) Let $q - r$, $s - t$, $u - v$ be elements of $\mathbb{Q}$. Then

$$\begin{aligned}(q-r)((s-t)+(u-v)) &= (\alpha_{qr}-\beta_{qr})((\alpha_{st}+\alpha_{uv})-(\beta_{st}+\beta_{uv}))\\ &= (\alpha_{qr}\alpha_{st}+\alpha_{qr}\alpha_{uv}+\beta_{qr}\beta_{st}+\beta_{qr}\beta_{uv})\\ &\quad -(\alpha_{qr}\beta_{st}+\alpha_{qr}\beta_{uv}+\beta_{qr}\alpha_{st}+\beta_{qr}\alpha_{uv}),\\ (q-r)(s-t)+(q-r)(u-v) &= ((\alpha_{qr}\alpha_{st}+\beta_{qr}\beta_{st})-(\alpha_{qr}\beta_{st}+\beta_{qr}\alpha_{st}))\\ &\quad +((\alpha_{qr}\alpha_{uv}+\beta_{qr}\beta_{uv})-(\alpha_{qr}\beta_{uv}+\beta_{qr}\alpha_{uv}))\\ &= (\alpha_{qr}\alpha_{st}+\beta_{qr}\beta_{st}+\alpha_{qr}\alpha_{uv}+\beta_{qr}\beta_{uv})\\ &\quad -(\alpha_{qr}\beta_{st}+\beta_{qr}\alpha_{st}+\alpha_{qr}\beta_{uv}+\beta_{qr}\alpha_{uv}).\end{aligned}$$

(x) Let $q-r$ and $s-t$ be elements of $\mathbb{Q}$ such that $q-r < s-t$, i.e., $q+t < r+s$. Take $u = r + s$, $v = q + t$. Then $(q - r) + (u - v) = (q + u) - (r + v) = (q + (r + s)) - (r + (q + t)) = ((q + r) + s) - ((q + r) + t) \overset{\mathbf{2.}}{=} s - t$.

Conversely, let $(q-r)+(u-v) = s-t$ and $0-0 < u-v$, i.e., $q+u+t = r+v+s$ and $v < u$. From $(q+u+t)-v = (r+v+s)-v$ or $((q+t)+u)-v = ((r+s)+v)-v$, applying $(3')$ and $(19')$ it follows that $(q + t) + (u + v) = r + s$. Hence, by (ix), *List 2*, $q + t < r + s$, i.e., $q - r < s - t$.

(xi) Let $q - r$ and $s - t$ be arbitrary elements of $\mathbb{Q}$. By property (x) of $\mathbb{Q}_+$, one of the following relations must hold: $q+t < s+r$, $q+t = s+r$, or $s+r < q+t$. Hence, one of the relations $q - r < s - t$, $q - r = s - t$, $s - t < q - r$ is true.

(xii) Let $q - r$, $s - t$ and $u - v$ be arbitrary elements of $\mathbb{Q}$. Then

$$\begin{aligned} q-r<s-t &\iff \alpha_{qr}-\beta_{qr}<\alpha_{st}-\beta_{st} \iff \alpha_{qr}+\beta_{st}<\beta_{qr}+\alpha_{st}\\ &\iff (\alpha_{qr}+\beta_{st})+(\alpha_{uv}+\beta_{uv})<(\beta_{qr}+\alpha_{st})+(\alpha_{uv}+\beta_{uv})\\ &\iff (\alpha_{qr}+\alpha_{uv})+(\beta_{st}+\beta_{uv})<(\beta_{qr}+\beta_{uv})+(\alpha_{st}+\alpha_{uv})\\ &\iff (\alpha_{qr}+\alpha_{uv})-(\beta_{qr}+\beta_{uv})<(\alpha_{st}+\alpha_{uv})-(\beta_{st}+\beta_{uv})\\ &\iff (\alpha_{qr}-\beta_{qr})+(\alpha_{uv}-\beta_{uv})<(\alpha_{st}-\beta_{st})+(\alpha_{uv}-\beta_{uv})\\ &\iff (q-r)+(u-v)<(s-t)+(u-v).\end{aligned}$$

(xiii) Let $q - r$, $s - t$ and $u - v$ be elements of $\mathbb{Q}$, with $0 - 0 < u - v$. Then $\alpha_{uv} > 0$, $\beta_{uv} = 0$, hence

$$\begin{aligned} q-r<s-t &\iff \alpha_{qr}-\beta_{qr}<\alpha_{st}-\beta_{st} \iff \alpha_{qr}+\beta_{st}<\beta_{qr}+\alpha_{st}\\ &\iff (\alpha_{qr}+\beta_{st})\alpha_{uv}<(\beta_{qr}+\alpha_{st})\alpha_{uv}\\ &\iff \alpha_{qr}\alpha_{uv}+\beta_{st}\alpha_{uv}<\beta_{qr}\alpha_{uv}+\alpha_{st}\alpha_{uv}\\ &\iff \alpha_{qr}\alpha_{uv}-\beta_{qr}\alpha_{uv}<\alpha_{st}\alpha_{uv}-\beta_{st}\alpha_{uv}\\ &\iff (\alpha_{qr}-\beta_{qr})(\alpha_{uv}-0)<(\alpha_{st}-\beta_{st})(\alpha_{uv}-0)\\ &\iff (\alpha_{qr}-\beta_{qr})(\alpha_{uv}-\beta_{uv})<(\alpha_{st}-\beta_{st})(\alpha_{uv}-\beta_{uv})\\ &\iff (q-r)(u-v)<(s-t)(u-v).\end{aligned}$$

When property (iv) is taken off *List 3*, systems $\mathbb{Q}_+$ and $\mathbb{Q}$ have identical sets of basic properties (up to the letters denoting variables). Thus, all properties deduced from these sets are also identical. In particular, all basic properties of $\mathbb{Q}_+$ and all those deduced from them are equally valid in $\mathbb{Q}$. We write $(1'')$, $(2'')$, $(3'')$, … to denote properties $(1')$, $(2')$, $(3')$, … from Section 3, imagining them as being transferred to $\mathbb{Q}$, i.e., taking their form completely unchanged except that the variables a, b, c, … are used.

We state some additional properties that follow from the existence of additive inverse.

$(29'')$ *For each $a \in \mathbb{Q}$, the additive inverse is unique.*

This can be shown similarly as in the case of multiplicative inverse in $(25')$ (Section 3).

$(30'')$ *If $a, b \in \mathbb{Q}$, then the equation $a + x = b$ has a unique solution in $\mathbb{Q}$.*

Since $a + (a^- + b) = (a + a^-) + b = 0 + b = b$, then $a^- + b$ is a solution for $a + x = b$. Conversely, if $a + x = b$, then $(a + x) + a^- = b + a^-$, implying that $x = b + a^-$.

$(31'')$ *The difference of numbers a and b from $\mathbb{Q}$ can be expressed as $a - b = a + b^-$.*

$(32'')$ *Let x^- and 1^- be additive inverses of x and 1, respectively. Then $x^- = 1^- \cdot x$.*

It follows from $x + 1^- \cdot x = 1 \cdot x + 1^- \cdot x = (1 + 1^-)x = 0 \cdot x = 0$ and $(30'')$ that $1^- \cdot x$ is the additive inverse of x.

The set of all $a \in \mathbb{Q}$ satisfying $0 < a$ is denoted by $\mathbb{Q}^+$ and called the set of *positive rational numbers*. Similarly, the set of all $a \in \mathbb{Q}$ satisfying $a < 0$ is denoted by $\mathbb{Q}^-$ and called the set of *negative rational numbers*. By property (xi),

$$a < 0, \quad \text{or} \quad a = 0, \quad \text{or} \quad 0 < a$$

holds. Thus, set $\mathbb{Q}$ is a disjoint union $\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$.

5. (a) *The product of two positive or two negative numbers is positive.*

(b) *The product of a positive and a negative number is negative.*

Indeed, if $q > 0$ and $r > 0$, then $qr > 0$ and

$$(q - 0)(r - 0) = (qr - 0), \qquad (0 - q)(0 - r) = (qr - 0),$$

and (a) is proved. Also,

$$(q - 0)(0 - r) = (0 - qr),$$

and (b) is proved.

6. *If $a, b, c \in \mathbb{Q}$ and $c < 0$, then $a < b \iff bc < ac$.*

Let $a = \alpha_{qr} - \beta_{qr}$, $b = \alpha_{st} - \beta_{st}$ and $c = \alpha_{uv} - \beta_{uv}$, where $\alpha_{uv} = 0$ and

$\beta_{uv} > 0$. Then

$$\begin{aligned}
a < b &\iff \alpha_{qr} - \beta_{qr} < \alpha_{st} - \beta_{st} \iff \alpha_{qr} + \beta_{st} < \beta_{qr} + \alpha_{st} \\
&\iff (\alpha_{qr} + \beta_{st})\beta_{uv} < (\beta_{qr} + \alpha_{st})\beta_{uv} \\
&\iff \alpha_{qr}\beta_{uv} + \beta_{st}\beta_{uv} < \beta_{qr}\beta_{uv} + \alpha_{st}\beta_{uv} \\
&\iff \beta_{st}\beta_{uv} - \alpha_{st}\beta_{uv} < \beta_{qr}\beta_{uv} - \alpha_{qr}\beta_{uv} \\
&\iff (\alpha_{st} - \beta_{st})(0 - \beta_{uv}) < (\alpha_{qt} - \beta_{qr})(0 - \beta_{uv}) \\
&\iff (\alpha_{st} - \beta_{st})(\alpha_{uv} - \beta_{uv}) < (\alpha_{qr} - \beta_{qr})(\alpha_{uv} - \beta_{uv}) \\
&\iff ac < bc.
\end{aligned}$$

5. A more general view on number systems

In the process of extending number systems, the Peacock's principle of the permanence of equivalent forms plays a role of a guiding principle. Being undoubtedly very suggestive, this principle lacks a precision which could not be attained at the Peacock's time when the abstract conception of mathematics did not exist yet and which, half a century later, the ideas of set and mathematical structure have brought.

Let us look at $\{\mathbb{N}, +, \cdot, <\}$, $\{\mathbb{Q}_+, +, \cdot, >\}$ and $\{Q, +, \cdot, <\}$ as abstract structures: $\mathbb{N}$, $\mathbb{Q}_+$, $\mathbb{Q}$ are sets with two operations and the order relation and the conditions on *List 1*, *List 2* and *List 3 are taken* to be axioms of these structures, respectively, which we call provisionally N-structure, Q_+-structure and Q-structure (which is standardly called the ordered field). The system of natural numbers satisfies the axioms of N-structure, the system of positive rational numbers the axioms of Q_+-structure and the system of rational numbers axioms of Q-structure. The existence of these systems (constructing them and verifying their properties) ensures that the axioms of these structures are not contradictory. The least restrictive are the axioms of N-structure and all following systems: natural numbers with 0, positive rational numbers with 0, rational and real numbers (being the continuous ordered field) are the examples of N-structure. Therefore, *all basic operative properties of the system of natural numbers and those deduced from them are also the properties of positive rational numbers with* 0*, rational numbers and real numbers, when they are transcribed writing the corresponding letters to denote variables of these extended systems.* This is a precise and logically justified form of the Peacock's principle of permanence. (Let us remark that the system of real numbers is characterized as a continuous ordered field but the existence of that field is ensured by a construction (Dedekind cuts, Cauchy sequences, infinite decimal fractions) or, of course, not very rigorously as it is done in school, interpreting real numbers geometrically on the number axis).

Let $\{S, +, \cdot, <\}$ be an example of N-structure. Now we define inductively a sequence N^* in S, starting with $a_0 = 0_S$ and $a_1 = 1_S$. Supposing that a_n has been defined, we take $a_{n+1} = a_n + 1_S$.

Let us prove that $a_{n+m} = a_n + a_m$. This relation is true when $m = 1$ and let us suppose that $a_{n+m} = a_n + a_m$ (this is the inductive hypothesis). Then, $a_{n+(m+1)} = a_{(n+m)+1} = a_{n+m} + 1_S = (a_n + a_m) + 1_S = a_n + (a_m + 1_S) = a_n + a_{m+1}$, what was to be proved.

Let us also prove that $a_{n\cdot m} = a_n \cdot a_m$. For $m = 1$, we have $a_{n\cdot 1} = a_n = a_n \cdot 1_S = a_n \cdot a_1$. Supposing that $a_{n\cdot m} = a_n \cdot a_m$, we have $a_{n(m+1)} = a_{n\cdot m+n} = a_{n\cdot m} + a_n = a_n \cdot a_m + a_n = a_n \cdot a_m + a_n \cdot 1_S = a_n(a_m + 1_S) = a_n \cdot a_{m+1}$.

Since for each n, $a_n < a_{n+1}$, the sequence N^* is increasing. Thus, $n < m$ implies $a_n < a_m$.

Taking the above facts into account, we see that the mapping $n \mapsto a_n$ is an isomorphism and N^* is an isomorphic image of the system of natural numbers.

Since the system $\mathbb{N}$ of natural numbers is an example of N-structure and each example of N-structure contains, as a subsystem, a copy of the system $\mathbb{N}$, we conclude that *the system* $\mathbb{N}$ *of natural numbers is the smallest system satisfying the axioms of the* N*-structure*. This is a characterization of the system of natural numbers with 0 analogous to that one which characterizes the system of rational numbers as the smallest ordered field.

We should notice that H. Grassmann in 1861 exploited a mapping similar to $n \mapsto a_n$ to define addition and multiplication in the set of natural numbers and to establish their basic properties (see, for example, [1]).

REFERENCES

[1] J. Dieudonné, *Pour l'honneur de l'esprit humain*, Hachette, 1987.

[2] Euclid, *Euclid's Elements of Geometry*, Edited and Provided with a Modern English Translation by Richard Fitzpatrick, 2003.

[3] C. Kieran, J. S. Pang, D. Schilter and S. F. Ng, *Early Algebra*, ICME-13, Topical Surveys, Springer, Berlin.

[4] M. M. Marjanović, *A survey of the system N of natural numbers assigned to primary teachers*, Open Access Library Journal **4** (2017), 1–18.

[5] M. M. Marjanović, *Extension of the system N of natural numbers assigned to primary teachers*, Open Access Library Journal **5** (2018), 1–11.

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