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RELATING PROCEDURAL AND CONCEPTUAL KNOWLEDGE

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Abstract. Relating procedural and conceptual mathematical knowledge is a very important educational goal that is difficult to attain. However, research has evidenced that some progress towards achieving this goal can be made. This contribution briefly reviews some of the main outcomes of research in this area, focusing on relating these knowledge types with technology, particularly that based upon a computer algebra system.

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1. Introduction

Students' mathematical knowledge and competencies have usually not been acquired to a degree expected by educators and employers. To illustrate this inappropriate state, one may, for example, just refer to two key results from PISA 2012 (involving around half a million 15-year old students in 65 countries and economies around the world): (1) only 13% of students from 34 OECD member countries could "develop and work with models for complex situations, and work strategically using broad, well-developed thinking and reasoning skills", and (2) 32% of all tested students could not "extract relevant information from a single source and use basic algorithms, formulae, procedures or conventions to solve problems involving whole numbers" (OECD, 2014; p. 4). Had these percentages been exchanged, the two results would be viewed as appropriate.

An often present failure to teach mathematics may primarily be caused by the fact that teachers do not take into account sufficiently that mathematical proficiency has many interwoven and interdependent competences, such as reasoning, problem posing and solving, modelling, representing, handling symbols and formalism, communicating, and utilizing tools and aids (e.g. Jaworski, 2015). (The reader may recall here the five widely known NCTM process standards: problem solving, reasoning & proof, communication, connections, and representation.) We should also bear in mind that various interlinked motives and activities have contributed to the philogenesis of mathematical knowledge, such as play, calculate, apply, construct, evaluate, argue, find, and order (Haapasalo & Zimmermann, 2015).

Despite this very important feature of interconnectedness, many mathematics teachers tend to focus on one aspect at a time, wrongly hoping that other aspect(s) would (spontaneously) develop as a consequence (extrapolated from Kilpatrick & Swafford, 2002). Even when teachers use good problems (e.g. those that focus on concepts and connections among mathematical ideas), they can, as Stigler and Hiebert (2004) underlined, implement them in a wrong or inadequate way (e.g. as problems that call for procedures and basic computational skills). A major challenge of mathematics education is thus related to developing and interrelating different aspects, competences, motives, and activities. An important part of this challenge is concerned with developing and relating conceptual and procedural mathematical knowledge.

2. Procedural and conceptual knowledge

Terminological clarifications

Because of different research frameworks and the fact that procedural and conceptual knowledge are not easy to define precisely (e.g. Carpenter, 1986), many views of procedural (P) and conceptual (C) knowledge can be found in the literature. Having examined some twenty such views in the end of 1990s, Haapasalo and Kadijevich (2000), proposed the following P-C distinction:

- "Procedural knowledge denotes dynamic and successful utilization of particular rules, algorithms or procedures within relevant representation form(s). This usually requires not only knowledge of the objects being utilized, but also knowledge of format and syntax for the representational system(s) expressing them.
- Conceptual knowledge denotes knowledge of and a skillful 'drive' around particular networks, the elements of which can be concepts, rules (algorithms, procedures, etc.), and even problems (a solved problem may introduce a new concept or rule) given in various representation forms." (p. 141)

They also noted that (1) procedural knowledge usually requires automated and unconscious steps, whereas conceptual knowledge typically calls for conscious thinking, and (2) procedural knowledge may also involve some conscious thinking (e.g. when a student skilfully combines two rules without knowing why they work).

This distinction suggests that procedural knowledge may involve some conceptual knowledge, and vice versa (this involvement is emphasized by Baroody, Feil & Johnson, 2007, for example). Having in mind Star (2005), it can be said that the distinction also reflects a frequently held view that procedural knowledge is poor in connections, whereas conceptual knowledge is rich in them. However, as he elaborated, there may be two other possibilities: procedural knowledge that is rich in connections, and conceptual knowledge that is poor in connections. His account clearly calls for considering knowledge in terms of both type (e.g. knowledge of concepts vs. knowledge of procedures) and quality (e.g. knowledge rich in connections vs. knowledge poor in connections). However, his position that one type of knowledge (e.g. procedural) can be rich in connections without some support of other knowledge type was questioned by Baroody and colleagues (2007). They also underlined that apart from connectedness, there are other important dimensions of knowledge quality (e.g. completeness, structuredness) that should be taken into account as well.

The terminological clarifications in question get more complex when we realize that each type of knowledge may be defined in different ways. Regarding conceptual knowledge, these ways may, for example, be conceptual knowledge as connection knowledge, general principle knowledge, or knowledge of principles underlying procedures, the first of which is the most common (Crooks & Alibali, 2014). (Of course, some definitions may fell into multiple categories.) Also, conceptual knowledge may be implicit or explicit, enabling educators to make, for example, a distinction between basic conceptual knowledge that is implicit, and advanced conceptual knowledge that is explicit (used in Abramovich, 2015).

Despite these complexities, there seems to be a consensus today on how procedural and conceptual knowledge should be defined. According to Rittle-Johnson and Schneider (2015), procedural knowledge denotes the knowledge of procedures used in problem solving, whereas conceptual knowledge is basically the knowledge of concepts whose degree of connectedness reflects a person's expertize.

Distinction of procedural and conceptual tasks

Because one type of knowledge may (and often does) call for some degree of the other, it is hard to develop conceptual test items that are procedurally free, and vice versa (Silver, 1986). Thus, we can only use mathematical tasks that primarily assess one type of knowledge or the other (Rittle-Johnson & Schneider, 2015). Developing P-C pairs of mathematically isomorphic problem solving tasks may be quite complex. Consider the following two tasks on uniform motion taken from Haapasalo and Kadijevich (2000; p. 143).

- Procedural task – A car and a truck started simultaneously from towns that are 150 km apart. After what time did they meet each other if their speeds were 80 km/h and 60 km/h, respectively?
- Conceptual task – A mountaineer started his trip in the morning arriving at a mountain house in the evening. Having spent the night there, the mountaineer started down the next morning by using the same trail. Is there a point on the trail where he was at the same place at the same time each day? Give a detailed explanation.[1]

This procedural task, as other procedural tasks, involves fully quantified objects requiring exact computations. On the other hand, this kind of conceptual tasks, which clearly call for genuine understanding of the underlying domain, involve objects that are not fully quantified and require little computation (for such tasks, see Dreyfus & Eisenberg, 1990; Kadijevich, 1999a; for example). A particular challenge is to develop a set of such conceptual tasks that are mathematically isomorphic. An example of this set is given below.

[1]This P-C operationalization is usually learner, content, and context dependent. Because of the quality and organization of the learning process, it is clear that one task can be routine for one learner, but challenging for other. These two tasks should be given to students who are familiar with (1) solving traditional meeting and overtaking problems, and (2) using quantitative and qualitative graphs to represent piecewise uniform motion of one and two objects.

- Two markers cost more than three pencils. Do 5 markers cost more than 7 pencils (no discount offered)?
- The circumference of an equilateral triangle is larger than that of a rhombus. Three of these triangles are used to form an isosceles trapezium, whereas two of those rhombuses are combined to form an arrow shaped figure. Which of these two such formed figures has a greater circumference?
- At a mathematical contest each solver correctly solved at least five tasks. The jury observed a curious fact: each task was correctly solved by exactly four students. Were there more solvers or tasks to be solved at this contest?[2] (These three tasks are taken from Kadijevich & Marinkovic, 2006; p. 37.)

Despite their relevance, P-C minded researchers have rarely used this kind of conceptual tasks, requiring little computation by using objects that are not fully quantified. Having in mind a range of tasks that can be applied in assessing conceptual knowledge (described in Rittle-Johnson & Schneider, 2015), these conceptual tasks may belong to 'explain judgments' task type.[3]

3. Relation between procedural and conceptual knowledge

Views of causal relations between two types of knowledge

There are four views of causal relations between procedural and conceptual knowledge (e.g. Haapasalo & Kadijevich, 2000; Rittle-Johnson & Schneider, 2015).

A summary of four views, proposed in the 1980s and 1990s, is given in Haapasalo and Kadijevich (2000), for example. They examined possible relations between procedural and conceptual knowledge that might be supported by empirical data, and then focused on those relations that could be supported by recent theoretical or empirical studies. Four views remained.

- Inactivation view: Procedural and conceptual knowledge are not related.
- Simultaneous activation view: Procedural and conceptual knowledge are based upon each other (or, in propositional terms, p is necessary and sufficient for c).
- Dynamic interaction view: Procedural knowledge is based upon conceptual knowledge (i.e. c is necessary but not sufficient for p).
- Genetic view: Conceptual knowledge is based upon procedural knowledge (i.e. p is necessary but not sufficient for c).

One of recent summaries of four views, based upon research in the 2000s and early 2010s, can be found in Rittle-Johnson and Schneider (2015), for example. The two authors describe these views in the following way:

- Procedural and conceptual knowledge develop independently.
- Learn concepts first, i.e. procedural knowledge develops on the basis of acquired conceptual knowledge.

[2] A question that clarifies the mathematics behind these three tasks may be: "If $3x > 4y$ for some $x, y > 0$, is $7x$ greater than $9y$?" [yes; $7x > (28/3)y > 9y$]

[3] Other suitable task types are examined in Kadijevich (2003), and Haapasalo (2013a), for example.

- Learn procedures first, i.e. conceptual knowledge develops on the basis of acquired procedural knowledge.
- Procedural and conceptual knowledge develop in an iterative way: increases in one of them lead to succeeding increases in the other.

These authors underline that the last view – which incorporates the second and the third view as further learning may start from one type of knowledge or the other – is widely accepted today. Furthermore, they underline that apart from considerable evidence that the iterative view usually applies, there is evidence that the influence of conceptual knowledge on procedural knowledge may be stronger than the reverse. The authors note that an iterative way was not discussed in early research on procedural and conceptual knowledge. However, the reader may find its roots in the above-mentioned simultaneous activation view (proposed by Byrnes & Wasik, 1991), according to which one type of knowledge is based upon the other, and vice versa.

Means and ways to relate two types of knowledge

Various means and ways to promote relations between procedural and conceptual knowledge (i.e. link the two types of knowledge) can be found in the literature. Regarding the means, a brief summary of four such means are given below.

- Links from procedural to conceptual knowledge may be established through the elaboration and coordination of several microworlds (whereby the learner 'slices the reality), within which different procedures are usually used in problem solving (Papert, 1987).
- By applying some general problem solving productions (i.e. if-then rules), links from conceptual to procedural knowledge may be established through the development of task-specific productions reflecting available conceptual knowledge of that task (Anderson, 1983).
- By using the notion of procept (i.e. "a combined mental object consisting of a process, a concept produced by that process, and a symbol which may be used to denote either of both"), Gray and Tall (1993; p. 8) proposed that the two knowledge types are related through utilizing "procedures where appropriate and symbols as manipulable objects where appropriate."
- Procedural and conceptual knowledge may be unconnected, or sparsely, somewhat, well or richly connected, and it is big ideas (e.g. equal partitioning) that, applied as overarching concepts, connect concepts and procedures from one or several topics (Baroody et al., 2007).

Note that while the first two positions assume that one type of knowledge is based upon the other, the remaining two do not assume particular knowledge dependency.

Regarding the ways, a short summary of some of them follows.

- The links in question can be promoted through replicating solutions with technology on the basis of technology-generated partial solutions, whose steps are given in direct or reverse orders (Zehavi, 1997).

- Problem solving through the development of expert system knowledge bases comprising if-then rules (the so-called knowledge engineering) would relate procedural and conceptual knowledge (Kadijevich, 1999b).
- Relating different problem representations would establish links between procedural and conceptual knowledge (e.g. Schwarz, Dreyfus & Bruckheimer, 1990; Haapasalo, Zimmermann & Rehlich, 2004; Haapasalo, 2013b).
- Using comparisons (e.g. comparing methods whereby problems are solved; comparing problems solved with the same procedures) may be a way to promote links between procedural and conceptual knowledge (extrapolated from Schneider, Rittle-Johnson & Star, 2011; Rittle-Johnson, Star & Durkin, 2012; Star et al., 2015).

4. Relating procedural and conceptual knowledge with technology

Many studies have examined whether the use of technology can aid the development of procedural and conceptual knowledge. Recent summaries of research findings in that respect can, for example, be found in Tall, Smith and Piez (2008), and Olive and Makar (2009), and, in brief, the findings are opposing.

The findings are also opposing with respect to the effects of technology on relating the two types of knowledge. Nevertheless, research studies that examine these effects are quite rare (Kadijevich, 2007), particularly studies that report the overall relations attained in terms of correlations or other appropriate means. Two such studies, with significant positive correlations in question, are reported in Kadijevich and Haapasalo (2001). One study was based upon production rules utilization, whereas the other made use of multiple representations transformation.

A number of studies in the last fifteen years examined how procedural and conceptual knowledge may be connected when technology is used (e.g. Haapasalo, 2003; Drijvers, 2004; Haapasalo et al., 2004; Trouche, 2005; Kadijevich, 2007; Ehmke, Pesonen & Haapasalo, 2010; Kieran, 2013; Abramovich & Connell, 2015). These studies made use of various technologies (e.g. Java applets, computer algebra systems, custom software) at different educational levels (mostly secondary and tertiary).

What kind of technology-based instruction may be applied in general?

It seems that a procedural (i.e. from procedures to concepts) approach should usually be combined with a conceptual (i.e. from concepts to procedures) approach (e.g. Haapasalo, 2003; Ehmke et al., 2010). In other words, in terms of Haapasalo and Kadijevich (2000), the developmental approach should be combined with the educational approach.[4]

[4]The developmental approach is supported by the genetic or simultaneous activation view, whereas the educational approach is supported by the dynamic interaction or simultaneous activation view. The former approach reflects the philogenesis of mathematical knowledge, whereas the latter one fulfils educational needs that require a large body of knowledge to be understood and applied successfully. Their possible instructional interpretations may be "utilize procedural knowledge and reflect on the outcome", and "build meaning for procedural knowledge before mastering it", respectively.

What may basic means to relate procedural and conceptual knowledge be?

Mathematics can be viewed as the science of making and updating links among representing, operating, and interpreting (Peschek & Schneider, 2001). In general, the use of technology allows more time for representing, interpreting, and reflecting on the three and their relations. By accepting representing as the foundation of understanding in school mathematics, the central issues of that understanding are thus related to selecting, using, and coordinating different representations (Heid, 2002). Such a position is in accord with Rittle-Johnson, Siegler and Alibali (2001), who found that a change of problem representation influences the relation in question. To avoid one-side uses of representations (i.e. using one of them ignoring the other(s)), instruction should provide appropriate prompts for use of multiple representations (e.g. Renkl, Berthold, Grosse & Schwonke, 2013).

Using computer algebra environments

Although it has been very important to uncover how different technologies (i.e. learning opportunities of different technologies) affect links between procedural knowledge and conceptual knowledge (Kaput, 1992), very little has been done in this important research area.[5] Several kinds of technological environments have been used (e.g. spreadsheets, interactive geometry software, computer algebra systems, and statistical packages), among which environments based upon computer algebra systems (CASs) are probable the most widely studied in general (suggested by Heid & Blume, 2008, and Hoyles & Lagrange, 2009; for numerical facts about CAS use, see Buteau, Jarvis & Lavicza, 2014). Such a dominant CAS orientation might partly be the results of a number of research activities within the international CAME (Computer Algebra in Mathematics Education) group during the 2000s, which resulted in five CAME symposia.

Research has evidenced (e.g. Gjone 2004; Trouche, 2005; Roger Brown, personal communication, July 21, 2006) that students often use CAS-based environments (CASEs) in one-sided ways (e.g. do not coordinate procedural and conceptual solutions; use just one problem representation; reason within the tool used). This unfavorable state has been the result of (a) using inappropriate tasks (e.g. not focusing on relating procedural and conceptual issues, and without explicit requirements for expected solutions concerning the work in different representations), (b) facing technical and personal challenges in using CASE (e.g. changing CAS representations systematically; having no difficulties with CAS answers; using CAS in strategic, functional, and pedagogical ways; Pierce & Stacey, 2004), and (c) having more strategies for solving problems that introduce difficulties for both students and their teachers (Brown, 2003). Because of these one-sided ways, links between procedural and conceptual knowledge are usually missing.

To reduce one-side approaches in solving tasks in CASE (and with technology in general), and promote links between procedural and conceptual knowledge,

[5]That this research area has been underrepresented for computer algebra environments is evidenced by, for example, a summary made by Heid and colleagues (Heid, Thomas & Zbiek, 2013).

students should be required to (a) produce, whenever possible, solutions based upon the application of both procedural (process) approach and conceptual (object) approach, and (b) work with different problem representations (Kadijevich, 2007). And, to promote appropriate reasoning that goes beyond the technology applied, the student may be continuously reminded of the following: when applying mathematics with technology, dont forget available affordances of that technology; when utilizing these affordances, don't forget the underlying mathematics (Kadijevich, Haapasalo & Hvorecky, 2005). Of course, appropriate instructional prompts concerning all these issues may be given to students.

Clearly, carefully developed tasks are means to relating techniques and theories (suggested by Artigue, 2002).[6] And, with the requirements given above, it is less likely that tasks designed to be conceptual would be solved in solely procedural ways (as happens in the traditional education as well; for interesting examples, see Engelbrecht, Bergsten & Kågesten, 2009). To confirm identities (e.g. $\sin 3x = 3\sin x - 4\sin^3 x$), apart from using procedural approach (applying several rows of eligible transformations) and conceptual approach (testing the equality of the two expressions), the transformations in question may be coupled with comparisons of graphs of the underlying functions, and the verifications of the equalities of two expressions through solving appropriate equations. In solving equation $e^x = \sqrt{x}-3$, for example, apart from comparing graphs of the two underlying functions ($f(x) = e^x$, $g(x) = \sqrt{x} - 3$) or utilizing built-in CAS commands, the student may apply a "don't forget" argument such as $x - 3 < x < e^x$ (possibly validated in CASE).

Not only are suitable tasks (with requirements how to solve them) critical for promoting links between procedural and conceptual knowledge, but the quality of the tool used and the adequacy of the scaffolding offered also considerably influence this promotion.

Regarding the tool quality, the success in relating the two types of knowledge in CASE, as is probably the case of other technologies, is mainly conditioned by the congruence between a paper-and-pencil technique and its CAS version, and the transparency of that version (Drijvers, 2004).[7] Having in mind various limitations of CASEs (e.g. Böhm, 2009; Kadijevich, 2009), the quality of CASE may be improved with appropriately designed user-defined commands (Gjone, 2009; Kadijevich, 2009). This CASE enrichment is attained through a process called instrumentalization (for the twin processes of instrumentation and instrumentalization, see Trouche, 2005). To benefit from it to a larger extent, educators need (1) a better CASE with a control of auto-simplification and a full linkage of different representations; (2) a better traditional instruction that unravels procedural,

[6] It is techniques that relate tasks and theories. Techniques have not only a pragmatic, but also an epistemic function. Through solving tasks techniques fulfill their pragmatic function. Through building concepts they accomplish their epistemic function. It may be said that techniques are means to link procedures and concepts (Langrage, 2005).

[7] An approach that requires students to compare their solutions with CASE generated solutions may need considerable scaffolding (suggested by Tnisson, 2013, and Tönisson & Lepp, 2015). A more efficient approach would be to require students to compare correct paper-and-pecil solutions with (more or less similar) CASE generated solutions with some modest scaffolding offered.

conceptual, and other relevant issues in explicit ways; and (3) an improved programming practice that, when appropriate, also takes into account the epistemic aspects of the developed programs (Kadijevich, 2014).

Regarding the scaffolding adequacy, it is not only instructional prompts regarding the requirements how tasks should be solved that matter. Suitable descriptions of acceptable solutions (e.g. to clarify the result undefined, you may use visual arguments) and scaffolds to help students cope with the limitations of the tool (e.g. to realize the solution, replace 0 with 1 and compare the two outcomes) are also needed (Kadijevich, 2014). By using these prompts, descriptions, and scaffolds[8] initial designs of technology-based tasks (e.g. Berger, 2011) would be improved.

The relation between two types of knowledge may also be affected by the learner's P-C profile, and his/her view of the way technology is basically used. Regarding the profile, the learner may have a persistent approach to learning (e.g. a preference for from-concepts-to-skills, from-skills-to-concepts, or skills-only approach, Järvel & Haapasalo, 2005; a primarily procedural or conceptual approach because that knowledge is stronger, Hallett, Nunes, Bryant & Thorpe, 2012).[9] Distinct students profiles may refer to students' different relationships with technology (e.g. thinker-er, experimentalist; examined by Trouche cited in Artigue, 2002). Regarding the view of basic technology usage, although technology should primarily be viewed in the Vygotskian sense as a tool that usually expands our mental functions (Ivic, 1989), students may, nevertheless, hold various views promoting different kinds of learning (e.g. technology as a master, servant, partner, or an extension of self; Galbraith, 2002). Because of that, instruction should take into account these profiles and views, and, if needed, help students overcome positions that would limit their learning. Through doing that students would also become more confident in their abilities to learn mathematics with technology.

5. Closing remarks

Although at certain points of time conceptual knowledge may be based upon procedural, and/or vice versa (e.g. Kadijevich & Haapasalo, 2001; Arslan, 2010; Lauritzen, 2012), there is an agreement that the two types of knowledge usually develop iteratively, one influencing the other (Rittle-Johnson & Schneider, 2015). Future research may thus focus on modelling such dependence for a particular problem area. (Technology may or may not be used.) One such model, for the area of fractions without utilizing technology, can be found in Bailey et al. (2015). A more general model that may be (with or without technology) applied to several areas including fractions can be found in the papers of Haapasalo and colleagues (Haapasalo, 2003; Haapasalo et al., 2004; Ehmke et al., 2010).

[8] All these aids, like the whole instruction, may be implemented in the so-called minimalist fashion (Haapasalo, 2013b).

[9] As some students tend to proceduralize or conceptualize knowledge items, developing the two knowledge types and their links may be examined in terms of students' learning/thinking styles (Kadijevich, Maksic & Kordonis, 2003). This proposal was empirically supported by Kadijevich and Krnjaic (2004), who found that the higher field-independence the student demonstrated, the stronger links between procedural and conceptual knowledge he/she established.

As a result of this P-C dependence, instruction should not solely use a procedural or conceptual approach (for comparing the effectiveness of the two approaches, see, for example, Kadijevich, 2002), but rather a combination of the two. In other words, an instruction that goes from procedures to concepts should be combined with an instruction that goes from concepts to procedures through, for example, "iterating between lessons on concepts and procedures" (Rittle-Johnson, Schneider & Star, 2015; p. 593). Further research may thus aim at finding the most effective sequencing of lessons on concepts and procedures, taking into account critical variables that influence this sequencing (related to tasks, learners, and technology if applied). To attain a solid understanding of target concepts and procedures, this sequencing may be more of a conceptual than procedural nature (Rittle-Johnson, Fyfe, & Loehr, 2016), because, as already mentioned, the influence of conceptual knowledge on procedural knowledge may be stronger than the reverse (Rittle-Johnson & Schneider, 2015).

Although procedural and conceptual tasks may be suitable for measuring the two types of knowledge, conceptual tasks may be proceduralized (e.g. Hallett et al., 2012), whereas, though much less often, procedural tasks may be conceptualized. With a requirement to solve each task with technology in both procedural and conceptual ways (Kadijevich, 2007), the same task can be used to measure the two types of knowledge. This approach was used (technology was not utilized) by Chinnappan and Forrester (2014), and Kadijevich and Krnjajic (2004), for example. Because reliable and valid measurements of procedural and conceptual knowledge need to be developed (Rittle-Johnson & Schneider, 2015), further research may focus on this development, especially when technology is utilized.

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