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# Notes for Starlikeness Conditions of Analytic Functions

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## 1. Introduction

Let $A$ denote the class of functions $f$ which are analytic in the unit disc $D=\{z: |z|<1\}$, with

(1.1) $$f(0)=0 \text{ and } f'(0)=1.$$

And let $S^*$ be the usual class of starlike functions in $D$, i.e.

$$S^*=\left\{f\in A: \mathrm{Re}\frac{zf'(z)}{f(z)}>0,\ z\in D\right\}.$$

On the starlikeness condition for functions of $A$, we now investigate the following well known problems:

**Problem 1.** Find the maximum value of $r$ for which $f\in A$ and

(1.2) $$|f'(z)-1|<1,\ z\in D$$

imply that $f$ is starlike in

(1.3) $$D_r = \{z : |z| < r\}.$$

**Problem 2.** Find the maximum value of $r$ for which $f \in A$ and

(1.4) $$|f'(z) - 1| < r, \quad z \in D$$

imply that $f \in S^*$.

We denote by $r_1$ and $r_2$ the maximum value of the first problem and of the second problem, respectively.

For $r_1$, T. MacGregor [1] showed that

$$r_1 \geqq 2/\sqrt{5} = 0.894\cdots.$$

Recently, M. Nunokawa in [3] proved that $r_1 > 0.926\cdots$, and in [4] Nunokawa et all improved, that is $r_1 > 0.933\cdots$. On the other hand, P. Mocanu [2] showed that the function

(1.5) $$f(z) = -2z + z^2 + 6\log\frac{2+z}{2}$$

belongs to $A$ and satisfies the condition (1.2), but does not belong to $S^*$. Therefore $r_1 < 1$. For $r_2$, in the same article Mocanu also proved that

$$r_2 \geqq 2/\sqrt{5} = 0.894\cdots.$$

In this short paper, modifying the Mocanu's example, we show that

(1.6) $$r_1 < 0.9982$$

and

(1.7) $\qquad r_2 < 0.9962.$

## 2. Examples

**Example 1.** Let

$$(2.1) \qquad f(z) = (2-a^2)z + \frac{a}{2}z^2 + a(a^2-1)\log\frac{z+a}{a}, \quad z \in D,$$

where $\log 1 = 0$. Since

$$(2.2) \qquad f'(z) = 2 - a^2 + az + \frac{a(a^2-1)}{z+a} = 1 + z\frac{az+1}{z+a},$$

we easily deduce

$$f(0) = f'(0) - 1 = 0$$

and

$$|f'(z) - 1| < 1, \quad z \in D.$$

If we let

$$f(z) = s + it, \quad zf'(z) = u + iv$$

and $z = re^{i\theta}$, then we have

$$s = (2-a^2)r\cos\theta + \frac{a}{2}r^2\cos 2\theta + \frac{a(a^2-1)}{2}\log(r^2 + a^2 + 2ar\cos\theta)$$
$$- a(a^2-1)\log a,$$

$$t = (2-a^2)r\sin\theta + \frac{a}{2}r^2\sin 2\theta + a(a^2-1)\mathrm{Tan}^{-1}\frac{r\sin\theta}{a + r\cos\theta},$$

$$u = (2-a^2)r\cos\theta + ar^2\cos 2\theta + a(a^2-1) - \frac{a^2(a^2-1)(r\cos\theta + a)}{r^2 + a^2 + 2ar\cos\theta}$$

and

$$v=(2-a^2)r\sin\theta+ar^2\sin 2\theta+\frac{a^2(a^2-1)r\sin\theta}{r^2+a^2+2ar\cos\theta}.$$

And it is evident that $\mathrm{Re}[zf'/f]>0$ if and only if

(2.3) $$su+tv>0.$$

But by putting

(2.4) $$a=1.1,\quad r=0.9982,\quad \theta=2.743,$$

we deduce $su+tv=-9.13\times10^{-5}<0$, which shows that $f$ is not starlike in $D_r$, i.e. $r_1<0.9982$.

**Example 2.** Let

(2.5) $$g(z)=(1-\frac{b}{a})z+\frac{b}{2(a^2-1)}z^2+b\log(z+a)-b\log a.$$

Since

(2.6) $$g'(z)=1+\frac{b}{a(a^2-1)}\cdot\frac{z(az+1)}{z+a},$$

evidently $g\in A$, and

(2.7) $$|g'(z)-1<k=\left|\frac{b}{a(a^2-1)}\right|,\quad z\in D.$$

Let

$$g(z)=s+it,\quad zf'(z)=u+iv$$

and $z=re^{i\theta}$. Then we deduce

$$s=(1-\frac{b}{a})r\cos\theta+\frac{b}{2(a^2-1)}r^2\cos 2\theta+\frac{b}{2}\log(r^2+a^2+2ar\cos\theta)-b\log a,$$

$$t=(1-\frac{b}{a})\,r\sin\theta+\frac{b}{2(a^2-1)}r^2\sin 2\theta+b\,\mathrm{Tan}^{-1}\frac{r\sin\theta}{a+r\cos\theta},$$

$$u=(1-\frac{b}{a})\,r\cos\theta+\frac{b}{a^2-1}r^2\cos 2\theta+\frac{b(r^2+a\,r\cos\theta)}{r^2+a^2+2a\,r\cos\theta},$$

$$v=(1-\frac{b}{a})\,r\sin\theta+\frac{b}{a^2-1}r^2\sin 2\theta+\frac{a\,b\,r\sin\theta}{r^2+a^2+2a\,r\cos\theta}.$$

Substituting

(2.8) $a=1.1,\quad b=0.2301,\quad r=1,\quad \theta=2.743$

we obtain $su+tv=-4.36\times 10^{-5}<0$, and

(2.9) $k=0.9961038\cdots,$

which yield $r_2<0.9962$.

## References

[1] T. H. MacGregor, A Class of univalent functions, Proc. Amer. Math. Soc., 15(1964), 311-317.

[2] P. T. Mocanu, Some starlikeness conditions for analytic functions, Rev. Roumaine Math. Pures Appl., 33(1988), 117-124.

[3] M. Nunokawa, On the starlike boundary of univalent functions, Sûgaku, 31(1979), 255-256 (Japanese).

[4] M. Nunokawa, S. Fukui, S. Owa, H. Saitoh and T. Sekine, On the starlikeness bound of univalent functions, Math. Japonica, 33, No. 5(1988), 763-767.