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HEIGHT OF p-ADIC HOLOMORPHIC FUNCTIONS AND APPLICATIONS(*)

by Ha Huy Khoai (Hanoi)

Dedicated to Professor S. Kobayashi on his 60-th birthday

§1. Introduction

Let me start the talk by explaining why study p-adic Nevanlinna Theory. In the famous paper "De la métaphysique aux mathématiques" ([W]) A. Weil discussed about the role of analogies in mathematics. For illustrating he analysed a "metaphysics" of Diophantine Geometry: the resemblance between *Algebraic Numbers* and *Algebraic Functions*. However, the striking similarily between Weil's theory of heights and Cartan's Second Main Theorem for the case of hyperplanes is pointed out by P. Vojta only after 50 years! P. Vojta observed the resemblance between *Algebraic Numbers* and *Holomorphic Functions*, and gave a "dictionary" for translating the results of Nevanlinna Theory in the one-dimensional case to Diophantine Approximations. Due to this dictionary one can regard Roth's Theorem as an analog of Nevanlinna Second Main Theorem. P. Vojta has also made quantitative conjectures which generalize Roth's theorem to higher dimensions. One can say that P. Vojta proposed a "new metaphysics" of Diophantine Geometry: Arithmetic Nevanlinna Theory in higher dimensions. On the other hand, in the philosophy of Hasse-Minkowski principle one hopes to have an "arithmetic result" if one have had it in p-adic cases for all prime numbers p, and in the real and complex cases(**). Hence, one would naturally have interest to determine how Nevanlinna Theory would look in the p-adic case.

§2. Heights of p-adic holomorphic functions

One of most essential differences between complex holomorphic functions and p-adic ones is that the modulus of a p-adic holomorphic function depends only on the modulus of arguments, except on a "critical set". This fact led us to introduce the

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(**)The results of F. Fujimoto can be regarded as a real Nevanlinna Theory (see the talk of F. Fujimoto in this volume) notion of heights of a p-adic holomorphic function. Using the height one can reduce in many cases the study of the zero set of a holomorphic function to the study a real convex parallelopiped. This makes it easier to prove p-adic analogues of statements of Nevanlinna Theorey.

2.1. Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers, and $\mathbb{C}_p$ the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. The absolute value in $\mathbb{Q}_p$ is normalized so that $|p| = p^{-1}$. We further use the notion $v(z)$ for the additive valuation on $\mathbb{C}_p$ which extends ord_p.

Let $f(z_1, ..., z_k)$ be a holomorphic function in C_p^k represented by the convergent series

$$f(z_1, ..., z_k) = \sum_{|m|=0}^{\infty} a_{m_1...m_k} z_1^{m_1} ... z_k^{m_k}. \tag{1}$$

We set:

$$a_m = a_{m_1...m_k},$$
$$z^m = z_m^{m_1} ... z_k^{m_k},$$
$$|m| = m + 1 + ... + m_k,$$
$$mt = m_1 t_1 + ... + m_k t_k.$$

Then for every $(t_1, ..., t_k) \in R^k$ we have:

$$\lim_{|m| \to \infty} \{v(a_m) + mt\} = \infty.$$

Hence, there exists an $(m_1, .., m_k) \in \mathbb{N}^k$ such that $v(a_m) + mt$ is minimal.

2.2. **Definition**. *The height* of the function $f(z_1, ..., z_k)$ is defined by

$$H_f(t_1, ..., t_k) = \min_{o \leq |m| < \infty} \{v(a_m) + mt\}.$$

We use also the notation $H_f(z_1, ..., z_k) = H_f(v(z_1), ..., v(z_k))$.

2.3. Let us now give a geometric interpretation of heights. For every $(m_1, ..., m_k)$ we construct the graph $\Gamma_{m_1...m_k}$ representing $v(a_m z^m)$ as function of $(t_1 ...., t_k)$ where $t_i = v(z_i)$. Then we obtain a hyperplane in R^{k+1}:

$$\Gamma_{m_1...m_k} : t_{k+1} = v(a_m) + mt.$$

Since $\lim_{|m|\to\infty}\{v(a_m)+mt\} = \infty$ for every $(t_1,...,t_k) \in R^k$ there exists a hyperplane realizing

$$t_{k+1}(\Gamma_{m_1...m_k}) \leq t_{k+1}(\Gamma_{m'_1..m'_k})$$

for all $\Gamma_{m'_1...m'_k}$. We denote by H the boundary of the intersection in $R^k \times R$ of half-spaces of R^{k+1} lying under the hyperplane $\Gamma_{m_1...m_k}$. It is easy to show that if $(t_1,...,t_k,t_{k+1})$ is a point of H, then $t_{k+1} = H_f(t_1,...,t_k)$.

2.4. To study of the zero set of a holomorphic function we need the following definition of local heights.

We set:

$$I_f(t_1,...,t_k) = \{(m_1,...,m_k) \in N^k, v(a_m) + \sum_{j=1}^{k} m_i t_i = H_f(t_1,...,t_k)\}$$

$$n_i^+(t_1,...,t_k) = \min\{m_i | \exists (m_1,...,m_i,...,m_k) \in I_f(t_1,...,t_k)\}$$

$$n_i^-(t_1,...,t_k) = \max\{m_i | \exists (m_1,...,m_i,...,m_k) \in I_f(t_1,...,t_k)\}$$

It is easy to see that there exists a number T such that for $(t_1,...,t_k) \geq (T,...,T)$ (this means $t_i \geq T$ for all i), the numbers $n_i^+(t_1,...,t_k)$ and $n_i^-(t_1,...,t_k)$ are constants. Then we set:

$$h_i^+(t_1,...,t_k) = n_i^+(t_1,...,t_k)(T-t_i)$$

$$h_i^-(t_1,...,t_k) = n_i^-(t_1,...,t_k)(T-t_i)$$

$$h_i(t_1,...,t_k) = h_i^-(t_1,...,t_k) - h_i^+(t_1,...,t_k)$$

$$h_f(t_1,...,t_k) = \sum_{i=1}^{k} h_i(t_1,...,t_k).$$

2.5. **Definition**. $h_f(t_1,...,t_k)$ is said to be the *local height* of the function $f(z_1,...,z_k)$ at $(t_1,...,t_k) = (v(z_1),...,v(z_k))$.

2.6. One can prove basic properties of the height and local height by using the geometric interpretation 2.3. For our purpose we need some of them, namely, the following.

2.7. H is the boundary of a convex polyhedron in R^{k+1}.

2.8. If we denote by $\Delta(H)$ the set of the edges of the polyhedron H then the set of the critical points is exactly the image of $\Delta(H)$ by the projection:

$$\pi_k : R^k \times R \to R^k.$$

2.9. We can show that for every finite parallelopiped in $R^{k+1}, P = \{-\infty < r_i < t_i < +\infty, i = 1, ..., k+1\}, H \cap P \times R$ consists of parts of a finite number of hyperplanes $\Gamma_{m_1 ... m_k}$. Indeed, these are the hyperplanes such that at least for an index i we have $m_i = n_i^+(t_1, ..., t_k)$ or $m_i = n_i^-(t_1, ..., t_k)$ for a point $(t_1, ..., t_k) \in P$.

2.10. For every finite parallelopiped and every hyperplane L in general position with respect to $H, L \cap H \cap P$ is a part of a hyperplane of dimension $k-1$.

2.11. If for $i \leq k$ the hyperplane $t_i = s_i =$ const is not in general position, then the hyperplane $t_i = s_i \pm \epsilon$ are in general position for small enough ϵ. Moreover we have:

$$\lim_{\epsilon \to 0} H_f(..., s_i \pm \epsilon, ...) = H_f(..., s_i, ...).$$

2.12. The set of critical points $\pi_k \Delta(H)$ is an union of hyperplanes of dimensions less or equal $k-1$.

2.13. Suppose that $S = S_1 \cap ... \cap S_{k-1}$, where S_i is the hyperplane $t_i = s_i, i = 1, ..., k-1$. Replacing S_i by $S_i^{\pm\epsilon} : t_i = s_i \pm \epsilon$ if necessary, one can suppose that the hyperplanes S_i are in general position. Then the intersection $S \cap \pi_k \Delta(H) \cap P$ is a finite set of points.

Note that we are using "general position" in an evident sense.

2.14. By using the above remarks we can formulate and prove an analogue of the Poisson-Jensen formula. For any $(t_1, ..., t_k) \in R^k$ we set:

$$h_f(t_1, ..., t_i^{\pm}, ..., t_k) = \lim_{\epsilon \to 0} h_f(t_1, ..., t_i \pm \epsilon, ..., t_k)$$

and for two points $(t_1, ..., t_k)$ and $(T_1, ..., T_k)$:

$$\begin{aligned}\delta_i = &\, h_i^{-\epsilon_i}(t_1^{\epsilon_1}, ..., t_{i-1}^{\epsilon_{i-1}}, T_i^{\epsilon_i}, ..., T_k^{\epsilon_k}) \\ &- h_i^{\epsilon_i}(t_1^{\epsilon_1}, ..., t_{i-1}^{\epsilon_{i-1}}, t_i, T_{i+1}^{-\epsilon_{i+1}}, ..., T_k^{-\epsilon_k}) \\ &+ \sum_{s_i} h_i^{\epsilon_i}(t_1^{\epsilon_1}, ..., t_{i-1}^{\epsilon_{i-1}}, s_i, T_{i+1}^{-\epsilon_{i+1}}, ..., T_k^{-\epsilon_k})\end{aligned}$$

where $\epsilon_i =$ sign $(T_i - t_i)$ and the sum takes all $s_i \in (T_i, t_i)$. Note that by 2.4 the h_i are vanishing, except possibly on a finite set of values s_i, and δ_i does not depends on the choice of T.

2.15. **Theorem.** *(The Posson-Jensen formula).*

$$H_f(T_1, ..., T_k) - H_f(t_1, ..., t_k) = \sum_{i=1}^{k} \epsilon_i \delta_i.$$

2.16. **Remark**. Formula 2.15 is analogous to the classical Poisson-Jensen formula. In fact, suppose that $k = 1, t_0 = \infty, f(0) \neq 0$ and t is not a critical point of the function $f(z)$. Then we have $H_f(t_0) = -\log_p |f(0)|, H_f(t) = \log_p |f(z)|$, where the sum extends over all the zeros z_i of the function $f(z)$ in the disc $|z| \leq p^{-t}$. Then the formula 2.15 takes the following form:

$$\log_{v(z)=t} |f(z)| - \log_p |f(0)| = \sum - \log_p |z_i|.$$

Recall that the classical Poisson-Jensen formula is the following:

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta - \log |f(0)| = \sum_{a \in D, a \neq 0} -(\mathrm{ord}_a f) \log |a|,$$

where D is the unit disc in C and $\mathrm{ord}_a f$ is the order of $f(z)$ at a.

2.17. **Remark**. The formula 2.15 is not symmetric in variables $t_1, ..., t_k$, and then one obtains a number of formulas of the height via local heights. Then it follows many equalities relating local heights. This fact has an analogue in the case of holomorphic functions of two complex variables (see [Ca]).

2.18. **Remark**. In [Ro] Robba gave an "approximation formula", from which follows the Schwarz lemma for p-adic holomorphic functions of severed variables. One can also obtain the Schwarz lemma by using 2.15.

Let us finish this section with the following important theorem, the proof of which is easy by using the geometric interputation of height.

2.19. **Theorem**. Every non-constant holomorphic function on $\mathbb{C}_p^k$ is a surjective map onto $\mathbb{C}_p$.

By using the notion of height we obtain in the one-dimensional case the analogue of two Main Theorems of Nevanlinna Theory (see [H-M], [Ha3]). However in higher dimensions the problem is still open.

§3. Lelong number

It is well-known that the Lelong number plays an important role in the theory of complex entire functions. In this paper I define the Lelong number of a p-adic entire function of several variables. In the p-adic case I do not know how to define an analogue of the "volume element", and I use here the notion of local heights.

3.1. **Definition**. The *Lelong number* of a holomorphic function $f(z_1, ..., z_k)$ at the point $(z_1, ..., z_k)$ is defined by:

$$\nu_f(z_1,...,z_k) = \sum_{i=1}^{k}\{n_i^-(t_1,...,t_k) - n_i^+(t_1,...,t_k)\},$$

where $t_i = v(z_i)$.

3.2. **Example**. In the case of $n = 1, \nu_f(z)$ is the number of zeros of f at $v(z) = t$ with counting multiplicity (see [Ma]).

3.3. **Remark**. The Lelong number of a holomorphic function $f(z)$ depends only on the modulus of the arguments.

3.4. **Lemma**. *$\nu_f(z_1,...,z_k) \neq 0$ if and only if $v(z_1,...,z_k) \in \pi_k\Delta_{H(f)}$ is the projection of $\Delta_H(f) \subset R^k \times R$ on R^k.*

Proof. In fact, suppose $\nu_f(z_1,...,z_k) \neq 0$ and denote $t_i = v(z_i)$. Then for every $i, n_i^+(t_1,...,t_k) = n_i^-(t_1,...,t_k)$ and there exists an unique n_i such that the set $\{(m_1,...,m_k) \in I_f, m_i = n_i\}$ is not empty. From this it follows that I_f contains an unique element $(n_1,...,n_k)$, and we have $H_f(t_1,...,t_k) = v(a_n) + nt, |f(z_1,...,z_k)| = p^{-H_f(t_1,...,t_k)}$. Hence, $(t_1,...,t_k) \notin \pi_k\Delta_{H(f)}$.

Conversely, suppose $\nu_f(z_1,...,z_k) \neq 0$. Then there exist at least one indexe i such that $n_i^-(t_1,...,t_k) \neq n_i^+(t_1,...,t_k)$. Therefore by using Remark 2.9 one can see that there exist at least two faces of $H(f)$ containing the point $(t_1,...,t_k)$. This means that $v(z_1,...,z_k) \in \pi_k\Delta_{H(f)}$. Lemma 3.4 is proved.

3.5. **Theorem**. *A holomorphic function $f(z_1,...,z_k)$ is a polynomial if and only if the Lelong number $\nu_f(z_1,...,z_k)$ is constant for large enough $||z||$.*

Proof. From the properties 2.7-2.13 of height one can show that $\nu_f(z_1,...,z_k) =$ const for large enough $||z||$ if and only if there exist finitely many hyperplane $\Gamma_{m_1...m_k}$ appear in the construction of H_f. This is equivalent to that f is a polynomial.

3.6. **Remark**. In the case of functions of one variable $\nu_f(z) =$ const is equivalent to that $\nu_f(z) = 0$ for large enough $|z|$.

§4. Hyperbolicity

There are interesting relations between the value distribution theory, Diophantine problems and hyperbolic geometry. Some of them are deep results of Faltings, Vojta, Noguchi and others, while many statements are still conjectural (see [La1], [La2], No1], [No2], [Vo]). In the p-adic case, because of the total discontinuity it is difficult to define an analogue of the Kobayashi distance(*). In this paper we propose a definition of p-adic hyperbolicity in the sense of Brody. Namely, a domain X in the projective space $P^n(C_p)$ is called hyperbolic if every holomorphic map from C_p to X is constant. We shall prove some theorems of Borel type on maps with the image lying in the complement of hyperplanes and algebraic hypersurfaces. Our purpose is only to examine in p-adic case some properties of hyperbolic spaces described in Lang's book [La3].

4.1. **Definition**. A subset X of the projective space $P^n(C_p)$ is called *hyperbolic* if every holomorphic map from C_p into $P^n(C_p)$ with the image in X is constant.

Note that by a holomorphic map from C_p into $p^n(C_p)$ we mean a collection $f = (f_0, f_1, ..., f_n)$ where $f_i(z)$ are holomorphic functions having no zeros in common.

4.2. **Examples**. 4.2.1. The unit disc $D \in C_p$ is hyperbolic. Indeed, every holomorphic function on C_p with values in D is a bounded entire function, and therefore, is constant (Theorem 2.19).

4.2.2. If X, Y are hyperbolic, the $X \times Y$ is hyperbolic. Hence, a polydisc $D \times ... \times D$ in $P^n(C_p)$ is hyperbolic.

4.2.3. From Theorem 2.19 it follows that the set $C_p \setminus \{$ one point $\}$ and $P^1 \setminus \{$ two points $\}$ are hyperbolic.

4.3. **Remark**. For any hyperbolic set $X \in C_p^n$, $C_p^n \setminus X$ is not bounded. Indeed, if $C_p^n \setminus X$ is bounded, then $C_p^n \setminus X \subset B_r$ for a ball of radius r. For a constant a with $|a| > r$ the following map

$$f : C_p \to C_p^n, \quad z \mapsto (z, z + a, ..., z + a)$$

has the image lying in $C_p^n \setminus B_r$, and hence X is not hyperbolic.

4.4. H_k, $(k = 0, 1, ..., m)$ be hyperplanes of $P^n(C_p)$, then they said to be *in general position* if any $l (l \leq n + 1)$ these hyperplanes are linearly independent.

4.5. **Theorem**. *The complement in $P^n(C_p)$ of $n + 1$ hyperplanes in general position is a hyperbolic space.*

Indeed, let $f : C_p \to P^n$ be a holomorphic map with image lies in the complement of $n + 1$ hyperplanes in general position. Let $(x_0, ..., x_n)$ be the coordinates of $P^n(C_p)$.

(*) I suppose that the "p-adic hyperbolic distance" and the "arithmetic hyperbolic distance" proposed by L. Weng and J. Noguchi would work in this case (see their talks in this volume)

Then there is a projective change of coordinates such that these hyperplanes are defined by the equations $x_0 = 0, ..., x_n = 0$. Now we can write f in homogeneous coordinate

$$f = (f_0, ..., f_n).$$

By the hypothesis the functions $f_0, ..., f_n$ are non-zero entire functions in C_p, and then they are constant.

4.6. **Theorem.** *Let $X_1, ..., X_{n+1}$ be $n+1$ hyperplanes in $P^n(C_p)$ in general position. Let*

$$X = X_1 \cup X_2 \cup ... \cup X_{n+1}$$

be their union. Then

1) $P^n(C_p) \setminus X$ is hyperbolic.

2) for every $\{i_1, ..., i_k, j_1, ..., j_k\} = \{1, ..., n+1\}$ the space

$$X_{i_1} \cap ... \cap X_{i_k} \setminus (X_{j_1} \cup ... \cup X_{j_r})$$

is hyperbolic.

Proof. 1) Theorem 4.5.

2) Let

$$f : C_p \to X_{i_1} \cap ... \cap X_{i_k} \setminus X_{j_1} \cup ... \cup X_{j_r}$$

be a holomorphic map. Since the hyperplanes are in general position, $X = X_{i_1} \cap ... \cap X_{i_k}$ can be identified with P^{n-k}. Then $\{X_{j_m} \cap X\}$ are in general position in X. We have $r = (n-k)+1$, and 2) is a corollary of Theorem 4.5.

4.7. **Theorem.** *Let $X \to Y$ be a holomorphic map of p-adic analytic spaces. Suppose that Y is hyperbolic, and for every $y \in Y$ there exists a neighbourhood U of y such that $\pi^{-1}(U)$ is hyperbolic. Then X is a hyperbolic space.*

Proof. Let $f : C_p \to X$ be a holomorphic map. Then $\pi.f$ is holomorphic, and is constant, since Y is hyperbolic. We set $y_0 = \pi.f(C_p)$. Let U_0 is a neighbourhood of y_0 such that $\pi^{-1}(U_0)$ is hyperbolic. Since the image of f lies in $\pi^{-1}(U_0), f$ is constant.

4.8. **Theorem.** *Let f be a holomorphic map from C_p into $P^n(C_p)$ with image lies in the complement of $k \geq 2$ different hypersurfaces. Then there exist proper algebraic subspaces $X_1, ..., X_m, m = \dfrac{k(k-1)}{2}$, such that the image of f lies in the intersection of $X_1, ..., X_m$.*

Proof. Let $P_1, ..., P_k$ be the homogeneous polynomials defining the hypersurfaces $Y_1, ..., Y_k$. For every $i, 1 \leq i \leq k, P_i.f$ is constant. We can find numbers α_i such that $\alpha_i(P_i.f) - \alpha_j(P_j.f) \equiv 0$ on C_p. We set

$$Q_{ij} = \alpha_j P_i - \alpha_j P_j.$$

Then Q_{ij} are homogeneous polynomials, which define the algebraic subspaces $X_1, ..., X_m$, $m = \dfrac{k(k-1)}{2}$. Note that X_i's are proper algebraic subspaces, and the image of f lies in their intersection.

4.9. **Remark**. The theorem can be regarded as an analogue of the Green theorem in the complex case (see [La3]).

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