数理解析研究所講究録
第 808 巻 1992 年 91-146

Local monodromy on the fundamental groups of algebraic curves along a degenerate stable curve

MAMORU ASADA, MAKOTO MATSUMOTO, AND TAKAYUKI ODA
(朝田衛) (松本眞) (織田孝幸)

Introduction.

The purpose of this paper is to prove some result on the local monodromy representation on the fundamental groups for a universal degenerating family of punctured algebraic curves.

Let us explain a typical case in a more precise way, i.e. the case of no puncture. We start with a most degenerate stable curve C_0 of genus $g \geq 2$. For such a curve, we can associate the dual graph Y whose vertices correspond to the irreducible components of C_0 and edges to double points. Consider a local universal deformation $f : \mathcal{C} \to \mathcal{D}$ of C_0 in the category of stable curves. Let $\mathcal{D}^\circ$ be the open subset of $\mathcal{D}$, on which the fibers of f are smooth. Let $\mathbf{t}$ be a point on $\mathcal{D}^\circ$. Then we obtain the monodromy map on the fundamental group $\pi_1(C_{\mathbf{t}}, *)$

$$\rho_{C_0} : \pi_1(\mathcal{D}^\circ, \mathbf{t}) \to Out\ \pi_1(C_{\mathbf{t}}, b).$$

Here b is a base point in $C_{\mathbf{t}}$.

We can consider the weight filtration on the fundamental group of curves, which is preserved by the monodromy homomorphism. The main target of this paper is to describe the relation between the monodromy homomorphism and the weight filtration for the local universal deformation of a most degenerate stable curve. The weight filtration coincides with the lower central series for the fundamental group of a complete curve.

Here is a description of the main result: Let I_Y be the image of the injective homomorphism ρ_{C_0} which is a free abelian group of rank $3g-3$, and let $\{I_Y^{(m)}\}_{m=0,1,2,\ldots}$ be the induced filtration on I_Y derived from the lower central filtration on $\pi_1(C_{\mathbf{t}}, b)$. Put

$$r_m(Y) = rank_{\mathbb{Z}}\ I_Y^{(m)}/I_Y^{(m+1)} \text{ for all } m\ (m = 0, 1, 2, \ldots).$$

Then the main result tells

$$r_m(Y) = 0, \text{ if } m \geq 3,\ r_2(Y) = s_2(Y),\ r_1(Y) = s_1(Y),$$
$$\text{and } r_0(Y) = 3g - 3 - s_1(Y) - s_2(Y).$$

Here $s_2(Y)$ is the number of bridges in the graph Y, and $s_1(Y)$ is also another geometric invariant of Y related with the connectivity (cf. Subsection 1.4 for a precise definition). We also note here the equality $r_0(Y) = 3g - 3 - s_1(Y) - s_2(Y)$ is due to Brylinski [**Br**].

The first motivation was to generalize the transcendental part of the previous paper [**O**] by one of the authors, in which we discussed a similar problem when the base $\mathcal{D}$ is one-dimensional, and the graph of C_0 is a tree. Similarly to that paper, we expect that these results have some applications to l-adic setting.

Now let us explain the outline of the contents of this paper. In Section 1, we recall some basic notions on stable curves and stable n-pointed curves, and their associated graphs. Defining some combinatorial invariants for graphs, we formulate the main result of this paper. In Section 2, we recall basic facts on the graph of group by Bass and Serre [**S**]. We define the notion of edge twists, which is used to describe Dehn twists in an algebraic language. Section 3 is the corner stone of this paper. In this section, we translate the problem of the local monodromy on the fundamental group into a completely algebraic and combinatorial language of the graph of groups. We start with a special case of the Seifert-van Kampen theorem. The key proposition here is the non-abelian Picard-Lefschetz formula (Theorem (3.2)).

In Section 4, we discuss the algorithm to compute Dehn twists for the monodromy explicitly. Some examples are discussed for the low genus cases. These examples also serve as the initial step of the inductive proof of the main result in Sections 5 and 6.

In Sections 5 and 6, we give an inductive proof of the main result. In the first place, we discuss the case of no puncture which is simpler compared with the general case. After that the general case is reduced to this former special case by a simple idea.

Though we do not discuss, our results have purely topological interpretation in terms of Dehn twists associated to pants decomposition of punctured Riemann surfaces.

By the results of J. Morgan and R. Hain, we can equip the Malcev Lie algebras of the fundamental groups of algebraic varieties with mixed Hodge structures. It seems an interesting problem to push forward our result toward this direction.

We thank H. Nakamura for pointing out a clue for proving our main results. We also thank Y. Ihara for valuable and stimulating discussion on the theme of this paper, and S. Morita for informing us of the literature on low dimensional topology.

1. Formulation of the main result.

1.1 *Stable n-pointed curves and their graph.*

Let us recall the definition of stable n-pointed curves [**Kn**, §1].

Definition 1.1 A stable n-pointed curve (C, S) of genus g is a pair (C, S) of a proper connected curve C over the complex number field $\mathbb{C}$ and a subset of n-distinct smooth points on C satisfying the following conditions:

(i) C has only ordinary double points as singularities. C_{sing} denotes the locus of singularities. Let $p : C^* \to C$ be the normalization of C. Then we set $C^*_{sing} = p^{-1}(C_{sing})$ and identify $p^{-1}(S)$ with S via p.

(ii) (stability) On the normalization D^* of each irreducible component D of C which is isomorphic to $\mathbb{P}^1$, the sum of numbers of $D^* \cap C_{sing}$ and $D^* \cap S$ is at least 3.

When $n = 0$, the above definition gives the notion of stable curves [**DM**].

A graph of a stable n-pointed curve

For each stable n-pointed curve (C, S), we can associate the (dual) graph Y in the following manner [**DM**], [**N**].

Definition 1.2

(1) Each vertex P of the graph Y corresponds uniquely to an irreducible component C_P of C. Or equivalently, each vertex P corresponds uniquely to a connected component of the normalization C^* of C.

(2) A pair $\{y, \bar{y}\}$ of mutually inverse (oriented) edges of Y corresponds uniquely to a singular point $q_{\{y,\bar{y}\}}$ of C. If necessary, we refer to the pair $\{y, \bar{y}\} = |y|$ as a geometric edge associated with y or with $\bar{y}$. We also denote $q_{\{y,\bar{y}\}}$ by q_y, $q_{\bar{y}}$, or $q_{|y|}$. The set of geometric edges is denoted by $Edge(Y)_{geom}$.

(3) For each edge y, its two extremities are given by the vertices P_1, P_2 so that

$$q_y = C_{P_1} \cap C_{P_2} \qquad (\text{if } P_1 \neq P_2),$$

$$q_y = C_{P_1} \cap C_{sing} \qquad (\text{if } P_1 = P_2).$$

(4) There is a function

$$v : Vert(Y) \to \mathbb{Z} \times \mathbb{Z}$$

from the set of vertices $Vert(Y)$ of Y to the product of the set of non-negative integers defined by $v(P) = (g_P, n_P)$. Here g_P is the genus of the normalization C^*_P of C_P, and n_P is the cardinality of the set $S \cap C_P$.

For each edge y, we denote by $o(y)$ the origin and by $t(y)$ the terminus of y, respectively. Choose one edge from each geometric edge $|y| = \{y, \bar{y}\}$, and form a subset $Edge(Y)_+$. Then $\#(Edge(Y)_+) = \#(Edge(Y)_{geom}) = \frac{1}{2}\#(Edge(Y))$. We have the following condition for the above connected graph Y with function v.

Proposition (1.1).

$$g = \sum_{P \in Vert\ Y} g_P + h^1(Y),$$

where $h^1(Y) = \#(Edge(Y)_{geom}) - \#(Vert(Y)) + 1$. *Moreover*

$$n = \sum_{P \in Vert(Y)} n_P.$$

If $g_P = 0$, *then*

$$n_P + \#\{y \in Edge(Y) \mid P = o(y)\ \} \geq 3.$$

Remark 1.1 The graph (Y, v) determines the homotopy type of $C - S$.

The following is easy to prove.

Lemma (1.2). *Let* (Y, v) *be the graph of a stable* n*-pointed curve of genus* g. *Then*

$$\#(Vert(Y)) \leq 2(g-1) + n; \qquad \#(Edge(Y)_{geom}) = \frac{1}{2}\#(Edge(Y)) \leq 3(g-1) + n.$$

The most degenerate case

Definition 1.3 A stable n-pointed curve (C, S) is called most degenerate, if it has no deformation with the same homotopy type.

In this case, any irreducible component of C is of genus 0. Moreover the graph (Y, v) of (C, S) satisfies the following conditions.

Lemma (1.3). *If* (Y, v) *is the graph of a most degenerate stable* n*-pointed curve* (C, S) *of genus* g. *Then*

$$\#(Vert(Y)) = 2(g-1) + n; \quad \#(Edge(Y)_{geom}) = \frac{1}{2}\#(Edge(Y)) = 3(g-1) + n.$$

In particular,

$$h^1(Y) = \#(Edge(Y)_{geom}) - \#(Vert(Y)) + 1 = g.$$

For each $P \in Vert\ Y$,

$$n_P = 3 - \#\{\ y \in Edge(Y) \mid t(y) = P\ \}.$$

Since Y *is connected,* $n_P = 0$, *1, or 2. When* $n_P = 2$, P *is a terminal point of* Y.

This is very easy to prove and more or less well-known. We omit a proof.

Example. For $g = 2$ and $n = 1$, there are three types of the graphs of most degenerate stable n-pointed curves. The pictures are the following

(*a*) (*b*) (*c*)

Here the assigned number denotes n_P for each P if $n_P > 0$.

1.2 *Weight filtration on the fundamental groups and induced filtration on the automorphism groups.*

A group isomorphic to the fundamental group of a compact Riemann surface of genus g is called a surface group of genus g. The fundamental group of an n-punctured Riemann surface is a free group, if $n > 0$. On these groups, we can define the weight filtration in the following way.

1.2.1 *The weight filtration.*
Let π_1 be the surface group of genus g. Then we can introduce the weight filtration $\{W_{-m}(\pi_1)\}_{m\geq 1}$ on it, by the lower central series

$$W_{-m}(\pi_1) = \Gamma_m \pi_1 \text{ for each m} \geq 1.$$

Here the higher commutators $\Gamma_m \pi_1$ are defined inductively by

$$\Gamma_1 \pi_1 = \pi_1, \text{ and } \Gamma_{m+1}\pi_1 = [\Gamma_m \pi_1, \pi_1] \text{ for each } m \geq 1.$$

The case of the fundamental group of a punctured Riemann surface is slightly more complicated (*cf.* Kaneko [**K**]).

Let C be a compact Riemann surface of genus g, and S be a finite subset of C with cardinality n. Choose a base point $*$ in $C - S$. When n is arbitrary, the weight filtration on $\pi_1 = \pi_1(C - S, *)$ is defined as follows. Let N be the kernel of the canonical surjection

$$\pi_1(C - S, *) \to \pi_1(C, *)$$

which is a normal subgroup of π_1 generated by the homotopy classes which correspond to the puncture.

We set $W_{-1}(\pi_1) = \pi_1$ the whole group, and $W_{-2}(\pi_1) = [\pi_1, \pi_1]N$. Then the weight filtration $\{W_{-n}(\pi_1)\}_{n\geq 1}$ is defined as the fastest decreasing central filtration.

Note that the quotient group $\pi_1(C - S, *)/W_{-1}(\pi_1)$ is isomorphic to the 1-st homology group $H_1(C, \mathbb{Z})$.

1.2.2 *The induced filtration.*
Now we consider the induced filtration on the outer automorphism group of π_1 and its subgroup.

Let $Aut_S\ \pi_1$ be the subgroup of the automorphism group $Aut\ \pi_1(C-S, *)$ consisting elements which preserve the normal subgroup N. Also by $Aut_S^+\pi_1$ the subgroup of $Aut_S\ \pi_1$ given as the kernel of the composition of the canonical homomorphisms

$$Aut_S\ \pi_1(C - S, *) \to Aut\pi_1(C) \to Aut\ H_2(\pi_1(C), \mathbb{Z}).$$

When $g = 0$, $Aut_S^+\pi_1 = Aut_S\ \pi_1$, and when $g > 1$, $Aut_S^+\pi_1$ is an index 2 subgroup of $Aut_S\ \pi_1$.

Notation 1.1 We denote by $\tilde{\Gamma}_{g,n}$ the group $Aut_S^+\pi_1$, and by $\Gamma_{g,n}$ the group $Out_S^+\pi_1$.

Remark 1.2
By a classical theorem of Nielsen, $\Gamma_{g,n}$ is isomorphic to a mapping class group or a Teichmüller group (cf. [**ZVC**], §§5.7).

The weight filtration on $\pi_1(C-S,*)$ canonically induces a filtration on $\tilde{\Gamma}_{g,n}$ by

$$\tilde{\Gamma}_{g,n}[k] = \{\sigma \in \tilde{\Gamma}_{g,n} | \text{ for any } l \geq 1, \text{ and any } \; x \in W_{-l}(\pi_1),\ \sigma(x)x^{-1} \in W_{-k-l}(\pi_1)\}.$$

Passing to the quotient $\Gamma_{g,n} = \tilde{\Gamma}_{g,n}/Inn(\pi_1(C-S,*))$, we can define the induced filtration on $\Gamma_{g,n}$, by the image of the canonical homomorphism:

$$\Gamma_{g,n}[k] = Image(\tilde{\Gamma}_{g,n}[k] \to \Gamma_{g,n})$$

for each k. Then we have the following

Proposition (1.4).
(1) $\Gamma_{g,n}[0] = \Gamma_{g,n}$, *and*

$$[\Gamma_{g,n}[k], \Gamma_{g,n}[l]] \subset \Gamma_{g,n}[k+l] \textit{ for any } k,\ l \geq 0;$$

(2) *The quotient* $\Gamma_{g,n}/\Gamma_{g,n}[1]$ *is isomorphic to the Siegel modular group* $Sp(g;\mathbb{Z})$*;*
(3) *For any* m $(m \geq 1)$*, the quotient group* $\Gamma_{g,n}[m]/\Gamma_{g,n}[m+1]$ *is a free abelian group of finite rank.*

Proof. The statements (1) and (2) are well-known. When $n=0$, (3) is proved by Asada [**A**]. In the case of $n>0$, a pro-l analogy is proved by Kaneko [**K**]. Although the discrete case can be treated almost in the same way, we shall give a proof for the sake of completeness. Also the case of $m=2$ is not explicitly stated in [**K**].

For simplicity, we write $\tilde{\Gamma}$ and Γ instead of $\tilde{\Gamma}_{g,n}$ and $\Gamma_{g,n}$, respectively. And for each $m \geq 0$, we write $\tilde{\Gamma}[m]$ and $\Gamma[m]$ for $\tilde{\Gamma}_{g,n}[m]$ and $\Gamma_{g,n}[m]$, respectively. We write $gr_m(\pi_1) = W_{-m}(\pi_1)/W_{-m-1}(\pi_1)$ for each $m \geq 1$.

First, we define a group homomorphism

$$\tilde{h}_m : \tilde{\Gamma}[m]/\tilde{\Gamma}[m+1] \to gr_{m+1}(\pi_1)^{\oplus 2g} \times gr_m(\pi_1)^{\oplus(n-1)}$$

as follows. For $\sigma \in \tilde{\Gamma}$, put $s_i(\sigma) = \sigma(\alpha_i)\alpha_i^{-1}$, $s_{g+i}(\sigma) = \sigma(\beta_i)\beta_i^{-1}$ $(1 \leq i \leq g)$, and let t_j be an element of π_1 such that $\sigma(\gamma_j) = t_j\gamma_j t_j^{-1}$ $(1 \leq j \leq n-1)$. Since π_1 is a free group of rank $2g+n-1>1$, the centralizer of γ_j is an infinite cyclic group generated by γ_j. Hence, if $m \neq 2$, t_j is uniquely determined. If $m=2$, we normalize t_j as follows. Since $gr_2(\pi_1)$ is a free $\mathbb{Z}$-module with a basis

$$\begin{aligned}
&[\alpha_i, \alpha_j],\ [\beta_i, \beta_j] \quad (1 \leq i < j \leq g);\\
&[\alpha_i, \beta_j] \quad (1 \leq i, j \leq g,\ (i,j) \neq (g,g)); \quad \gamma_j \quad (1 \leq j \leq n-1),
\end{aligned}$$

we can normalize t_j uniquely in such a way that the coefficients of γ_j is 0 when $\{t_j \bmod W_{-3}(\pi_1)\}$ is expressed as a $\mathbb{Z}$-linear combination of this basis. Now, for $\sigma \in \tilde{\Gamma}[m]$, we define

$$\tilde{h}_m(\bar{\sigma}) = (s_i(\sigma) \bmod W_{-m-2}(\pi_1))_{1\leq i\leq 2g} \times (t_j(\sigma) \bmod W_{-m-1}(\pi_1))_{1\leq j\leq n-1}$$

($\bar{\sigma}$ denotes the class of σ). The fact that $\tilde{\Gamma}[m]$ acts trivially on $gr_{m+1}(\pi_1)$ and the formula

$$s_i(\sigma\tau) = \tau(s_i(\sigma))s_i(\tau) \qquad \sigma,\ \tau \in \tilde{\Gamma}$$

implies that $\tilde{h}_m$ is a homomorphism. Obviously, $\tilde{h}_m$ is injective.

For each positive integer m, set

$$Int_{\pi_1}(W_{-m}(\pi_1)) = \{\ \sigma \in Int(\pi_1) \mid \sigma = Int(g) \text{ with } g \in W_{-m}(\pi_1)\ \}.$$

Here $Int(g)$ is the inner automorphism of π_1 induced from the transform by g : $Int(g)(x) = gxg^{-1}$ ($x \in \pi_1$). Let us consider the following two homomorphisms:

$$\begin{aligned} \iota : gr_m(\pi_1) &\to Int_{\pi_1}(W_{-m}(\pi_1))/Int_{\pi_1}(W_{-m-1}(\pi_1)); \\ \bar{t} &\to \text{ the class of } Int(t) \end{aligned}$$

$$\begin{aligned} h : gr_m(\pi_1) &\to (gr_{m+1}(\pi_1))^{\oplus 2g} \times (gr_m(\pi_1))^{\oplus(n-1)}. \\ \bar{t} &\to (\overline{[t, x_i]})_{1\leq i\leq 2g} \times (\bar{t}_j)_{1\leq j\leq n-1} \end{aligned}$$

Then, since the Lie algebra $gr^W(\pi_1) = \oplus_{m=1}^{\infty} gr_m(\pi_1)$ has trivial center, it follows that ι is an isomorphism, h is injective, and

$$\tilde{\Gamma}[m] \cap Int(\pi_1) = Int_{\pi_1}(W_{-m}(\pi_1)) \qquad \text{for all } m \geq 1$$

[**A**, Lemma 4]. Hence we have the following commutative diagram:

$$\begin{array}{c} 0 \to Int_{\pi_1}(W_{-m}\pi_1)/Int_{\pi_1}(W_{-m-1}\pi_1) \to \tilde{\Gamma}[m]/\tilde{\Gamma}[m+1] \to \Gamma[m]/\Gamma[m+1] \to 0 (\text{exact}) \\ \iota \uparrow \qquad\qquad\qquad\qquad \downarrow \tilde{h}_m \\ gr_m(\pi_1) \xrightarrow{h} (gr_{m+1}(\pi_1)^{\oplus 2g} \times (gr_m(\pi_1))^{\oplus(n-1)}. \end{array}$$

Since $\tilde{h}_m$ is injective, to prove Proposition, it suffices to show that the cokernel of h is a free $\mathbb{Z}$-module of finite rank. Now, $gr_m(\pi_1)$ and $gr_{m+1}(\pi_1)$ are both free $\mathbb{Z}$-module of finite rank, h is injective, and $h \otimes_{\mathbb{Z}} \mathbb{F}_p$ is also injective for all prime number p since $gr^W(\pi_1) \otimes_{\mathbb{Z}} \mathbb{F}_p$ has trivial center. Therefore, by Lemma 4 in [**A**], the cokernel of h is a free $\mathbb{Z}$-module of finite rank. (q.e.d)

1.3 *The non-abelian monodromy homomorphism.*

Let $(C_0, S_0) = (C, S)$ be a most degenerate stable n-pointed curve of genus g with the graph Y. Consider the local universal deformation of (C_0, S_0). For each geometric edge $e = |y|$ $(y \in Edge(Y))$, let

$$u_e v_e = 0 \qquad (\text{ in } (u_e, v_e) \in \mathbb{C}^2)$$

be the local defining equation of the singularity associated with e. Let

$$u_e v_e = t_e \qquad (\text{ in } (u_e, v_e, t_e) \in \mathbb{C}^3)$$

be the local universal deformation of the above singularity [**DM**, §1] [**Kn**, §2]. For each e, we can associate a small complex disk $\mathcal{D}_e = \{ t_e \in \mathbb{C} \mid |t_e| < \varepsilon \}$. Then over the polydisk $\mathcal{D} = \prod_{e \in Edge(Y)_{geom}} \mathcal{D}_e$, we have a local universal family

$$f : \mathcal{C} \to \mathcal{D}, \qquad \mathcal{S} : \{1, \ldots, n\} \times \mathcal{D} \to \mathcal{C}.$$

If $\mathbf{t} = (t_e)_{e \in Edge(Y)_{geom}}$ satisfies $t_e \neq 0$ for any $e \in Edge(Y)_{geom}$, the fiber $f^{-1}(\mathbf{t}) = C_{\mathbf{t}}$ is a smooth proper curve of genus g, and $\mathcal{S}(\mathbf{t}) = S_{\mathbf{t}}$ is a set of n distinct points on $C_{\mathbf{t}}$. Let $\mathcal{D}^0$ be the open subset of $\mathcal{D}$ consisting of such points. Choose such a point $\mathbf{t}_0$ in $\mathcal{D}^0$. Let

$$\pi_1(C_{\mathbf{t}_0} - S_{\mathbf{t}_0}, *)$$

be the fundamental group of the n-punctured Riemann surface $C_{\mathbf{t}_0} - S_{\mathbf{t}_0}$ with a base point $*$. Then we have the non-abelian monodromy homomorphism

$$\rho_{(C_0,S_0)} : \pi_1(\mathcal{D}^0, \mathbf{t}_0) \cong \mathbb{Z}^{3g-3+n} \to Out\ \pi_1(C_{\mathbf{t}_0} - S_{\mathbf{t}_0}, *).$$

By using a transcendental result, we can assure that the monodromy homomorphism $\rho_{(C_0,S_0)}$ is injective [**BLM**].

Now we want to see the fact that this monodromy homomorphism is compatible with the weight filtration. In fact,

Proposition (1.5). *The monodromy homomorphism $\rho_{(C_0,S_0)}$ preserves the weight filtration on $\pi_1(C_{\mathbf{t}_0} - S_{\mathbf{t}_0}, *)$. In particular, for any σ of $\pi_1(\mathcal{D}^0, \mathbf{t}_0)$, we have $\sigma(N) = N$, where N is the kernel of the canonical surjective homomorphism*

$$\pi_1(C_{\mathbf{t}_0} - S_{\mathbf{t}_0}, *) \to \pi_1(C_{\mathbf{t}_0}, *).$$

The proof of above proposition is given in Subsection 3.4.

1.4 *Bridges, cut systems, and invariants s_1, s_2 in a graph.*

Definition 1.4

(1) An edge y is called a *bridge* , if the subgraph $Y - \{|y|\}$ is not connected.
(2) A pair $\{|y_1|, |y_2|\}$ of geometric edges is called a *cut pair*, if neither $|y_1|$ nor $|y_2|$ is a bridge, and the subgraph $Y - \{|y_1| \cup |y_2|\}$ is not connected.

The following is easy to prove.

Lemma (1.6). *Let $\{|y_1|, |y_2|\}$ be a cut pair, and $\{|y_2|, |y_3|\}$ $(|y_3| \neq |y_1|)$ be another cut pair. Then $\{|y_1|, |y_3|\}$ is also a cut pair.*

Definition 1.5. We call a set E of geometric edges a *maximal cut system*, if

(1) it contains at least two distinct geometric edges;
(2) any pair of two distinct geometric edges $|y|$, $|y'|$ in E is a cut pair;
(3) and no edge y'' outside E makes a cut pair with an edge in E.

Now we define two invariants of a graph Y which is used to describe the main result of this paper.

Definition 1.6

(1) Let $s_2(Y)$ be the number of bridges in the graph Y.
(2) Put $s_1(Y) = \sum_E \{|E| - 1\}$, where E runs over the maximal cut systems in Y.

1.5 *Main results.*

Let (Y, v) be a graph of a most degenerate n-pointed stable curve of genus g. Recall the monodromy homomorphism $\rho_{(C_0,S_0)}$ in Subsection 1.3.

Definition 1.7 We denote by I_Y the image of the monodromy homomorphism

$$\rho_{(C_0,S_0)} : \pi_1(\mathcal{D}^0, \mathbf{t}_0) \cong \mathbb{Z}^{3g-3+n} \to Out\ \pi_1(C_{\mathbf{t}_0} - S_{\mathbf{t}_0}, *).$$

in $\Gamma_{g,n} = Out_S^+(\pi_1)$.

Since $\rho_{(C_0,S_0)}$ is injective, I_Y is a free abelian subgroup of rank $3g - 3 + n$. Let $I_Y^{(m)} = I_Y \cap \Gamma_{g,n}[m]$ for each $m \geq 1$, and define the numbers $\{r_m(Y)\}_{m\geq 0}$ by

$$r_m(Y) = rank_{\mathbb{Z}} I_Y^{(m)}/I_Y^{(m+1)} \qquad \text{for each } m \geq 0.$$

Note here each $I_Y^{(m)}/I_Y^{(m+1)} \subset \Gamma_{g,n}[m]/\Gamma_{g,n}[m+1]$ is a free abelian group of finite rank by Proposition 1.4, if $m \geq 1$. We will see later that $I_Y^{(0)}/I_Y^{(1)}$ is also a free $\mathbb{Z}$-module (Subsection *.*).

Here is the main result of this paper.

Theorem (1.7). *Let Y be an associated graph with a most degenerate stable pointed curve of type (g, n). Then*

(1) $r_0(Y) = 3g - 3 + n - s_1(Y) - s_2(Y)$;
(2) $r_1(Y) = s_1(Y)$;
(3) $r_2(Y) = s_2(Y)$;
(4) $I_Y^{(3)} = \{0\}$.

Remark 1.3 The first statement (1) is due to Brylinski [**Br**, Prop. 5].

Corollary (1.8).

(1) *When $n = 0$, the naturally induced homomorphism*

$$\rho_{(C_0,S_0)}(mod\ 3) : \pi_1(\mathcal{D}^0, \mathbf{t}_0) \cong \mathbb{Z}^{3g-3+n} \to Out(\pi_1(C_{\mathbf{t}_0} - S_{\mathbf{t}_0}, *)/W_{-4}\pi_1)$$

is injective;

(2) *When $n > 0$, the homomorphism*

$$\rho_{(C_0,S_0)}(mod\ 4) : \pi_1(\mathcal{D}^0, \mathbf{t}_0) \cong \mathbb{Z}^{3g-3+n} \to Out(\pi_1(C_{\mathbf{t}_0} - S_{\mathbf{t}_0}, *)/W_{-5}\pi_1)$$

is injective.

2. Graph of groups and edge twists.

In this section, we give a preparatory result for a combinatorial description of Dehn twist. In the next section, we specialize the results of this section to the graph of surface groups, and apply them to describe the local monodromy for the fundamental group associated with a given degenerate stable n-pointed curve. We recall the basic contents of Serre's book [**S**] in Subsection 2.1. The notion of edge twist does not seem to be found in the literature.

As in the previous section, Y denotes a connected non-empty graph, with oriented edges. For each $y \in Edge(Y)$, $\bar{y} \in Edge(Y)$ is the inverse edge of y, $o(y)$ and $t(y)$ are the origin and the terminus of y, respectively.

2.1 *The fundamental groupoid of a graph of groups.*

Definition 2.1 (graph of groups)

A graph of groups (G, Y) is

(1) a group G_P assigned for each vertex $P \in Vert(Y)$.

(2) a group G_y assigned for each edge $y \in Edge(Y)$, with a monomorphism

$$G_y \to G_{t(y)}, \quad \text{denoted by } a \to a^y.$$

We impose $G_y = G_{\bar{y}}$ for any $y \in Edge(Y)$.

Let (G, Y) be a graph of groups. Then Serre [**S**,§5] defines an auxiliary group $F(G, Y)$. Let us recall its definition. Let F_Y be the free group generated over $Edge(Y)$. Then $F(G, Y)$ is the quotient group of the free product

$$F_Y * (\mathop{*}_{P \in Vert(Y)} G_P)$$

by the subgroup normally generated by the relations:

$$y\bar{y} = 1 \quad (y \in Y); \qquad y a^y \bar{y} = a^{\bar{y}}, \text{ for } y \in Edge(Y),\ a \in G_y.$$

Here $*$ is the product symbol for free product.

Words of $F(G, Y)$.

Let c be a path in Y whose origin is a vertex P_0. We let $y_1, \ldots, y_n$ denote the edges of c, where $n = l(c)$ is the length of c, and put

$$P_i = o(y_{i+1}) = t(y_i).$$

Definition 2.2 A word of type c in $F(G, Y)$ is a pair (c, μ) where $\mu = (r_0, \ldots, r_n)$ is a sequence of elements $r_i \in G_{P_i}$. The element

$$|c, \mu| = r_0 y_1 r_1 y_2 \ldots y_n r_n \text{ of } F(G, Y)$$

is said to be associated with the word (c, μ). When $n = 0$, we have $|c, \mu| = r_0$. An element of $F_Y * (\mathop{*}_{P \in Vert(Y)} G_P)$ is *admissible* if it has the form of $|c, \mu|$ for some c, μ. One says that (c, μ) is *reduced* if it satisfies the following condition: if $n = 0$ then one has $r_0 \neq 1$; if $n \geq 1$ then one has $r_i \notin G_{y_i}^{y_i}$ for each index i such that $y_{i+1} = \bar{y}_i$, where $G_{y_i}^{y_i}$ denotes the image of the monomorphism $G_{y_i} \to G_{t(y_i)}$.

Fundamental groupoid.

Let us consider composable paths c_1, c_2, i.e. $t(c_1) = o(c_2)$. Let $c_1 * c_2$ be the concatenation of c_1 and c_2. Two words $(c_1, \mu_1), (c_2, \mu_2)$ are said composable, if c_1, c_2 are composable. We define the concatenation $(c_1, \mu_1) * (c_2, \mu_2)$ by

$$(c_1 * c_2, \mu_1 * \mu_2), \text{ where } l(c_1)\text{-th element in } \mu_1 * \mu_2 \text{ is given by } r_n^{(1)} r_0^{(2)} \in G_{t(c_1)}.$$

We write $\pi_1(G, Y; P_0, P_1)$ for the set of elements of $F(G, Y)$ of the form $|c, \mu|$ with $o(c) = P_0$, $t(c) = P_1$. The sets $\{ \pi_1(G, Y, ; P_0, P_1) \mid P_0, P_1 \in Vert(Y) \}$ form a groupoid. In particular,

$$\pi_1(G, Y; P_0, P_0) = \pi_1(G, Y; P_0)$$

is the fundamental group of the graph of groups (G, Y) with the base point P_0.

Another realization of the fundamental group

Let us recall another realization of the fundamental group of a graph of groups, i.e. realization as a quotient group of the ambient group $F(G, Y)$.

Let us choose a spanning (or maximal) tree T in Y. Then we define the group $\pi_1(G, Y, T)$ as the quotient group of $F(G, Y)$ by the subgroup normally generated by the elements

$$y \quad (y \in Edge(T)).$$

It is shown in Serre[S] (Chap. I, §5, Prop. 20) that this group is isomorphic to the fundamental group $\pi_1(G, Y, P_0)$ by the composition of the canonical homomorphisms

$$\pi_1(G, Y, P_0) \to F(G, Y) \to \pi_1(G, Y, T).$$

2.2 *Edge Twist.*

We choose an edge $y \in Edge(Y)$, and an element d in the center $Z(G_y)$ of the group G_y. Let $D_{y,d}$ be the endomorphism of $F_Y * (\underset{P \in Vert(Y)}{*} G_P)$ defined by

$$\begin{array}{ll} D_{y,d}(y) = y d^y, & D_{y,d}(\bar{y}) = \bar{y}(d^{\bar{y}})^{-1}, \\ D_{y,d}(y') = y' & \text{for other edges } y' \notin \{y, \bar{y}\} \end{array}$$

and

$$D_{y,d}(x) = x \qquad \text{for any element } x \in \underset{P \in Vert(Y)}{*} G_P.$$

Then, since $D_{y,d} D_{\bar{y},d} = 1$, $D_{y,d}$ is an automorphism of $F_Y * (\underset{P \in Vert(Y)}{*} G_P)$.

Lemma (2.2). *$D_{y,d}$ induces an automorphism of $F(G,Y)$.*

Proof. We have to check that the defining relation is preserved under the map $D_{y,d}$. In fact, the relation $y\bar{y} = 1$ is mapped to

$$yd^y\bar{y}(d^{\bar{y}})^{-1} = \{yd^y\bar{y}\}(d^{\bar{y}})^{-1} = d^{\bar{y}}(d^{\bar{y}})^{-1} = 1.$$

Also $ya^y\bar{y} = a^{\bar{y}}$ is mapped to

$$(yd^y)a^y(\bar{y}d^{\bar{y}}) = y(da)^y\bar{y}(d^{\bar{y}})^{-1} = (da)^{\bar{y}}(d^{\bar{y}})^{-1} = (dad^{-1})^{\bar{y}},$$

and since d belongs to the center of G_y, $dad^{-1} = a$. Here we use the assumption that d belongs to the center of G_y. Since the other relators are preserved trivially by $D_{y,d}$, this settles the proof of our proposition.

Definition 2.3 By an abuse of notation, we denote by the same symbol $D_{y,d}$ the automorphism of $F(G,Y)$ induced from $D_{y,d} \in Aut(F_Y * (\mathop{*}_{P \in Vert(Y)} G_P))$, and call it the *edge twist* associated with (y,d).

The following is immediate from the above lemma.

Proposition (2.3).

(1) *The automorphism $D_{y,d}(w)$ induces a bijection*

$$\pi_1(G,Y;P_0,P_1) \tilde{\to} \pi_1(G,Y;P_0,P_1)$$

for each P_0 and P_1, compatible with composition of groupoid. In other words, $D_{y,d}$ defines an automorphism of the fundamental groupoid of (G,Y). In particular, $D_{y,d}$ defines an automorphism of the fundamental group $\pi_1(G,Y;P_0)$.

(2) *For any pair $d \in Z(G_y)$ and $d' \in Z(G_{y'})$, the twists $D_{y,d}$ and $D_{y',d'}$ commute.*

Thus we can define a homomorphism

$$\prod_{y \in Edge(Y)_+} Z(G_y) \to Aut\ \pi_1(G,Y;P_0).$$

Applying the above construction to a graph of surface groups, we can obtain an algebraic description of Dehn twists in the next section.

3. Non-abelian Picard-Lefschetz formula.

3.1 *Graph of surface groups.*

For each graph of a stable n-pointed curve of genus g, we can assign a graph of groups naturally, and recover the fundamental group of an n-punctured Riemann surface of genus g as the fundamental group of the graph of groups.

Let (Y, v) be the graph of a stable n-pointed curve of genus g. For such a graph, we consider the following more specialized version of graph of groups.

Definition 3.1 (graph of surface groups)

(1) For each vertex P, G_P is the fundamental group of $C_P - C_P \cap (C_{sing} \cup S)$.

(2) For each edge y, G_y is an infinite cyclic group with an assigned generator ι_y.

We put $\iota_{\bar{y}} = \iota_y^{-1}$. The monomorphism

$$G_y \to G_{t(y)}$$

is defined by mapping ι_y to x in $G_{t(y)}$ which is free-homotopically equivalent to a closed path encircling the deleted point q_y in counter-clockwise.

Choose one vertex P of Y. If $v(P) = (g_P, n_P)$ and let Y_P be the subset of edges y in Y such that $t(y) = P$. We fix some order on the set Y_P. Then the group G_P has a presentation:

$$< \alpha_1, \beta_1, \dots, \alpha_{g_P}, \beta_{g_P}, \gamma_1, \dots, \gamma_{n_P}, \gamma_y \ (y \in Y_P) \mid \\ [\alpha_1, \beta_1] \cdots [\alpha_{g_P}, \beta_{g_P}] \gamma_1 \cdots \gamma_{n_P} \prod_{y \in Y_P} \gamma_y = 1 > .$$

For $y \in Y_P$, the image of the generator ι_y of G_y is an element γ'_y which is conjugate to γ_y in $G_{t(y)} = G_P$.

Remark 3.1 In the above definition of graph of surface groups, the choice of $\iota_y \mapsto x \in G_{t(y)}$ has ambiguity, since only the conjugacy class of x is specified. However, this ambiguity does not affect the definition in the following sense.

Let (G, Y) be a graph of groups. Let (G, Y') be a graph of groups obtained from (G, Y) by "changing the choice of x in the same conjugacy class in $G_{t(y)}$". Then, there is an isomorphism between $F(G, Y)$ and $F(G, Y')$ compatible with edge twists.

To be precise, let us fix $s_y \in G_{t(y)}$ for each $y \in \mathrm{Edge}(Y)$. Let (G, Y') be the graph of groups defined as follows. The graph Y' is isomorphic to Y, with $\mathrm{Vert}(Y) = \mathrm{Vert}(Y')$ and $\mathrm{Edge}(Y) \cong \mathrm{Edge}(Y')$ by $y \mapsto y'$. The groups G_P on $P \in \mathrm{Vert}(Y')$ are identical with the ones in (G, Y), and $G_{y'} = G_y$. We define the monomorphisms $G_{y'} \to G_{t(y')}$ by

$$a \mapsto a^{y'} := s_y a^y s_y^{-1}.$$

The isomorphism $F(G,Y) \to F(G,Y')$ is defined on generators by $g \mapsto g$ for $g \in G_P$ and

$$y \mapsto s_{\bar{y}}^{-1} y' s_y$$

for $y \in \mathrm{Edge}(Y)$. Then relators are mapped as

$$y\bar{y} = 1 \mapsto s_{\bar{y}}^{-1} y' s_y s_y^{-1} \bar{y}' s_{\bar{y}} = 1,$$

and

$$y a^y \bar{y} \mapsto s_{\bar{y}}^{-1} y' s_y a^y s_y^{-1} \bar{y}' s_{\bar{y}} = s_{\bar{y}}^{-1} y' a^{y'} \bar{y}' s_{\bar{y}} = s_{\bar{y}}^{-1} a^{\bar{y}'} s_{\bar{y}} = a^{\bar{y}}.$$

This isomorphism is compatible with $D_{y,d} \mapsto D_{y',d}$, since we have

$$D_{y,d}(y) = y d^y \mapsto s_{\bar{y}}^{-1} y' s_y d^y = s_{\bar{y}}^{-1} y' d^{y'} s_y = D_{y',d}(s_{\bar{y}}^{-1} y' s_y)$$

and

$$D_{y,d}(\bar{y}) = \bar{y}(d^{\bar{y}})^{-1} \mapsto s_y^{-1} \bar{y}' s_{\bar{y}} (d^{\bar{y}})^{-1} = s_y^{-1} \bar{y}' s_{\bar{y}} (s_{\bar{y}}^{-1} d^{\bar{y}'} s_{\bar{y}})^{-1} = D_{y',d}(s_y^{-1} \bar{y}' s_{\bar{y}}).$$

Hence, we do not specify the image of ι_y but specify its conjugacy class only.

3.2 *Recovery of surface groups, or Seifert-van Kampen theorem.*

In this section, we confirm that the fundamental group of a graph of surface groups gives the fundamental group of the generic punctured Riemann surface.

Theorem (3.1). *(Seifert-van Kampen) Let (G,Y) be a graph of surface groups of a stable n-pointed curve of genus g. Then the fundamental group of (G,Y) is isomorphic to the fundamental group of an n-punctured Riemann surface of genus g.*

Remark 3.2 Moreover, we can describe an algorithm to obtain a canonical system of generators. The algorithmic part of the above theorem is discussed in the next section.

Proof. For each vertex P of Y, let C_P^* be a closed subset of the puncture Riemann surface $C_P - C_P \cap S$, obtained from $C_P - C_P \cap S$ by deleting a very small open disk D_x around each point x in $C_P \cap C_{sing}$. Then the Riemann surface with boundary C_P^* is a deformation retract of $C_P - C_P \cap (C_{sing} \cup S)$. Hence $\pi_1(C_P^*, b) \cong G_P$, with b being a base point in C_P^*. The union $\bigcup_{x \in C_P \cap C_{sing}} \partial \bar{D}_x$ is the boundary of C_P^*, where $\bar{D}_x$ is the closure of D_x, and $\partial \bar{D}_x$ its boundary.

Let I be the unit interval $[0,1]$ and S_1 the 1-dimensional circle. Put $A_y = S_1 \times I$ for each edge y, and identify it with $A_{\bar{y}}$ via a mapping $(\theta, t) \to (\theta, 1-t)$ $(\theta \in S_1,\ t \in I)$. Fix an orientation on $S_1 \times I$, and induce it to A_y.

Consider the disjoint union $(\bigcup_{P \in Vert(Y)} C_p^*) \cup (\bigcup_{|y| \in Edge(Y)_{geom}} A_y)$, and patch each boundary $\{(\theta, 1) | \theta \in S_1\}$ of A_y with $\partial \bar{D}_x$ such that the orientation of A_y and

C_p^* are compatible. Then we obtain a Riemann surface R with no boundary of genus g and n punctures.

We have to show $\pi_1(R,*) \cong \pi_1(G,Y,P)$ which is nothing but a variant of van Kampen theorem. Since we could not find a good reference, we discuss how to reduce our claim to a simpler well-known case.

Choose a maximal tree T in Y, and consider the surface R_T which is the image of $(\bigcup_{P\in Vert(Y)} C_p^*) \cup (\bigcup_{|y|\in \mathrm{Edge}(T)_{geom}} A_y)$, in R with respect to the natural map. Let $G_{|T}$ be the restriction of G to T. Then the usual van Kampen theorem implies $G_T = lim_{\rightarrow}(G_{|T},T)$ is isomorphic to $\pi_1(R_T,*)$.

Let $Y' = Y/T$ be the graph obtained from Y by contracting every edges in T to a point. Then Y' is a graph with a unique vertex P'. Define a function v' on $Vert(Y')$ by $v'(P') = (g,n)$. Then setting $G_{P'} = G_T \cong \pi_1(R_T,*)$, we obtain a graph of surface groups (G',Y').

By the definition of the fundamental group of a graph of groups, it is easy to check that there is a canonical isomorphism $\pi_1(G,Y,P) \cong \pi_1(G',Y',P')$. The surface R is obtained from R_T by attaching g handles A_y ($|y| \in Edge(Y)_{geom} - \mathrm{Edge}(T)_{geom}$). Meanwhile $\pi_1(G',Y',P')$ is g times iterated HNN-extension of $G_{P'}$. It is well known that the isomorphism $G_{P'} \cong \pi_1(R_T,*)$ implies $\pi_1(G',Y',P') \cong \pi_1(R,*)$. (q.e.d)

3.3 *Non-abelian Picard-Lefschetz formula.*

Let Y be a graph of a stable n-pointed curve (C_0, S_0) of genus g. Then we consider the graph of surface groups G, naturally associated to Y:a free group of rank 2 with a set of three assigned generators for each vertex, and an infinite cyclic group for each edge.

There are n-generators corresponding to the n-assigned points in S. The fundamental group of (G, Y) is isomorphic to the fundamental group of an n-punctured Riemann surface of genus g. Then n-generators of assigned points are free-homotopically equivalent to the simple curves which bound small disks centered at n punctures, respectively.

Let P_0 be a vertex of Y, and $\pi_1(G, Y, P_0)$ be the fundamental group of the graph of groups with base point P_0. Then for each edge y of Y, we can associate the edge twist D_{y,ι_y}, where ι_y is a canonical generator of the free cyclic group G_y.

Remark 3.3 Let $\bar{y}$ be the inverse edge of y. Then we put $\iota_{\bar{y}} = \iota_y^{-1}$ with respect to the identification $G_{\bar{y}} = G_y$. Then we have $D_{y,\iota_y} = D_{\bar{y},\iota_{\bar{y}}}$.

Hence we may consider D_{y,ι_y} depends only on geometric edge $|y|$. Thus we denote it by $D_{|y|}$, and call it the edge twist associated to $|y|$. We also denote by the same symbol $D_{|y|}$ the induced element in $Out\ \pi_1(G, Y, P_0)$.

Let us consider the local universal deformation of (C_0, S_0) in the category of stable n-pointed curves $f : \mathcal{C} \to \mathcal{D}$, where the base space $\mathcal{D}$ is a $3g - 3 + n$ dimensional polydisk with coordinates $\{(t_i)\}_{1 \le i \le 3g-3+n}$. Moreover for the parameters t_i, we may assume that the first $\#(Edge(Y)_{geom})$-parameters are the parameters of the local universal deformation of the singularities on C_0.

Let $\mathcal{D}_e$ be the complex disk associated to a geometric edge e of Y with coordinates t_e. Put $\mathcal{D}_Y = \prod_{e \in \mathrm{Edge}(Y)_{geom}} \mathcal{D}_e$. Then $\mathcal{D}$ has a product decomposition $\mathcal{D}_Y \times \mathcal{D}'$ (non-canonical). Here $\mathcal{D}'$ is a polydisk of dimension $3g - 3 + n - \#(Edge(Y)_{geom})$. For each punctured disk $\mathcal{D}_e^0 = \{\, t \in \mathbb{C} |\ |t| < \varepsilon,\ t \neq 0\}$, we denote by γ_e the associated generator of $\pi_1(\mathcal{D}_e^0, t_{e0})$ $(t_{e0} \neq 0)$, which encircle the origin in counter-clockwise. Then for $\mathcal{D}_Y^0 = \prod_{e \in \mathrm{Edge}(Y)_{geom}} \mathcal{D}_e$, $\pi_1(\mathcal{D}_Y^0, \mathbf{t}_0')$ is generated by $\{\gamma_e \mid e \in Edge(Y)_{geom}\}$.

Let $\mathcal{D}^0$ be the open subset of $\mathcal{D}$ consisting of points whose fibers are smooth. Then $\mathcal{D}^0 = \mathcal{D}_Y^0 \times \mathcal{D}'$ and $\pi_1(\mathcal{D}^0, \mathbf{t}_0) \cong \oplus_{e \in Edge(Y)_{geom}} \mathbb{Z}$.

The following result plays a crucial role to reduce the proof of the main result to a combinatorial problem for graph of groups.

Theorem (3.2). *(non-abelian Picard-Lefschetz formula) We have a commutative diagram*

$$\begin{array}{ccc} \pi_1(\mathcal{D}^0, \mathbf{t}_0) & \xrightarrow{\rho} & Out\ \pi_1(\mathcal{C}_{\mathbf{t}_0} - \mathcal{S}_{\mathbf{t}_0}, *) \\ \downarrow & & \downarrow \\ \prod_{e \in Edge(Y_{geom})} \mathbb{Z}D_e & \longrightarrow & Out\ \pi_1(G, Y, P_0) \end{array}$$

Here the left vertical arrow is defined by mapping each γ_e to the corresponding edge twist D_e, and the right vertical arrow is induced from $\pi_1(\mathcal{C}_{\mathbf{t}_0} - \mathcal{S}_{\mathbf{t}_0}, *) \cong \pi_1(G, Y, P_0)$ obtained in the previous theorem, which is unique up to inner automorphisms.

Proof. Assume that $n = 0$, i.e. S_0 is empty. Then the proof is a generalization of Main Lemma (1.7) of the transcendental part of the previous paper [**O**].

Let $\pi : \tilde{C_0} \to C_0$ be the normalization of C_0. Then

$$\tilde{C_0} = \bigcup_{P \in \mathrm{Vert}(Y)} C_P \qquad \text{(disjoint)}$$

and we can number the singularities of C_0 by $\{p_e\}_{e \in \mathrm{Edge}(Y)_+}$.

Let $f : \mathcal{C} \to \mathcal{D}$ be the local universal deformation of C_0. Let $\{P, Q\}$ be two vertices of an edge e. Then using the parameter of deformation t_e of each double point p_e of C_0, the local defining equation of the smooth analytic space $\mathcal{C}$ at p_e is written as

$$u_{P,e} u_{Q,e} = t_e \qquad \text{in } (t_e, u_{P,e}, u_{Q,e}) \in \mathbb{C}^3$$

with certain local coordinates $u_{P,e}$ and $u_{Q,e}$. Moreover at $t_e = 0$, we may assume that $u_{P,e} = 0$ is the local defining equation of the component C_P at p_e, and $u_{Q,e} = 0$ the local defining equation of C_Q at p_e.

For each edge e, choose a sufficiently small positive real number ε_e. For any $\varepsilon \in (0, \varepsilon_e)$ we define a chart

$$U_e(\varepsilon) = \{(u_{P,e}, u_{Q,e}) \in \mathbb{C}^2; |u_{P,e}| < \varepsilon, |u_{Q,e}| < \varepsilon\}$$

of a neighbourhood of p_e in $\mathcal{C}$, which is identified with that neighbourhood in $\mathcal{C}$.

Let $\mathbf{t} = (t_e)_{e \in \mathrm{Edge}(Y)_+}$ be a point in $\mathcal{D}^0$. Set $\varepsilon = \varepsilon_e/2$ and put $A_{e,\mathbf{t}} = U_e(\varepsilon_e/2) \cap f^{-1}(\mathbf{t})$ for each $e \in \mathrm{Edge}(Y)_+$. Then each $A_{e,\mathbf{t}}$ is an annulus in the Riemann surface $C_{\mathbf{t}} = f^{-1}(\mathbf{t})$, and the complement $B_{\mathbf{t}} = C_{\mathbf{t}} - \bigcup_{e \in \mathrm{Edge}(Y)_+} A_{e,\mathbf{t}}$ consists of $\#(\mathrm{Vert}(Y))$ connected components, each of them corresponding to a unique vertex P of Y, and a deformation retract of $C_P^0 = C_P - \{\text{double point}\}$. We denote this component by $B_{P,\mathbf{t}}$ for each vertex $P \in V(Y)$.

Put

$$B^*_{P,\mathbf{t}} = B_{P,\mathbf{t}} \cup \bigcup_{e \in St(P)} \{(u_{P,e}, u_{Q,e}) \in U_e(\varepsilon_e/2) \cap C_{\mathbf{t}}; |u_{P,e}| \geq \eta\}$$

for a sufficiently small positive real number η, smaller than $|t_e|^{1/2}$ for each e. Here $St(P)$ is the set of edges with vertex P.

Then $B^*_{P,\mathbf{t}}$ has $\#(St(P))$ boundary components. The curve $C_{\mathbf{t}}$ is written as a union

$$C_{\mathbf{t}} = \bigcup_{P \in \mathrm{Vert}(Y)} B^*_{P,\mathbf{t}} \cup \bigcup_{e \in \mathrm{Edge}(Y)_{geom}} A_{e,\mathbf{t}}.$$

Each $B^*_{P,\mathbf{t}}$ is a deformation retract of $B_{P,\mathbf{t}}$, which is homotopically equivalent to C^0_P. Therefore, the C^∞-fibration $\cup_{\mathbf{t}\in\mathcal{D}^0} B^*_{P,\mathbf{t}} \to \mathcal{D}^0$ is homotopically equivalent to a product $pr_2 : C^0_P \times \mathcal{D}^0 \to \mathcal{D}^0$. Thus in order to describe the Deck transformation with respect to γ_e, it suffices to see its action on $A_{e,\mathbf{t}}$'s and the change of the patching condition with $B^*_{P,\mathbf{t}}$.

Choose a point $\mathbf{t}_0$. For each P, we choose a base point b_P in $B^*_{P,\mathbf{t}_0}$, and for each tube $A_{e,\mathbf{t}_0}$, we fix a base point b_e. When the vertex P is on the edge e, we connect the base points b_P and b_e by an oriented arc $c_{P,e}$ emanating from b_P. If we consider the graph with vertices b_P's and b_e's and with edges $c_{P,e}$, then this is canonically identified with the barycentric subdivision of the geometric graph Y_{geom}.

For each oriented edge y with $o(y) = P$ and $t(y) = Q$, we associate an oriented arc $c_y = c_{P,|y|}c^{-1}_{Q,|y|}$ starting from b_P and ending at b_Q.

Let us choose a vertex P_0 of Y and a base point b_P in $B^*_{P,\mathbf{t}_0}$. Then we can regard b_P as a point on $C_{\mathbf{t}_0}$. If we fix the arcs $c_{P,e}$ once for all, then we have a canonical isomorphism

$$\pi_1(C_{\mathbf{t}_0}, b_P) \cong \pi_1(G, Y, P_0).$$

Via the above isomorphism of the fundamental groups, any element of $\pi_1(C_{\mathbf{t}_0}, b_P)$ is written as a product

$$u_0 c_{y_1} u_1 c_{y_2} \dots u_{n-1} c_{y_n} u_n.$$

Here $y_1, \dots, y_n$ is a loop of the graph Y, such that

$$o(y_1) = t(y_n) = P_0; \qquad t(y_i) = o(y_{i+1}) \quad \text{for each } i \ (1 \le i \le n-1).$$

For each i $(0 \le i \le n)$, u_i is an element of $\pi_1(B^*_{P_i,\mathbf{t}_0}, b_{P_i})$, with $P_i = t(y_i)$ for $1 \le i \le n$.

Let $t_e = r_e e^{2\pi i\theta_e}$ be the polar coordinates of t_e for each geometric edge e of Y. We may assume that $\mathbf{t}_0 = (r_e)_{e\in \mathrm{Edge}(Y)_{geom}}$. Then by the relation $u_{P,e}u_{Q,e} = r_e e^{2\pi i\theta_e}$, $B^*_{P,\mathbf{t}}$ and $B^*_{Q,\mathbf{t}}$ are patched along the two annuli

$$\{u_{P,e} \in \mathbb{C} | \eta \le |u_{P,e}| \le \frac{r_e}{\eta}\} \text{ and } \{u_{Q,e} \in \mathbb{C} | \eta \le |u_{Q,e}| \le \frac{r_e}{\eta}\}$$

in $A_{e,\mathbf{t}}$. The increase of θ_e from 0 to 1 rotates the patching condition of two annuli. Hence the arc $c_y = c_{P,e}c^{-1}_{Q,e}$ is transformed to dc_y, where d is an element of $\pi_1(B^*_{P,\mathbf{t}_0}, b_P)$ which is free-homotopically equivalent to the generator of $\pi_1(A_{e,\mathbf{t}_0}, *)$.

Thus the proof is completed for the case $n = 0$.

Now let us discuss the general case. Let $f : \mathcal{C} \to \mathcal{D}$ and $s : \{1, 2, \dots, n\} \times \mathcal{D} \to \mathcal{C}$ be the local universal deformation of (C_0, S_0). Similarly to the case $n = 0$, we can define $A_{e,\mathbf{t}}$ and $B^*_{P,\mathbf{t}}$ for each edge e and vertex P. Define a subset $S_{\mathbf{t}} = \cup_{i=1}^n s(i, \mathbf{t})$ in $C_{\mathbf{t}}$ for each point $\mathbf{t}$. Then $B^*_{P,\mathbf{t}} - S_{\mathbf{t}}$ is homotopically equivalent to $C_P - C_P \cap S_0$. The rest of the proof proceeds completely similarly as the case $n = 0$. (q.e.d)

3.4 *Proof of Proposition (1.5).*

Since the weight filtration $\{W_{-m}(\pi_1)\}_{m\geq}$ is determined by N and the characteristic subgroups $\Gamma_m\pi_1$, it suffices to show that $\sigma(N) = N$ for any $\sigma \in Im\ \rho_{(C_0,S_0)} = I_Y$.

Let γ be an element in $\pi_1(C_{\mathbf{t}} - S_{\mathbf{t}}, *)$, free-homotopically equivalent to a small circle around a point $s \in S_{\mathbf{t}}$. Then via the isomorphism $\pi_1(C_{\mathbf{t}} - S_{\mathbf{t}}, *) \cong \pi_1(G, Y, P_0)$ of the previous subsection, γ is represented by an element which is conjugate to the image of some element γ'' in G_{P_1} corresponding to a puncture in the graph of groups (G, Y).

Therefore, there exists some path c from P_0 to P_1 such that γ is identified with $w\gamma''w^{-1}$ for some element $w = |c, \mu| \in \pi_1(G, Y; P_0, P_1)$. Then for any edge e, the twist D_e maps γ to its conjugation $D_e(w\gamma''w^{-1}) = D_e(w)\gamma''D_e(w)^{-1}$. This completes the proof of proposition (1.5), because N is normally generated by the elements of the form γ for various $s \in S_{\mathbf{t}}$.

4. An algorithm to compute Dehn twists and examples for the case of low genus.

The purpose of this section is twofold: one is to describe an algorithm to compute Dehn twists explicitly using the theorems of the previous section; another is to calculate some examples for the case when genus is 2 or 3, which also gives the starter of the inductive proof of the main result.

4.1 *Description of the algorithm.*

For simplicity, we consider the case when $n = 0$, and the curve C_0 is most degenerate. Then the graph Y is tri-valent. When Y is most degenerate, G_P is isomorphic to a free group of rank 2 for any $P \in Vert\, Y$. Let $y_1,\ y_2,\ y_3$ be the three edges such that $t(y_i) = P$. Corresponding to each y_i, we can consider the images $x_{P,y_i} = \iota_{y_i}^{y_i} \in G_P$. Changing x_{P,y_i} by its conjugate if necessary, we may assume that x_{P,y_i} satisfy the relation

$$x_{P,y_1} x_{P,y_2} x_{P,y_3} = 1.$$

Step 1 *Search of canonical generators.*

We choose a maximal tree T, and want to find a system of canonical generators in the surface group $\pi_1(G, Y, T)$ of genus g. We restrict the graph of surface groups G to T, and investigate the inductive limit $G_T = \lim_{\rightarrow}(G|_T, T)$ in the first place. We want to show that G_T is isomorphic to a free group of rank $2g - 1$. In order to prove the above fact by induction, we reformulate it for subtrees T' of T. Put $G_{T'} = \lim_{\rightarrow}(G|_{T'}, T')$.

Holes.

For each vertex $P \in Vert(T')$, we can consider $\{\ y \in Edge(Y) \mid t(y) = P\ \}$. We call the pair (P, y) *a hole.* The set of total holes of the graph Y is given by

$$\{\ (P, y) \mid P = t(y),\ P \in Vert(Y),\ y \in Edge(Y)\ \}$$

When $P \in Vert(T')$ and $y \notin Edge(T')$, then we call (P, y) is *an open hole* for T'. We denote by $h(T')$ the total number of open holes for T'. Then $h(T') = 3\#(Vert(T')) - \#(Edge(T')) = \#(Vert(T')) + 2$.

Lemma (4.1). *$G_{T'}$ is isomorphic to a free group of rank $h(T') - 1$. The generators are given by*

$$H(T') = \{\ x'_{P,y} \mid (P, y) \text{ open hole for } T'\ \}$$

with a relation

$$\prod_{(P,y)\in H(T')} x'_{P,y} = 1, \tag{4.1.1}$$

where the order of the product is considered appropriately. Here $x'_{P,y}$ are the images of $x_{P,y}$ via $G_P \to G_{T'}$.

The order of generators in the relation.

If one wants to specify the order of the product of (4.1.1), we can do it as follows. For each open hole (P, y), we can associate a dummy vertex $Q_{(P,y)}$ and an edge connecting P and $Q_{(P,y)}$. Let $\tilde{T}'$ be the extended tree. Then we can embed $\tilde{T}'$ in an oriented plane Π, so that the orientation of Π is compatible with the order (P, y_1), (P, y_2), (P, y_3) of three holes of P. Namely, the direction of the edges y_1, y_2, y_3 changes in a counter-clockwise for the orientation on Π for each P.

Tree-traversal search.

Let us start from a vertex P_0, and choose an edge y with $o(y) = P_0$.

(Case 1) If $(P_0, \bar{y})$ is not an open hole, we move to the adjacent vertex $P_1 = t(y)$. Write $y' = y$.

(Case 2) If $(P_0, \bar{y})$ is an open hole of T', we write $x_{P_0,\bar{y}}$ first in the product (4.1.1). Rotate the vector $\overrightarrow{o(y)t(y)}$ counter-clockwise with $o(y)$ fixed until to meet another edge y_1 with $o(y_1) = P_0$.

(Case 2-1) If $(P_0, \bar{y}_1)$ is also an open hole, then write $x_{P_0,\bar{y}_1}$ after $x_{P_0,\bar{y}}$ in the product (4.1.1). In this case, P_0 is a terminal vertex of T', and for the last edge y_2 with $o(y_2) = P_0$, the hole $(P_0, \bar{y}_2)$ is not open, unless T' consists of oee vertex P_0, the trivial case. We move to the adjacent vertex P_1 such that $t(y_2) = P_1$. Write $y' = y_2$.

(Case 2-2) If $(P_0, \bar{y}_1)$ is not an open hole, we set $P_1 = t(y_1)$, and write $y' = y_1$.

At P_1, we start scanning an adjacent edge y'' lying to the left of $\bar{y}'$, i.e. y'' is the first edge with $o(y'') = P_1$ which is meet if we rotate small vector in counter-clockwise starting from $\overrightarrow{o(\bar{y}')t(\bar{y}')} = \overrightarrow{P_1P_0}$.

Remark 4.1 The order of three generators $x_{P,y_1}, x_{P,y_2}, x_{P,y_3}$ for each edge is not essential. Even if we are given a relation of different order

$$x_{P,y_1}x_{P,y_3}x_{P,y_2} = 1,$$

we can rewrite it as

$$x_{P,y_1}x_{P,y_2}(x^{-1}_{P,y_2}x_{P,y_3}x_{P,y_2}) = 1,$$

and replace the generator x_{P,y_3} by its conjugate $x^{-1}_{P,y_2}x_{P,y_3}x_{P,y_2}$. Thus in the above determination of the order of elements in the relator (4.1.1) of Lemma (4.1), the embedding of $\tilde{T}'$ into an oriented plane Π is not essential.

Proof of Lemma.

We prove Lemma by induction on $\#(Vert(T'))$. If $\#(Vert(T')) = 1$, it is trivial.

Choose a terminal vertex P_0 of T', and let $\{y_0, \bar{y}_0\}$ be the edges with $t(y_0) = P_0$, and $o(\bar{y}_0) = P_0$. Let T'' be a tree

$$Vert(T'') = Vert(T') - \{P_0\};$$

$$Edge(T'') = Edge(T') - \{y_0, \bar{y}_0\}.$$

Then

$$G_{T'} = G_{T''} *_{G_{y_0}} G_{P_0}.$$

Put $P_1 = o(y_0)$. Then $(P_1, \bar{y}_0)$ is an open hole for T''. Rearranging the position of $x''_{P,y}$ in the product (4.1.1) by a cyclic rotation if necessary, we may assume that $x''_{P_1,\bar{y}_0}$ is the last element in the product (4.1.1). We take generators x_{P_0,y_0}, x_{P_0,y_1}, x_{P_0,y_2} satisfying

$$x_{P_0,y_0} x_{P_0,y_1} x_{P_0,y_2} = 1.$$

Then

$$x'_{P_1,\bar{y}_0} x'_{P_0,y_0} = 1 \qquad \text{in } G_{T'}.$$

Thus the presentation of $G_{T'}$ is given by

$$< \ x'_{P,y} \mid (P,y) \text{ open hole for } T;$$

$$\Big(\prod_{(P,y) \text{ open hole for } T'', (P,y) \neq (P_1,\bar{y}_0)} x'_{P,y} \Big) x'_{P_0,y_1} x'_{P_0,y_2} = 1 > .$$

The group $G_{T'}$ is a free group of rank $rank(G_{T''}) + 1$.

Construction of canonical generators.

We compute the quotient realization $\pi_1(G,Y,T)$ of the fundamental group of a graph of groups (G,Y) with respect to a maximal or spanning tree T in Y.

Since $h(T) = \#(Vert\ T) + 2 = \#(Vert\ Y) + 2 = 2g$, G_T is a free group of rank $2g - 1$ with generators $\{x'_{P,y} | (P,y) \in H(T)\}$. From now on we delete the "$'$" in the symbol $x'_{P,y}$ to simplify notation.

Consider the contracted graph $Y' = Y/T$, which has a unique vertex T/T and g geometric edges. Let $y_1, \ldots, y_g$ be g oriented edges which represent all g geometric edges (i.e. $|y_i| \neq |y_j|$, if $i \neq j$). Then for each edge y_i, two open holes $(o(y_i), y_i)$ and $(t(y_i), y_i)$ are associated. Now the ambient group $F(G,Y)$ is generated by G_T and $y_1, \ldots, y_g$ with relations

$$y_i x_{t(y_i),y_i} y_i^{-1} = x^{-1}_{o(y_i),\bar{y}_i}.$$

Decompose the word $\prod_{(P,y) \in H(T)} x_{P,y}$ into segments. Then it has a form

$$w_f x_{o(y_1),\bar{y}_1} w x_{t(y_1),y_1} w_t,$$

or

$$w_f x_{t(y_1),y_1} w x_{o(y_1),\bar{y}_1} w_t.$$

Reversing the orientation of the edge y_1 for the second case, we may discuss only the first case. Then we put $\alpha_1 = x_{o(y_1),\bar{y}_1}$ and $\beta_1 = y_1^{-1} = \bar{y}_1$. The original word is written as

$$w_f [\alpha_1, \beta_1] x^{-1}_{t(y_1),y_1} w x_{t(y_1),y_1} w_t,$$

and changing the order of words cyclically, we may assume that the relator is of the form

$$[\alpha_1, \beta_1] x_{t(y_1),y_1}^{-1} w x_{t(y_1),y_1} w_t w_f.$$

Now for each i $(2 \le i \le y)$, we want to rewrite the generators $x_{o(y_i),\bar{y}_i}$, $x_{t(y_i),y_i}$ and y_i as follows.

(i) If both $x_{o(y_i),\bar{y}_i}$ and $x_{t(y_i),y_i}$ are contained in the segment $w_t w_f$, then we keep them and y_i the same.

(ii) If both $x_{o(y_i),\bar{y}_i}$ and $x_{t(y_i),y_i}$ are contained in the segment w, then we replace them and y_i by their transforms with respect to $x_{t(y_1),y_1}^{-1}$. In this case, the relation

$$y_i x_{t(y_i),y_i} y_i^{-1} = x_{o(y_i),\bar{y}_i}^{-1}$$

is still valid.

(iii) If one of $x_{o(y_i),\bar{y}_i}$ and $x_{t(y_i),y_i}$ is contained in w and another in $w_t w_f$, then reversing the orientation of the edge y_i, we may assume that $x_{t(y_i),y_i}$ is contained in $w_t w_f$. Then we transform $x_{o(y_i),\bar{y}_i}$ by $x_{t(y_1),y_1}^{-1}$, and replace y_i by $x_{t(y_1),y_1}^{-1} y_i$. Then the relation

$$y_i x_{t(y_i),y_i} y_i^{-1} = x_{o(y_i),\bar{y}_i}^{-1}$$

is still valid.

Thus the segment after $[\alpha_1, \beta_1]$ is a product of new $x_{o(y_i),\bar{y}_i}$ and $x_{o(y_i),\bar{y}_i}$ $(2 \le i \le g)$. We can apply the above process for this shorter word of length $2g-2$. Iterating this process, we can reach the canonical relation

$$[\alpha_1, \beta_1] \dots [\alpha_g, \beta_g] = 1.$$

Step 2

The algorithm to pass from the quotient realization $\pi_1(G, Y, T)$ to a subgroup realization $\pi_1(G, Y, P)$ is described in the book of Serre [**S**] (§5, Prop. 20). Under this subgroup realization, apply the definition of edge twists in Section 2.

4.2 Examples in the case of genus 2.

Proposition (4.2). *The main theorem (1.5) is true when $g = 2$ and $n = 0$.*

Proof. There are two graphs corresponding to the most degenerate stable curves of genus 2.

One of the two graphs consists of two vertices P_1, P_2 with three edges y_i $(i = 1, 2, 3)$ so that $t(y_i) = P_2$ and $o(y_i) = P_1$ for any i $(i = 1, 2, 3)$. Other vertices are given by $\{\bar{y}_i\ (i = 1, 2, 3)\}$. We denote this graph by Y_A.

In this case , $s_1(Y_A) = s_2(Y_A) = 0$. Therefore the part (1) of the main theorem for $n = 0$, which is a result of [**Br**], implies that the homomorphism

$$I_{Y_A} \to Aut\ \pi_1(C_{\mathbf{t}}, *)^{ab} = Out\ \pi_1(C_{\mathbf{t}}, *)/W_{-2}\pi_1$$

is injective. This means $I_Y^{(1)} = \{0\}$. Hence $I_Y^{(3)} = \{0\}$ and $r_i(Y_A) = 0$ for any $i \geq 1$. Thus we can confirm the main theorem for the graph Y_A.

The other graph consists of two vertices P_1, P_2 with three edges y_i $(i = 1, 2, 3)$ such that $o(y_2) = t(y_2) = P_1$, $o(y_3) = t(y_3) = P_2$, and $t(y_1) = P_2$ and $o(y_1) = P_1$. Other edges are given by $\{\bar{y}_i\ (i = 1, 2, 3)\}$. We denote this graph by Y_B.

In order to compute Dehn twists, from now on, we use the following abridged convention to denote the elements in $F(G, Y)$. In place to write x_{P_i, y_j}, we simply write x_{ij}, when $t(y_j) = P_i$. Similarly for $x_{P_i, \bar{y}_j}$ with $o(y_j) = P_i$, we write $x_{i\bar{j}}$.

4.2.1 Computation of the edge twists of the graph Y_B.

Let us start with 9 generators:

$$x_{1\bar{1}},\ x_{1\bar{2}},\ x_{12},\ x_{21},\ x_{2\bar{3}},\ x_{23},\ y_i\ (i = 1, 2, 3)$$

with 5 relations:

$$x_{1\bar{1}}x_{1\bar{2}}x_{12} = 1; \qquad x_{21}x_{2\bar{3}}x_{23} = 1;$$

$$y_2 x_{12} y_2^{-1} = x_{1\bar{2}}^{-1}; \qquad y_3 x_{23} y_3^{-1} = x_{2\bar{3}}^{-1}; \qquad y_1 x_{21} y_1^{-1} = x_{1\bar{1}}^{-1}.$$

If we choose a tree $T = \{|y_1|\}$, then $y_1 = 1$, $x_{21} = x_{1\bar{1}}^{-1}$, $x_{1\bar{1}}x_{21} = 1$. Hence $x_{1\bar{2}}x_{12}x_{2\bar{3}}x_{23} = 1$ with relations:

$$x_{12} = \bar{y}_2 x_{1\bar{2}}^{-1} \bar{y}_2^{-1}; \qquad x_{23} = \bar{y}_3 x_{2\bar{3}}^{-1} \bar{y}_3^{-1},$$

which implies the canonical relation

$$[x_{1\bar{2}}, \bar{y}_2][x_{2\bar{3}}, \bar{y}_3] = 1.$$

Thus we should set

$$\alpha_1 = x_{1\bar{2}}; \quad \beta_1 = \bar{y}_2; \quad \alpha_2 = x_{2\bar{3}}; \quad \beta_2 = \bar{y}_3$$

in the group $\pi_1(G, Y_B, T)$.

Choose P_1 as a base point. Then, we have

$$\alpha_1 = x_{1\bar{2}}; \quad \beta_1 = \bar{y}_2; \quad \alpha_2 = y_1 x_{2\bar{3}} y_1^{-1}; \quad \beta_2 = y_1 \bar{y}_3 y_1^{-1}$$

in $\pi_1(G, Y_B, P_1)$. We note here that $x_{21} = (x_{2\bar{3}}x_{23})^{-1} = ([\alpha_2, \beta_2])^{-1}$.

Here is the computation of the Dehn twists D_{y_i} $(i = 1, 2, 3)$.

Computation (4.1). *We write D_i for D_{y_i}.*

(1) *D_1 keeps α_1 and β_1 invariant. $D_1(\alpha_2) = x_{21}\alpha_2 x_{21}^{-1} = [\alpha_2, \beta_2]^{-1}\alpha_2[\alpha_2, \beta_2]$, and $D_1(\beta_2) = x_{21}\beta_2 x_{21}^{-1} = [\alpha_2, \beta_2]^{-1}\beta_2[\alpha_2, \beta_2]$.*
(2) *D_2 keeps the canonical generators invariant except for β_1, and $D_2(\beta_1) = \beta_1\alpha_1$.*
(3) *D_3 keeps the canonical generators invariant except for β_2, and $D_3(\beta_2) = \beta_2\alpha_2$.*

It is clear that D_2 and D_3 act on $\pi_1(C_{\mathbf{t}}, *)^{ab}$ as mutually independent transvections.

Lemma (4.3). $D_1 \notin I_{Y_B}^{(3)}$.

Proof. The proof is completely the same as that of [**O**, Lemma (1.12)]. We omit it.

Hence we have $r_0(Y_B) = 2$, $r_1(Y_B) = 0$, and $r_2(Y_B) = 1$. Meanwhile, we find $s_1(Y_B) = 0$ and $s_2(Y_B) = 2$ by drawing the picture of Y_B. Thus we have confirmed the main theorem for Y_B. (q.e.d)

4.3 *One example of genus 3.*

In order to complete the inductive proof in Section 5, we have to discuss the case of graph Y_C given as follows. It consists of four vertices P_i ($i = 1, 2, 3, 4$), and six unoriented edges. The oriented edges y_i ($i = 1, \ldots, 6$) are defined by

$$o(y_1) = t(y_1) = P_1, \quad o(y_2) = P_1, \quad t(y_2) = P_2, \quad o(y_3) = P_4, \quad t(y_3) = P_3,$$

$$o(y_4) = t(y_4) = P_4, \quad o(y_5) = t(y_6) = P_2, \quad o(y_6) = t(y_5) = P_3.$$

The generators of the ambient group are

$$x_{1\bar{2}}, x_{11}, x_{1\bar{1}}, \quad x_{4\bar{3}}, x_{44}, x_{4\bar{4}}, \quad x_{22}, x_{26}, x_{2\bar{5}}, \quad x_{33}, x_{35}, x_{3\bar{6}} \text{ and } y_i \ (i = 1, \ldots, 6)$$

with relations:

$$x_{1\bar{2}}x_{11}x_{1\bar{1}} = 1, \qquad x_{4\bar{3}}x_{44}x_{4\bar{4}} = 1, \qquad x_{22}x_{26}x_{2\bar{5}} = 1, \qquad x_{33}x_{35}x_{3\bar{6}} = 1$$

and

$$\begin{aligned} y_1 x_{11} y_1^{-1} &= x_{1\bar{1}}^{-1}, & y_2 x_{22} y_2^{-1} &= x_{1\bar{2}}^{-1}, \\ y_3 x_{33} y_3^{-1} &= x_{4\bar{3}}^{-1}, & y_4 x_{44} y_4^{-1} &= x_{4\bar{4}}^{-1}, \\ y_5 x_{35} y_5^{-1} &= x_{2\bar{5}}^{-1}, & y_6 x_{26} y_6^{-1} &= x_{3\bar{6}}^{-1}. \end{aligned}$$

Choose $T = \{y_i, \ y_{\bar{i}} \ (i = 2, 3, 5)\}$ as a spanning tree. Then in the group $\pi_1(G, Y, T)$, we have

$$x_{1\bar{2}} = x_{22}^{-1}, \quad x_{35} = x_{2\bar{5}}^{-1}, \quad x_{33} = x_{4\bar{3}}^{-1}.$$

Eliminating the above 6 x_{ij} from the 4 relations between x_{ij}, we have the relation

$$x_{11}x_{1\bar{1}}x_{26}x_{3\bar{6}}x_{44}x_{4\bar{4}} = 1,$$

which in turn implies the canonical relation:

$$[x_{11}, y_1][x_{26}, y_6][x_{44}, y_4] = 1.$$

Naturally we should set

$$\alpha_1 = x_{11},\ \beta_1 = y_1,\ \alpha_2 = x_{26},\ \beta_2 = y_6, \alpha_3 = x_{44},\ \beta_3 = y_4.$$

Rewrite these in the fundamental group $\pi_1(G, Y, P_2)$ with base point P_2. Then we have

$$\alpha_1 = y_2^{-1}x_{11}y_2,\ \beta_1 = y_2^{-1}y_1y_2,\ \alpha_2 = x_{26},\ \beta_2 = y_5y_6,$$
$$\alpha_3 = y_5y_3^{-1}x_{44}y_3y_5^{-1},\ \beta_3 = y_5y_3^{-1}y_4y_3y_5^{-1}.$$

We write only the result of the computation of the Dehn twists, which is easy to check.

Computation (4.2). *We write D_i for D_{y_i}. Then D_i $(i = 1, \ldots, 6)$ are given as follows.*

(1) *D_1 keeps canonical generators invariant except for β_1. $D_1(\beta_1) = \beta_1\alpha_1$.*

(2) *D_2 keeps canonical generators invariant except for α_1, β_1.*

$$D_2(\alpha_1) = [\alpha_1, \beta_1]^{-1}\alpha_1[\alpha_1, \beta_1], \qquad D_2(\beta_1) = [\alpha_1, \beta_1]^{-1}\beta_1[\alpha_1, \beta_1].$$

(3) *D_3 keeps canonical generators invariant except for α_3, β_3.*

$$D_3(\alpha_3) = [\alpha_3, \beta_3]^{-1}\alpha_3[\alpha_3, \beta_3], \qquad D_3(\beta_3) = [\alpha_3, \beta_3]^{-1}\beta_3[\alpha_3, \beta_3].$$

(4) *D_4 keeps canonical generators invariant except for β_3. $D_4(\beta_3) = \beta_3\alpha_3$.*

(5) *D_5 keeps α_1, β_1, and α_2 invariant.*

$$D_5(\beta_2) = [\alpha_3, \beta_3]^{-1}\beta_2\alpha_2,$$
$$D_5(\alpha_3) = c_3^{-1}d_2\alpha_3d_2^{-1}c_3,$$
$$D_5(\beta_3) = c_3^{-1}d_2\beta_3d_2^{-1}c_3,$$

where

$$c_3 = [\alpha_3, \beta_3] \text{ and } d_2 = \beta_2\alpha_2\beta_2^{-1}.$$

(6) *D_6 keeps canonical generators invariant except for β_2. $D_6(\beta_2) = \beta_2\alpha_2$.*

Obviously, we have $D_5 \equiv D_6$ modulo $I_{Y_C}^{(1)}$.

Lemma (4.4). $D_5D_6^{-1} \notin I_Y^{(2)}$.

Proof. Let us compute $\delta = D_5D_6^{-1} \in I_{Y_C}^{(1)}$. Then

$$\delta(\alpha_1)\alpha_1^{-1} = 1,\ \delta(\beta_1)\beta_1^{-1} = 1,\ \delta(\alpha_2)\alpha_2^{-1} = 1,\ \delta(\beta_2)\beta_2^{-1} = [\alpha_3, \beta_3]^{-1},$$
$$\delta(\alpha_3)\alpha_3^{-1} \equiv [\alpha_2, \alpha_3] \text{ modulo } W_{-3}(\pi_1),$$
$$\text{and } \delta(\beta_3)\beta_3^{-1} \equiv [\alpha_2, \beta_3] \text{ modulo } W_{-3}(\pi_1).$$

Since there exists no element of weight -2 in π_1 such that the associated inner automorphism is equal to δ modulo $\Gamma_{g,n}[2]$, δ represents a non-zero element in $I_{Y_C}^{(1)}/I_{Y_C}^{(2)}$. (q.e.d)

5. Proof of Main Result.

5.1 *Restating Main Theorem.*

(5.1.1) Let (G, Y) be a graph of surface groups associated with a most degenerate stable n-pointed curve of genus g (see Definition 1.3 and Definition 3.1 if necessary). By Lemma 1.3, the number of edges in Y is $3g-3+n$. From now on, we simply write D_y for $D_{y,\iota_y} = D_{\bar{y},\iota_{\bar{y}}}$. The terms *bridges, cut pairs* imply *geometric* edges.

For the i-th puncture of (G, Y), $(i = 1, \ldots, n)$, we denote by Q_i the vertex on which the puncture lies, and denote by z_i the corresponding element of G_{Q_i}. It may happen that $Q_i = Q_j$ for distinct i, j.

We denote by $\pi_{g,n} := \pi_1(G, Y, P)$ the fundamental group with base point $P \in \mathrm{Vert}(Y)$. This group is uniquely determined by g and n up to isomorphism, that is,

$$\pi_{g,n} \cong < \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots \gamma_n | [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_n = 1 > .$$

We shall omit subscripts g, n in $\pi_{g,n}$ if they are clear.

Let us fix a spanning tree T in the graph Y as in Subsection 2.1. For each i, $(i = 1, \ldots, n)$, there exists a unique path from P to Q_i in T and let us denote it by q_i. Then, $q_i z_i q_i^{-1}$ is an element of $\pi_{g,n} := \pi_1(G, Y, P)$, corresponding to one of γ_j up to conjugacy. We denote $q_i z_i q_i^{-1}$ by c_i.

We equip $\pi_{g,n}$ with the central filtration $\pi_{g,n} = \pi_{g,n}(1), \pi_{g,n}(2), \ldots$ that decreases fastest with condition that $c_1, \ldots, c_n \in \pi_{g,n}(2)$. In other words, we define

$$\begin{aligned} \pi(1) &:= \pi \\ \pi(2) &:= << [\pi, \pi], c_1, \ldots, c_n >> \\ \pi(3) &:= << [\pi(1), \pi(2)] >> \\ \pi(4) &:= << [\pi(1), \pi(3)], [\pi(2), \pi(2)] >> \\ &\vdots, \end{aligned}$$

where $<<>>$ denotes the subgroup normally generated by the elements inside. It is easy to see that this filtration coincides with the one provided in 1.2.1; i.e., we have $\pi(m) = W_{-m}\pi$. We say as usual that $\gamma \in \pi$ has weight $-m$ if and only if $\gamma \in \pi(m) - \pi(m+1)$. It is known that $\cap_{m=1}^{\infty} \pi(m) = \{1\}$ holds, and we define the weight of 1 as $-\infty$(for a proof, see **[K]**, in which the pro-l case is proved, and the above follows immediately from the fact that π can be embedded into its pro-l completion preserving the weight.)

Let us recall the definition of the induced filtration on $\Gamma_{g,n}$ defined in 1.2.2.

Definition 5.1 We define a subgroup $\tilde{\Gamma}_{g,n}$ of $\mathrm{Aut}(\pi_{g,n})$ by

$$\tilde{\Gamma}_{g,n} := \{\sigma \in \mathrm{Aut}(\pi_{g,n}) | \sigma : \text{orientation preserving}, \sigma(c_i) \sim c_i \text{ for } i = 1, \ldots, n\},$$

where $\sim$ denotes conjugacy (see 1.2.2 for the meaning of *orientation preserving*). We equip $\tilde{\Gamma}_{g,n}$ with a filtration $\tilde{\Gamma}_{g,n}[m]$ by

$$\tilde{\Gamma}_{g,n}[m] := \{\sigma \in \tilde{\Gamma}_{g,n}| \\ \sigma(\eta)\eta^{-1} \in \pi_{g,n}(m+k) \text{ for any } k \geq 1 \text{ and any } \eta \in \pi_{g,n}(k)\}.$$

We define $\Gamma_{g,n}$, $\Gamma_{g,n}[m]$ to be the image of $\tilde{\Gamma}_{g,n}$, $\tilde{\Gamma}_{g,n}[m]$ in $\mathrm{Out}(\pi_{g,n})$ respectively.

It is not difficult to see that this definition does not change if we restrict η to be chosen from a fixed generating set of $\pi_{g,n}$.

Let I_Y denote the subgroup of $\mathrm{Out}(\pi_{g,n})$ generated by edge twists. It is known that I_Y is in fact a subgroup of $\Gamma_{g,n}$ isomorphic to $\mathbb{Z}^{\oplus 3g-3+n}$[**BLM**](§3).

In Definition 1.7, we equipped I_Y with a filtration by

$$I_Y^{(m)} := I_Y \cap \Gamma_{g,n}[m]$$

for $m = 0, 1, \ldots$.

Let H denote the set of bridges in Y. We denote by BRG the subset

$$\{D_y | y \in H\}$$

of I_Y, and denote by MCS the subset

$$\{D_{y_i} D_{y_i^j}^{-1} | i = 1 \ldots l, y_i^j \in S_i - \{y_i\}\}$$

of I_Y, where $S_1, \ldots, S_l$ are the maximal cut systems in Y and each y_i is an arbitrarily chosen element from S_i. Observe that $\#\mathrm{BRG} = s_2(Y)$ and $\#\mathrm{MCS} = s_1(Y)$ hold (see Definition 1.6 for s_1 and s_2).

In this formulation, we shall prove the next theorem from which Main Theorem 1.7 immediately follows by applying Theorem 3.2.

Theorem 5.1. *Let s_2 denote the number of bridges in Y and let s_1 denote the summation $\sum\{\#(S) - 1\}$ over all the maximal cut systems $S_1, S_2, \ldots, S_l$. Then we have*

(1) $\mathrm{rank}_{\mathbb{Z}}(I_Y/I_Y^{(1)}) = 3g - 3 + n - s_1 - s_2$,
(2) BRG *is a base of* $I_Y^{(2)}/I_Y^{(3)}$,
(3) MCS *is a base of* $I_Y^{(1)}/I_Y^{(2)}$,
(4) $I_Y^{(3)} = 0$.

We shall prove this theorem in the following manner.

Step 1. Prove that $\mathrm{BRG} \subset I_Y^{(2)}$ and that $\mathrm{MCS} \subset I_Y^{(1)}$.
Step 2. Prove (1) of Theorem.

Step 3. Prove that BRG is linearly independent modulo $I_Y^{(3)}$ and that MCS is linearly independent modulo $I_Y^{(2)}$.

When the above steps are completed, we have an inequality

$$\begin{aligned} 3g-3+n &= \mathrm{rank}_{\mathbb{Z}}(I_Y) \\ &\geq \mathrm{rank}_{\mathbb{Z}}(I_Y/I_Y^{(1)}) + \mathrm{rank}_{\mathbb{Z}}(I_Y^{(1)}/I_Y^{(2)}) + \mathrm{rank}_{\mathbb{Z}}(I_Y^{(2)}/I_Y^{(3)}) \\ &\geq 3g-3+n-s_1-s_2+\#(\mathrm{BRG})+\#(\mathrm{MCS}) \\ &= 3g-3+n, \end{aligned}$$

hence equality must hold. This implies that, when tensored with $\mathbb{Q}$, BRG, MCS are respectively bases of $I_Y^{(1)}/I_Y^{(2)}$, $I_Y^{(2)}/I_Y^{(3)}$ and that $I_Y^{(3)}=0$. Since each $I_Y^{(m)}/I_Y^{(m+1)}$ is a free $\mathbb{Z}$-module, we have

$$I_Y = I_Y/I_Y^{(1)} \oplus I_Y^{(1)}/I_Y^{(2)} \oplus I_Y^{(2)}/I_Y^{(3)} \oplus I_Y^{(3)}.$$

It is obvious that BRG $\cup$ MCS can be extended to a base of I_Y, hence their quotient is torsion free. It follows that (2), (3), and (4) hold.

The hardest part is Step 3. We shall treat this step in Section 6.

From now on, we shall use the following notation. For $y \in \mathrm{Edge}(Y)$, t_y denotes the element ι_y^y(see Definition 3.1). By definitions, we have the following

Lemma 5.2. *For any edge $y \in \mathrm{Edge}(Y)$, We have*

$$y t_y \bar{y} t_{\bar{y}} = 1.$$

The edge twist D_y maps

$$y \mapsto y t_y, \quad \bar{y} \mapsto \bar{y} t_{\bar{y}}$$

and leaves the other generators unchanged.

5.2 *Step 1-A.* BRG $\subset I_Y^{(2)}$.

Proposition 5.3. *Let y be a bridge of the graph Y. Then the Dehn twist D_y associated with the edge y belongs to $I_Y^{(2)}$. In particular, D_y acts trivially on the group $\pi/\pi(2)$.*

Proof. It is enough to show that $D_y \in \tilde{\Gamma}_{g,n}[2]$.

Put $Y-|y| = Y_1 \cup Y_2$, where Y_i (i=1,2) are both connected. Let us fix the orientation of y by $t(y) \in Vert\ Y_2$ and $o(y) \in Vert\ Y_1$. Choose $P_i \in Vert\ Y_i$ $(i=1,2)$, and form

$$G_i = \pi_1(G|Y_i, Y_i, P_i) \qquad (i=1,2).$$

Let $Y' = Y/(Y_1 \cup Y_2)$ be the graph obtained from Y by contracting both Y_i $(i = 1, 2)$ to a point Q_i $(i = 1, 2)$, respectively. Then Y' is a graph with two vertices Q_1, Q_2 and unique geometric edge $\{y, \bar{y}\}$.

There are canonically induced monomorphisms

$$G_y \to G_2, \text{ and } G_{\bar{y}} \to G_1.$$

Then $G' = \{\ G_i\ (i = 1, 2),\ G_y \cong G_{\bar{y}}, \text{and the above monomorphisms}\ \}$ is a graph of groups over Y'. Then we have a canonical isomorphism

$$\pi_1(G, Y, P_1) \cong \pi_1(G', Y', Q_1).$$

By Theorem 3.1, each G_i $(i = 1, 2)$ has the following presentation:

$$G_1 =< \alpha_1, \beta_1, \ldots, \alpha_i, \beta_i, \gamma_0, \gamma_1, \cdots, \gamma_j | (\prod_{k=1}^{i} [\alpha_k, \beta_k]) \gamma_0 \gamma_1 \cdots \gamma_j = 1 >;$$

$$G_2 =< \alpha_{i+1}, \beta_{i+1}, \ldots, \alpha_g, \beta_g, \gamma_{j+1}, \cdots, \gamma_{n+1} | (\prod_{k=i+1}^{g} [\alpha_k, \beta_k]) \gamma_{j+1} \cdots \gamma_{n+1} = 1 >$$

with $\gamma_0 = t_{\bar{y}}$ and $\gamma_{n+1} = t_y$, and each of the other γ_i corresponds to a puncture up to conjugacy.

Consider the presentation of $\pi_1(G', Y', Q_1)$ with base point Q_1. This group is generated by $\xi \in G_1$ and $\xi = y \eta y^{-1}$ for $\eta \in G_2$. It is enough to show that $D_y(\xi)\xi^{-1} \in \pi(2 + l)$ for these ξ with weight $-l$.

Dehn twist D_y has the following description.

$$\begin{cases} D_y(\xi) = \xi, \text{ if } \xi \in G_1; \\ D_y(y \eta y^{-1}) = y t_y \eta \bar{y} t_{\bar{y}} = t_{\bar{y}}^{-1} y \eta y^{-1} t_{\bar{y}} \text{ for } \eta \in G_2. \end{cases}$$

Since $t_{\bar{y}}^{-1} = \gamma_0^{-1} = \gamma_1 \cdots \gamma_j (\prod_{k=1}^{i} [\alpha_k, \beta_k]) \in \pi(2)$, we have

$$D_y(\xi)\xi^{-1} = \begin{cases} 1, \text{ if } \xi \in G_1; \\ [t_{\bar{y}}^{-1}, \xi] \in \pi(2 + l), \text{ if } \xi \in y G_2 y^{-1}. \end{cases}$$

(q.e.d.)

5.3 *Step 1-B.* MCS $\subset I_Y^{(1)}$.

Proposition 5.4. *If $\{|y_1|, |y_2|\}$ is a cut pair of edges, then the actions of Dehn twists D_{y_1} and D_{y_2} on the group $\pi/\pi(2)$ coincide.*

Proof. Let $Y - |y_1| \cup |y_2|$ have two connected components Y_1 and Y_2. Contract both Y_1 and Y_2 to points. Denote these points by P_1 and P_2, respectively. Then, by changing the orientation of edge if necessary, the quotient graph $Y' = Y/(Y_1, Y_2)$ is given by

$$Y' = \{ \qquad \}.$$

The vertex groups are given by

$$G_{P_1} =< \alpha_1, \beta_1, \ldots, \alpha_i, \beta_i, \gamma_1, \ldots, \gamma_j, t_{\bar{y}_1}, t_{\bar{y}_2} | (\prod_{k=1}^{i} [\alpha_k, \beta_k])\gamma_1 \cdots \gamma_j t_{\bar{y}_2} t_{\bar{y}_1} = 1 >;$$

$$G_{P_2} =< \alpha_{i+2}, \beta_{i+2}, \ldots, \alpha_g, \beta_g, \gamma_{j+1}, \ldots, \gamma_n, t_{y_1}, t_{y_2} | \\ (\prod_{k=i+2}^{g} [\alpha_k, \beta_k])\gamma_{j+1} \cdots \gamma_n t_{y_1} t_{y_2} = 1 >$$

with each γ_l corresponding to a puncture up to conjugacy for $l = 1, \ldots, n$. We also have

$$y_1 t_{y_1} y_1^{-1} = t_{\bar{y}_1}^{-1}; \qquad y_2 t_{y_2} y_2^{-1} = t_{\bar{y}_2}^{-1}.$$

Firstly, we choose y_1 as a spanning tree and consider $\pi := \pi_1(G, Y', T)$. Then, since $y_1 = 1$, we have

$$t_{\bar{y}_1} = t_{y_1}^{-1} \text{ and } t_{\bar{y}_2} t_{y_2} = y_2 [t_{y_2}^{-1}, y_2^{-1}] y_2^{-1}.$$

Thus, the defining equation of $\pi_1(G, Y', T)$ is

$$(\prod_{k=1}^{i} [\alpha_k, \beta_k])\gamma_1 \cdots \gamma_j y_2 [t_{y_2}^{-1}, y_2^{-1}] y_2^{-1} (\prod_{k=i+2}^{g} [\alpha_k, \beta_k])\gamma_{j+1} \cdots \gamma_n = 1,$$

and we have

$$\pi/\pi(2) \cong \mathbb{Z}\alpha_1 \oplus \cdots \mathbb{Z}\alpha_g \oplus \mathbb{Z}\beta_1 \oplus \cdots \mathbb{Z}\beta_g$$

with $\alpha_{i+1} = t_{\bar{y}_2}^{-1}$ and $\beta_{i+1} = y_2^{-1}$.

Now we choose P_1 as a base point, and see the presentation of $\pi_1(G, Y, P_1)$. This group is generated by elements of the following five types:

(1) $\xi \in G_{P_1}$,
(2) $\xi = y_1 \eta \bar{y}_1$, $\eta \in G_{P_2}$,
(3) $\xi = y_2 \eta \bar{y}_2$, $\eta \in G_{P_2}$,
(4) $\xi = y_1 \eta \bar{y}_2$, $\eta \in G_{P_2}$,
(5) $\xi = y_2 \eta \bar{y}_1$, $\eta \in G_{P_2}$.

By symmetry it is enough to prove for the cases (1), (2), and (4) that

$$D_{y_1} D_{y_2}^{-1}(\xi)\xi^{-1} \in \pi(1+l),$$

where $-l$ is the weight of ξ.

The case (1) is trivial. In the case of (2), D_{y_2} acts trivially, and

$$D_{y_1}(\xi)\xi^{-1} = y_1 t_{y_1} \eta t_{y_1}^{-1} \bar{y}_1 \xi^{-1} = [y_1 t_{y_1} \bar{y}_1, \xi] \in \pi(1+l).$$

In the case of (4), observe that the weight of ξ is -1 because in $\pi/\pi(2)$ we have

$$\xi = \eta + \bar{y}_2 = \eta + \beta_{i+1},$$

and η does not contain β_{i+1}-component under the canonical basis $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g$ of an abelian group $\pi/\pi(2)$, hence ξ is not zero in $\pi/\pi(2)$. Since

$$\begin{aligned} D_{y_1} D_{y_2}^{-1}(\xi)\xi^{-1} &= t_{\bar{y}_1}^{-1} y_1 \eta \bar{y}_2 t_{\bar{y}_2}^{-1} \xi^{-1} \\ &= t_{\bar{y}_1}^{-1} t_{\bar{y}_2}^{-1} [t_{\bar{y}_2}, \xi] \end{aligned}$$

and

$$t_{\bar{y}_1}^{-1} t_{\bar{y}_2}^{-1} = (\prod_{k=1}^{i} [\alpha_k, \beta_k] \gamma_1 \cdots \gamma_j)^{-1},$$

the right hand side is contained in $\pi(2) = \pi(1+l)$. (q.e.d.)

5.4 *Step 2.* $\mathrm{rank}_{\mathbb{Z}}(I_Y/I_Y^{(1)}) = 3g - 3 + n - s_1 - s_2$.

(5.4.1) The compact case of this equality was proved by Brylinski and Baclawski (see Proposition 5 in [**Br**]).

Theorem 5.5. *(Brylinski and Baclawski.)*
Suppose that $n = 0$. Then the group I_Y is a free abelian group of rank $3g - 3$ and

$$\mathrm{rank}_{\mathbb{Z}}(I_Y/I_Y^{(1)}) = 3g - 3 - s_1 - s_2$$

holds, where s_2 denotes the number of bridges of Y and s_1 denotes the summation $\sum\{\#(S) - 1\}$ over all the maximal cut systems $S_1, S_2, \ldots, S_l$.

The non-compact cases are reduced to the compact cases as follows.

Theorem 5.6. *The group I_Y is a free abelian group of rank $3g - 3 + n$ and*

$$\mathrm{rank}_{\mathbb{Z}}(I_Y/I_Y^{(1)}) = 3g - 3 + n - s_1 - s_2$$

holds.

In the rest of this subsection, we shall prove this theorem.

(5.4.2) Let (G^*, Y^*) denote the graph of surface groups obtained by "compactification" of (G, Y) as follows.

Let Q_1 be a vertex with at least one puncture(see subsection 5.1). If $n_{Q_1} = 1$, then we remove the vertex Q_1 and replace the two incident edges with an edge connecting the other ends of the two edges. If $n_{Q_1} = 2$, then we remove the vertex Q_1 and the unique incident edge, and increase n_P by one, where P is the other end of the removed edge.

We denote by (G', Y') the obtained graph. We also define a surjective partial map

$$* : \mathrm{Edge}(Y) \to \mathrm{Edge}(Y'), y \mapsto y^*$$

as follows. If $n_{Q_1} = 1$, then the both two edges in Y incident to Q_1 are mapped to the new added edge. If $n_{Q_1} = 2$, then the removed edge is removed from the defining domain of $*$. Note that this definition does not change the genus. We iterate this process as far as Y has at least two vertices, and denote by (G^*, Y^*) the obtained graph of surface groups. Let $* : \mathrm{Edge}(Y) \to \mathrm{Edge}(Y^*)$ be the composition of all the partial maps defined in each stage. Let s_1^*, s_2^* be the numbers $s_1(Y^*), s_2(Y^*)$, respectively. Let n^* denote the number of punctures in (G^*, Y^*).

There are three cases:

(1) (G^*, Y^*) is compact; i.e., $n^* = 0$.
(2) (G^*, Y^*) has $g = 0$, $n^* = 3$; i.e., consists of a unique vertex with three punctures and no edge.
(3) (G^*, Y^*) has $g = 1$, $n^* = 1$; i.e., consists of a unique vertex with one puncture and one loop.

The proof will be completed by showing three identities in

$$\mathrm{rank}_{\mathbb{Z}}(I_Y^{(0)}/I_Y^{(1)}) = \mathrm{rank}_{\mathbb{Z}}(I_{Y^*}^{(0)}/I_{Y^*}^{(1)}) = 3g - 3 + n^* - s_1^* - s_2^* = 3g - 3 + n - s_1 - s_2.$$

(5.4.3) First we shall treat the second identity. In the compact case (1), this is nothing else but Theorem 5.5. In the case of (2), both sides are trivially zero. In the case of (3), we have one edge twist. It is easy to compute that $\pi_1(G^*, Y^*)$ is a free group with generators α, β and the weight filtration coincides with the lower central filtration. The edge twist acts on $\pi/[\pi, \pi]$ by $\alpha \mapsto \alpha, \beta \mapsto \alpha + \beta$. This action has infinite order, hence both sides of the second identity are one.

(5.4.4) Now we shall treat the first identity. Let P be a vertex of Y^*. We denote by the same P the inverse image of P in Y.

Definition 5.2 We say that a homomorphism : $G \to G'$ between filtered groups *preserves* the filtration if the image of $G(m)$ is contained in $G'(m)$ for every m.

Lemma 5.7. *There is a filtration preserving surjective homomorphism*

$$\varphi : \pi_1(G, Y) \to \pi_1(G^*, Y^*)$$

inducing

$$\tilde{\Gamma}_{g,n} \to \tilde{\Gamma}_{g,0}.$$

By passing to quotient, we have a filtration preserving homomorphism

$$\Gamma_{g,n} \to \Gamma_{g,0}$$

and by restricting to I_Y *we get*

$$\begin{aligned} I_Y &\to I_{Y^*} \\ D_y &\mapsto D_{y^*}, \end{aligned}$$

where D_{y^*} *is defined to be identity if* y *is not contained in the defining domain of* $*$.

The kernel of φ *is generated by* $<< c_1, \ldots, c_j >>$ *as a normal subgroup, where each* $c_i = q_i z_i q_i^{-1}$ *corresponds to the removed puncture* z_i*(see (5.1.1)) for* $i = 1, \ldots, j$*. (Note that* $j = n, n-3, n-1$ *according to the cases (1), (2), (3) above, respectively.)*

Proof. It is enough to prove in each stage of compactification; i.e., for $Y^* = Y'$. We shall define a homomorphism $F(G, Y) \to F(G', Y')$. Note that we can take

$$\{y, t_y | y \in \mathrm{Edge}(Y)\} \cup \{z_1, \ldots, z_n\}$$

as generators of $F(G, Y)$.

Case 1. $n_{Q_1} = 1$.

Let y_1, y_2 be the two incident edges to Q_1 with $t(y_1) = Q_1 = o(y_2)$. Let y be the newly added edge in Y' with $o(y) = o(y_1), t(y) = t(y_2)$.

On generators, we define

$$\begin{array}{ccc} y_1 & \mapsto & y \\ \bar{y}_1 & \mapsto & \bar{y} \\ y_2, \bar{y}_2 & \mapsto & 1 \\ t_{y_1}, t_{y_2} & \mapsto & t_y \\ t_{\bar{y}_1} & \mapsto & t_{\bar{y}} \\ t_{\bar{y}_2} & \mapsto & t_y^{-1} \\ z_1 & \mapsto & 1, \end{array}$$

and the other generators are left unchanged.

Case 2. $n_{Q_1} = 2$.

Let y_1 be the unique edge with $t(y_1) = Q_1$, and let P be $o(y_1)$. We may assume $z_1, z_2 \in G_{Q_1}$ with $t(y_1)z_1z_2 = 1$. We map

$$\begin{array}{ccc} y_1 & \mapsto & 1 \\ z_1 & \mapsto & 1 \\ z_2 & \mapsto & t_{\bar{y}_1}. \end{array}$$

(Note that G_P in Y' is the same group with the one in Y, hence $t_{\bar{y}_1} \in G_P$ in Y makes sense.)

It is a tedious and simple task to prove that this map is a group homomorphism, inducing a homomorphism even restricted to π_1, that the kernel is generated by $q_1 z_1 q_1^{-1}$ (and $q_1 z_2 q_1^{-1}$ in the second case), that filtration is preserved, and that this map is compatible with

$$\begin{aligned} I_Y &\to I_{Y^*} \\ D_y &\mapsto D_{y^*}. \end{aligned}$$

The detailed proof goes in parallel with the proofs of Lemmas 6.7–6.12 in Subsection 6.5. Since this case is much simpler than these lemmas, we omit the proof. (q.e.d.)

The map $I_Y \to I_{Y^*}$ defined above is obviously surjective. For the first identity, it is enough to prove that the kernel of

$$I_Y \to I_{Y^*}/I_{Y^*}^{(1)}$$

coincides with $I_Y^{(1)}$. Trivially $I_Y^{(1)}$ is contained in this kernel. Conversely, if $\sigma \in I_Y$ is mapped to an element in $I_{Y^*}^{(1)}$, then

$$\sigma(\eta)\eta^{-1} \in << \pi_{g,n}(k+1), c_1, c_2, \ldots, c_j >>$$

holds for any $\eta \in \pi_{g,n}(k)$. We want to prove that $\sigma \in I_Y^{(1)}$. Since there is a generating set of $\pi_{g,n}$ consisting of elements in $(\pi_{g,n}(1) - \pi_{g,n}(2)) \cup \{c_1, \ldots, c_n\}$, it is enough to check the condition for η in this set.

If $\eta \notin \pi_{g,n}(2)$, then we have $\sigma(\eta)\eta^{-1} \in \pi_{g,n}(2) \mod << c_1, \ldots, c_j >>$, and since $c_k \in \pi_{g,n}(2)$ there is no problem. Otherwise $\eta = c_i$ for some $1 \leq i \leq n$. We have $\sigma(c_i) = s c_i s^{-1}$ for some $s \in \pi_{g,n}$ by definition of $\tilde{\Gamma}$, and $\sigma(c_i)c_i^{-1} = [s, c_i] \in \pi_{g,n}(3)$, proving that $\sigma \in \Gamma_{g,n}[1]$.

(5.4.5) The third equality is easily obtained by induction on n. Note that filling up one puncture decreases #BRG by one if the vertex containing the puncture is incident to a bridge, and decreases #MCS by one otherwise.

6. Linear independence of BRG and MCS.

First we shall establish the linear independence for compact cases, and next reduce the non-compact cases to the compact one. In compact cases, the proof is induction on #BRG and on #MCS. The induction depends on the graph reduction defined below.

The merit to restrict to compact cases is that the weight filtration of π_1 coincides with the usual lower central series and hence any homomorphism between two fundamental groups is always filtration preserving.

6.1 *Graph Reduction.*

In this subsection we assume $n = 0$. It is true that all results in this section are valid for the cases $n > 0$, but we don't use it here.

Definition 6.1 (Graph Reduction.)

Let (G, Y) be a graph of surface groups associated with a most degenerate stable curve. Suppose that two distinct edges e_1, e_2 in Y satisfy the following two conditions.

(1) The vertices $t(e_1)$ and $o(e_2)$ belong to the same connected component of $Y - |e_1| - |e_2|$.

(2) Let Y_d denote the above connected component, and let Y_r denote $Y - |e_1| - |e_2| - Y_d$(d for deletion and r for residue). Then, Y_r contains both $o(e_1)$ and $t(e_2)$.

Figure 1. Graph Reduction.

Then, we construct a new graph of surface groups (G', Y') called *the reduced graph of groups along the pair* (e_1, e_2) as follows. The graph Y' is obtained from Y_r by adjoining one edge e from $o(e_1)$ to $t(e_2)$. The new groups G' is the same one with G restricted to Y_r, equipped with a free group of one generator G_e on e and injections $G_e \to G_{o(e)}, G_{t(e)}$ induced from $G_{e_1} \to G_{o(e_1)}$ and $G_{e_2} \to G_{t(e_2)}$.

Note that the graph reduction along (e_1, e_2) is different from the one along (e_2, e_1).

Proposition 6.1. *(Graph Reduction.)*

Let $G, Y, G', Y', Y_r, Y_d, e_1, e_2$ be as above. Choose a simple path $\{z_1, \dots, z_k\}$ in Y_d from $o(e_1)$ to $t(e_2)$ (thus we have $z_1 = e_1$ and $z_k = e_2$). Let P be a vertex in Y_r. We can define a group homomorphism

$$\phi : \pi_1(G, Y, P) \to \pi_1(G', Y', P)$$

using z_i(see (6.5.1) for the precise definition).

The kernel of ϕ is stable under the action of I_Y, and consequently, ϕ induces a unique map

$$\varphi : I_Y \to \mathrm{Out}(\pi_1(G', Y', P))$$

satisfying the condition

$$\varphi(D_y) \cdot \phi(x) = \phi(D_y \cdot x).$$

The image of φ is contained in $I_{Y'}$; more precisely, φ maps the generator of I_Y as follows:

$$\begin{array}{lcll} D_y & \mapsto & D_y & \text{if } y \in \mathrm{Edge}Y_r \\ D_y & \mapsto & \mathrm{id} & \text{if } y \in \mathrm{Edge}Y_d - \{|z_1|, \dots, |z_k|\} \\ D_{z_i} & \mapsto & D_e. & \end{array}$$

Also, φ preserves the filtration, and induces a homomorphism

$$\varphi : I_Y^{(m)}/I_Y^{(m+1)} \to I_{Y'}^{(m)}/I_{Y'}^{(m+1)}.$$

This proposition will be proved in Subsection 6.5.

6.2 *Independence of* BRG *in compact cases.*

In this subsection we assume $n = 0$.

Proposition 6.2. BRG *is linearly independent in $I_Y^{(2)}/I_Y^{(3)}$.*

Proof. We proceed by induction on the cardinality of the set of the bridges H. Suppose that $H = \{|y_1|, |y_2|, \dots, |y_k|\}$. Let $A_1, A_2, \dots, A_{k+1}$ be the connected components of $Y - H$. It is clear that the graph obtained from Y by contracting every A_i to a vertex P_i is a tree. We call this tree *the skeleton* of Y.

The case $\#(H) = 1$. We have $H = \{|y|\}$ and there exist two connected components A_1, A_2 of $Y - |y|$.

It is enough to prove that D_y^n is nonzero in $I_Y^{(2)}/I_Y^{(3)}$ for $n > 0$. The skeleton of Y is shown in Figure 2. Since Y is tri-valent, there are two other edges e_1, e_2 incident to $t(y)$ other than y, with direction $o(e_1) = t(e_2) = t(y)$. If $e_1 = e_2$, then A_1 consists of a loop. Otherwise, using the graph reduction along (e_1, e_2), we have a new graph Y' with A_1' consisting of a loop, and a homomorphism

$$\begin{array}{cccc} \varphi : & I_Y^{(2)}/I_Y^{(3)} & \to & I_{Y'}^{(2)}/I_{Y'}^{(3)} \\ & D_y & \mapsto & D_y. \end{array}$$

Figure 2.

Applying the same operation on A_2 again, we may assume that Y' is the graph with two loops and one bridge; that is, both A_1' and A_2' are loops. For this graph, $g = 2$ holds, and from Proposition 4.2 it follows that D_y^n is trivial in $I_{Y'}^{(2)}/I_{Y'}^{(3)}$ if and only if $n = 0$. Thus, we have $n = 0$ if $D_y^n = 0$ in the original graph, by passing to Y'.

The case $\#(H) > 1$. Let A_1 be a *leaf* of Y; in other words, there exists only one $|y_1| \in H$ incident to A_1. We may assume $t(y_1) \in A_1$. There are two other edges e_1, e_2 incident to $o(y_1)$ with direction $t(e_1) = o(e_2) = o(y_1)$. Since $\#(H) > 1$, we have $e_1 \neq e_2$(Figure 3).

Figure 3.

Case 1. $|e_1|$ is not contained in H. In this case, it is clear that $|e_2|$ is not contained in H and that both $o(e_1)$ and $t(e_2)$ belong to the same connected component of $Y - |e_1| - |e_2|$. We apply the graph reduction along (e_1, e_2). Suppose that $\prod_{i=1}^k D_{y_i}^{n_i} = 0$ in $I_Y^{(2)}/I_Y^{(3)}$. Then, by passage to $I_{Y'}^{(2)}/I_{Y'}^{(3)}$, we have $\prod_{i=2}^k D_{y_i}^{n_i} = 0$ in $I_{Y'}^{(2)}/I_{Y'}^{(3)}$. By induction hypothesis, $n_i = 0$ holds for $i > 1$. Thus, we have $D_{y_1}^{n_1} = 0$ in $I_Y^{(2)}/I_Y^{(3)}$. Again applying the graph reduction along (e_2, e_1) (i.e., Y_r corresponds to $A_1 \cup \{o(y_1)\}$), we can reduce to the case $\#(H) = 1$.

Case 2. $|e_1| \in H$. In this case, it is obvious that $|e_2| \in H$. We may assume $y_2 = e_1$ and $y_3 = e_2$. Suppose again that $\prod_{i=1}^{k} D_{y_i}^{n_i} = 0$ in $I_Y^{(2)}/I_Y^{(3)}$. Then, by the graph reduction along (y_2, y_3), we have $\prod_{i=1}^{k} D_{y_i}^{n_i} = 0$ in $I_{Y'}^{(2)}/I_{Y'}^{(3)}$, that is, $D_e^{n_2+n_3} \prod_{i=4}^{k} D_{y_i}^{n_i} = 0$ in $I_{Y'}^{(2)}/I_{Y'}^{(3)}$, where e is the added edge to Y_r(see Proposition 6.1). From induction hypothesis, $n_2 + n_3 = 0$ and $n_i = 0$ for $i > 3$ follow. Similarly, by the graph reduction along (y_2, y_1) and $(\bar{y}_1, y_3)$, we have $n_1 + n_2 = 0$ and $n_1 + n_3 = 0$ respectively. (Observe that $D_{\bar{y}} = D_y$.) Solving these three equations, we have $n_i = 0$ for $i = 1, 2, 3$. (q.e.d.)

6.3 *Independence of* MCS *in compact cases.*

In this subsection we assume $n = 0$. From now on, a *pair* means a cut pair and a *system* means a maximal cut system.

Lemma 6.3. *Let (e_1, e_2), (e_3, e_4) be two disjoint cut pairs of Y. Then, at least one of the following holds.*

(1) *(e_1, e_2, e_3, e_4) belong to one system.*
(2) *Both e_3 and e_4 belong to one connected component of $Y - |e_1| - |e_2|$.*

Proof. Let Y_1, Y_2 denote the connected components of $Y - |e_1| - |e_2|$. Suppose that neither (1) nor (2) holds. We may assume that $e_3 \in Y_1$ and $e_4 \in Y_2$. Then, since (1) does not hold, both $Y_1 - |e_3|$ and $Y_2 - |e_4|$ are connected. Consequently, $Y - |e_3| - |e_4|$ is connected and this is a contradiction.

Lemma 6.4. *Let S be a system of Y. Then, the graph obtained from Y by contracting every connected component Y_i, $i = 1, 2, \ldots, n$ of $Y - S$ is a cycle (see Figure 4).*

Figure 4.

Proof. Clear.

By Lemma 6.3, for any system $S' \neq S$, there exists a Y_i which contains S'.

Lemma 6.5. *There exists a system S such that a connected component Y_1 of $Y - S$ contains all of the other systems.*

Proof. Take an arbitrary S, and decompose $Y - S$ as in Lemma 6.4. Suppose that a connected component Y_i other than Y_1 yet contains a system S'. Then, decompose $Y - S'$, and let Y_1' be the connected component of $Y - S'$ containing Y_1. By iterating this process, the size of $Y_1^{(k)}$ strictly increases, and this process stops when $Y_1^{(k)}$ contains all systems other than $S^{(k)}$.

Figure 5. Y_1 contains all systems other than S.

We now prove the independence of MCS by induction on #(MCS).

Proposition 6.6. MCS *is linearly independent in* $I_Y^{(1)}/I_Y^{(2)}$.

Proof.
The case #(MCS) $= 1$. In this case, Y is a graph of the form shown on the left in Figure 6.

Figure 6.

Figure 7.

We want to prove that $(D_{e_1}D_{e_2}^{-1})^n = 0$ in $I_Y^{(1)}/I_Y^{(2)}$ holds only if $n = 0$. It is known that $\pi_1(G, Y, P)$ is canonically isomorphic to $\pi_1(G', Y', p_1)$, where Y' is the graph shown on the right in Figure 6, $G_{p_i} := \pi_1(G|_{Y_i}, Y_i, q_i)$ with $q_i \in Y_i$, and $G_{e_i} \to G_{t(e_i)}$ is induced from the one in (G, Y).

As in Subsection 5.2, $\pi_1(G|_{Y_i}, Y_i, q_i)$ is isomorphic to the fundamental group of a Riemann surface of genus $g_i \geq 1$ with two punctures for $i = 1, 2$, and the edge twist D_{e_i} is compatible with this identification.

Thus, it is enough to prove the independence in the case that Y' is as above and

$$G_{p_1} =< \alpha_1, \beta_1, \dots, \alpha_{g_1}, \beta_{g_1}, x, y | [\alpha_1, \beta_1] \cdots [\alpha_{g_1}, \beta_{g_1}] xy = 1 >$$

$$G_{p_2} =< \alpha'_1, \beta'_1, \dots, \alpha'_{g_2}, \beta'_{g_2}, x', y' | [\alpha'_1, \beta'_1] \cdots [\alpha'_{g_1}, \beta'_{g_1}] x'y' = 1 > .$$

We have a projection

$$G_{p_1} \to \bar{G}_{p_1} :=< \alpha_1, \beta_1, x, y | [\alpha_1, \beta_1] xy = 1 >$$

$$G_{p_2} \to \bar{G}_{p_2} :=< \alpha'_1, \beta'_1, x', y' | [\alpha'_1, \beta'_1] x'y' = 1 >$$

inducing a group homomorphism

$$\pi_1(G', Y', p_1) \to \pi_1(\bar{G}, Y', p_1),$$

whose kernel is generated as a normal subgroup by

$$\{\eta \in \pi_1(G', Y', p_1) | \eta \in \mathrm{Ker}\{G_{p_1} \to \bar{G}_{p_1}\}$$
$$\text{or } \eta = y\eta_2 y^{-1}, \eta_2 \in \mathrm{Ker}\{G_{p_2} \to \bar{G}_{p_2}\}, y = e_1 \text{ or } \bar{e}_2\}.$$

It is easy to see that this kernel is stable under the action of D_{e_i}. (Skeptical readers may first read Section 6, in particular, the proof of Lemmas 6.8–6.10 in Subsection 6.5 and next return here.)

Thus, similarly to the case of BRG, it is enough to show the independence of $\{D_{e_1}D_{e_2}^{-1}\}$ in $I_Y^{(1)}/I_Y^{(2)}$ for $(\bar{G}, Y')$ the surface group of genus 3, as shown in Figure 7, which was settled in Lemma 4.4.

The case $\#(\mathrm{MCS}) > 1$. Fix a system S as in Lemma 6.5, and let Y_i, e_i for $1 \le i \le n$ be as shown in Figure 5. Let $S_1 = S, S_2, S_3, \dots, S_l$ be all the systems of Y. We choose $y_i \in S_i$ as in the definition of MCS. We may assume that $y_1 \in S_1$ equals e_i with $i \neq n$, by changing the direction of each e_j if it is necessary.

We may also assume $y_1^1 = e_{i+1}$(note that 1 is a superscript, not a power). Suppose that

$$\prod_{i=1}^{l} \prod_{j=1}^{\#(S_i)-1} (D_{y_i} D_{y_i^j}^{-1})^{n_{ij}} = 0$$

in $I_Y^{(1)}/I_Y^{(2)}$. Then, by applying the graph reduction along (e_i, e_{i+1}) and calculating the image of the LHS by φ, we have

$$\prod_{j=2}^{\#(S_1)-1} (D_{y_1} D_{y_1^j}^{-1})^{n_{1j}} \prod_{i=2}^{l} \prod_{j=1}^{\#(S_i)-1} (D_{y_i} D_{y_i^j}^{-1})^{n_{ij}} = 0$$

(see Proposition 6.1). By induction hypothesis, $n_{ij} = 0$ for $i \ge 2$ and $n_{1j} = 0$ for $j \ge 2$ hold.

Thus, we have

$$(D_{y_1} D_{y_1^1}^{-1})^{n_{11}} = (D_{e_i} D_{e_{i+1}}^{-1})^{n_{11}} = 0$$

in $I_Y^{(1)}/I_Y^{(2)}$. If S contains another edge e_{i-1} or e_{i+2}, then the induction hypothesis and the graph reduction along (e_{i+1}, e_{i-1}) or (e_{i+2}, e_i), respectively, imply that $n_{11} = 0$. If S contains only e_i and e_{i+1}, then Y_1 contains another system S', since $\#(\mathrm{MCS}) > 1$. Choose $e_1', e_2' \in S'$ so that in the graph reduction along (e_1', e_2'), Y_r contains Y_2(see Figure 8.)

Figure 8.

By induction hypothesis in the reduced graph along (e_1', e_2'), we have $n_{11} = 0$. (q.e.d.)

6.4 *Reducing non-compact cases to compact ones.* In this subsection we assume $n > 0$. Let (G, Y) be a graph of surface groups associated with a most degenerate stable n-pointed curve of genus g. We introduce another kind of compactification. We equip each puncture Q_i with a new vertex P_i with a loop L_i and put one edge y_i from Q_i to P_i. Let Y' denote the obtained graph. The graph of groups (G', Y') corresponds to the compact Riemann surface obtained from the original punctured Riemann surface by filling up each puncture with a handle (Figure 9).

Figure 9.

There exists a canonical injective homomorphism

$$\phi : \pi_{g,n} = \pi_1(G, Y) \cong \pi_1(G'|_Y, Y) \to \pi_{g+n,0} = \pi_1(G', Y').$$

By mapping D_y for $y \in \mathrm{Edge}Y$ to $D_y \in I_{Y'}$, we have a homomorphism

$$\varphi : I_Y \to I_{Y'},$$

since Y is a subgraph Y'. It is not difficult to see that ϕ preserves the filtration by checking generators. In fact, the only problem is whether c_i is mapped into $\pi(2)$ or not, but this is trivial since $c_i = [\alpha_{g+i}, \beta_{g+i}]$ holds in $\pi_1(G', Y')$. It is easy to see that $\varphi(\sigma)(\phi(\eta)) = \phi(\sigma(\eta))$ for any $\sigma \in I_Y, \eta \in \pi_{g,n}$.

We claim that φ is filtration-preserving. To show this, it is enough to prove for any $\sigma \in I_Y^{(m)}$ and for η of the form $q_i y_i d \bar{y}_i q_i^{-1}$ with $d \in < G_{P_i}, L_i >$ (see Subsection 5.1 for q_i) that

$$\varphi(\sigma)(\eta)\eta^{-1} \in \pi_{g+n,0}(m+1),$$

since these η of weight -1 together with the image of $\pi_1(G, Y)$ generate $\pi_1(G', Y')$.

Recall that $c_i = q_i z_i q_i^{-1}$. So we have

$$\varphi(\sigma)(c_i) = s c_i s^{-1}$$

and

$$\varphi(\sigma)(\eta) = s \eta s^{-1}$$

with the same $s = \sigma(q_i)q_i^{-1}$. Thus, our claim is reduced to proving that $s \in \pi_{g,n}(m)$.
To prove this, we have to know the structure of the associated Lie algebra

$$\mathcal{L} := \oplus \mathrm{gr}_m(\pi_{g,n}).$$

As mentioned in [**K**], we can embed $\mathcal{L}$ into a free non-commutative associative algebra A over $\mathbb{Z}$ with generators $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \dots, \gamma_{n-1}$, and if we equip A with a gradation by $\deg(\alpha_1) = \cdots = \deg(\beta_g) = 1$ and $\deg(\gamma_1) = \cdots = \deg(\gamma_{n-1}) = 2$, then this embedding preserves the gradation.

Let $-l$ be the weight of s. If $l \geq m$ including the case $l = \infty$, we have $s \in \pi_{g,n}(m)$. Assume $l < m$. Then, $[s, c_i] \in \pi_{g,n}(m+2) \subset \pi_{g,n}(l+2+1)$ holds since $\sigma \in I_Y^{(m)}$. Interpreting this in A, we have $s \cdot c_i - c_i \cdot s = 0$ in A. We may assume that c_i is one of γ_i, hence one of the canonical generators of A.

Since A is free, $s = k \cdot c_i$ in A holds for some integer k. However, we have $s = \sigma(q_i)q_i^{-1}$ and q_i was a path chosen from a fixed spanning tree T. We have

$$\pi_1(G|_{q_i}, q_i, P) \subset \pi_1(G|_T, T, P) \subset \pi_1(G, Y, P).$$

Then $s = \sigma(q_i)q_i^{-1}$ is in the left group, hence contained in the middle one, and can be written using only α_i(see Subsection 4.1). Since A is free on $\alpha_i, \beta_i, \gamma_i$, this implies $s = k \cdot c_i = 0$ in A, that is, $s = 1$, leading a contradiction.

We have proved that

$$\varphi : I_Y \to I_{Y'}$$

is filtration preserving. It is easy to see that bridges and maximal systems of cut pairs in Y are mapped to ones in Y' respectively. Hence, the independence of BRG, MCS in Y follows from the ones in Y', which was proved previously. (q.e.d.)

6.5 *Details on graph reduction.*
In this subsection we assume $n = 0$.

(6.5.1) Precise definition of graph reduction.
Let $G, Y, G', Y', Y_r, Y_d, e_1 = z_1, z_2, \dots, z_k = e_2, P$ be as in Proposition 6.1. We shall define a homomorphism

$$\phi : F(G, Y) \to F(G', Y'),$$

which restricts to

$$\phi : \pi_1(G, Y, P) \to \pi_1(G', Y', P).$$

We may consider $F(G, Y)$ as the group generated by

$$\{y, t_y | y \in \mathrm{Edge}(Y)\}$$

with relations

$$y\bar{y} = 1, \qquad yt_y\bar{y}t_{\bar{y}} = 1, \text{ and } t_{y_1}t_{y_2}t_{y_3} = 1,$$

for three edges with a common terminal vertex y_1, y_2, y_3 in the fixed order(see Subsection 4.1).

The edge twist D_y maps

$$y \mapsto yt_y, \bar{y} \mapsto \bar{y}t_{\bar{y}}$$

and leaves the other generators unchanged.

On generators, ϕ is defined as follows:

if $y \in \mathrm{Edge}(Y_r)$ then

$$y, t_y \mapsto y, t_y,$$

if $y \in \mathrm{Edge}(Y_d) - \{z_1, \dots, z_k, \bar{z}_1, \dots, \bar{z}_k\}$ then

$$y, t_y \mapsto 1,$$

if $y = z_i$ or $\bar{z}_i$ then for $i = 1$

$$\begin{array}{rcl} z_1 & \mapsto & e \\ \bar{z}_1 & \mapsto & \bar{e} \\ t_{z_1} & \mapsto & t_e \\ t_{\bar{z}_1} & \mapsto & t_{\bar{e}} \end{array}$$

and for $2 \le i \le k$,

$$\begin{array}{rcl} z_i, \bar{z}_i & \mapsto & 1 \\ t_{z_i} & \mapsto & t_e \\ t_{\bar{z}_i} & \mapsto & t_e^{-1}. \end{array}$$

We define

$$\phi_v : \mathrm{Vert}(Y) \to \mathrm{Vert}(Y')$$

by

$$\begin{array}{rcll} p & \mapsto & p & \text{if } p \in \mathrm{Vert}(Y_r) \\ p & \mapsto & t(e) & \text{otherwise.} \end{array}$$

It is easy to see that ϕ maps G_p into $G_{\phi_v(p)}$. It is a tedious but easy task to check that the relations in $F(G, Y)$ are compatible with the ones in $F(G', Y')$ through ϕ; that is, $\phi : F(G, Y) \to F(G', Y')$ is a group homomorphism. For this, it is enough to check that

$$\phi(y)\phi(\bar{y}) = 1,$$

$$\phi(y)\phi(t_y)\phi(y^{-1})\phi(t_{\bar{y}}) = 1,$$

$$\phi(t_{y_1})\phi(t_{y_2})\phi(t_{y_3}) = 1$$

for y_1, y_2, y_3 with the same target vertex. All are easy; for example, we check only the third identity. Suppose $p \in Y_r$. If $y_1, y_2, y_3 \in Y_r$, then $\phi(y_i) = y_i$ and there is no problem. Otherwise, $p = o(e_1)$ or $t(e_2)$. In the first case, suppose for example that $y_1 = \bar{e}_1$. Then

$$\phi(t_{y_1})\phi(t_{y_2})\phi(t_{y_3}) = t_{\bar{e}}t_{y_2}t_{y_3} = 1$$

holds. The latter case can be checked similarly. Next, suppose $p \in Y_d$. If p is not on the path $\{z_1, \ldots, z_k\}$, then $\phi(t_{y_i}) = 1$ holds and no problem exists. If p is on the path, suppose for example that $y_1 = z_i$ and $y_2 = \bar{z}_{i+1}$. It is clear that y_3 is different from any z_i, and thus we have

$$\phi(t_{y_1})\phi(t_{y_2})\phi(t_{y_3}) = t_e t_e^{-1} \cdot 1 = 1.$$

Now we shall check that

$$\phi(\pi_1(G, Y, P)) \subset \pi_1(G', Y', P).$$

Let x be an element of $\pi_1(G, Y, P)$. Then, x can be represented by an admissible word (see Definition 2.2)

$$w = r_0 y_1 r_1 y_2 \cdots r_{n-1} y_n r_n,$$

where $\{y_1, \ldots, y_n\}$ is a path from P to P, $r_i \in G_{o(y_{i+1})}$ for $i = 0, \ldots, n-1$, and $r_n \in G_{t(y_n)}$.

Regard $e_1, \bar{e}_2$ as left parentheses and $e_2, \bar{e}_1$ as right parentheses, and decompose w with respect to these parentheses as follows. Since $Y - |e_1| - |e_2|$ is not connected and $P \in Y_r$, it is clear that w decomposes as

$$w = A_1 B_1 A_2 B_2 \cdots A_s B_s A_{s+1},$$

where $A_i \in F(G|_{Y_r}, Y_r)$ and B_i is one of the following four types with $z \in F(G|_{Y_d}, Y_d)$:

$$(1)\ e_1 z \bar{e}_1,\ (2)\ \bar{e}_2 z e_2,\ (3)\ e_1 z e_2,\ (4)\ \bar{e}_2 z \bar{e}_1.$$

We call this decomposition *the decomposition of w along* (e_1, e_2).

Lemma 6.7. *Let B be an admissible word of one of the above four types. Let us denote by $s(B)$ the sum of the numbers of occurrence of t_{z_i} for $1 \leq i \leq k$ in B minus the one of $t_{\bar{z}_i}$ for $1 \leq i \leq k$. Then we have for each type:*

(1) $\phi(B) = e(t_e)^{s(B)}\bar{e} = t_{\bar{e}}^{-s(B)} \in G_{o(e)}$
(2) $\phi(B) = (t_e)^{s(B)} \in G_{t(e)}$
(3) $\phi(B) = e(t_e)^{s(B)}$
(4) $\phi(B) = (t_e)^{s(B)}\bar{e}$.

Proof. Straightforward.

It is easy to see that for an admissible word w as above, its image

$$\phi(w) = A_1\phi(B_1)A_2\phi(B_2)\cdots A_s\phi(B_s)A_{s+1}$$

is also admissible and belongs to $\pi_1(G', Y', P)$ under the identification $t_{\bar{e}_1} = t_{\bar{e}}$ and $t_{e_2} = t_e$. Thus, we have proved that

$$\phi(\pi_1(G, Y, P)) \subset \pi_1(G', Y', P).$$

(6.5.2) Stability of Ker(ϕ).

Lemma 6.8. *Let w be an admissible word representing an element in $\pi_1(G, Y, P)$. Suppose that w contains only*

$$t_{\bar{z}_1}, t_{z_k}, z_i, \bar{z}_i, \text{ and } y, \bar{y}, t_y, t_{\bar{y}} \text{ for } i = 1 \cdots k \text{ and } y \in \mathrm{Edge}(Y_r).$$

Then, $w \neq 1$ in $\pi_1(G, Y, P)$ implies $\phi(w) \neq 1$.

Proof. Let w be in the form of $w = r_0y_1r_1y_2\cdots r_{n-1}y_nr_n$ and suppose that $w \neq 1$. Suppose that this word is reducible; that is, this word contains a subword in the form of

$$y_i(t_{y_i})^l\bar{y}_i.$$

If $y_i \neq \bar{z}_1$ nor z_k, we may replace this part of w with $(t_{\bar{y}_i})^{-l}$ without influence on the assumption on w. We reduce w in this way as far as possible. If w is still reducible, then w contains

$$\bar{z}_1t^l_{\bar{z}_1}z_1$$

or

$$z_kt^l_{z_k}\bar{z}_k.$$

In the former case, since $Y - |z_1| - |z_k|$ is not connected and back-tracking inside $\{z_1, \ldots, z_k\}$ was already removed, w must contain the subword

$$\bar{z}_k\cdots\bar{z}_1t^l_{\bar{z}_1}z_1\cdots z_k,$$

which can be replaced with $t^{-l}_{z_k}$. In the latter case, w contains

$$z_1\cdots z_kt^l_{z_k}\bar{z}_k\cdots\bar{z}_1,$$

which can be replaced with $t^{-l}_{\bar{z}_1}$. By iterating these operations, we may assume that w is reduced and $w \neq 1$. We want to prove that $\phi(w) = \phi(r_0)\phi(y_1)\cdots\phi(y_n)\phi(r_n) \neq 1$.

Let us delete all the occurrence of $\phi(z_i)$ for $i = 2, \dots, k$ from the word $\phi(w)$ since they equal 1, and obtain a new word

$$\phi'(w) := A_0\phi(y_{i_1})A_1\phi(y_{i_2})\cdots\phi(y_{i_s})A_s$$

where $\{y_{i_1}, \dots, y_{i_s}\} = \{y_1, \dots, y_n\} - \{z_2, \bar{z}_2, \dots, z_k, \bar{z}_k\}$ and $A_j = \phi(r_{i_j}\cdots r_{i_{j+1}-1})$ for $j = 0, \dots, s$ with $i_0 = 0$ and $i_{s+1} = n+1$. Then we have $y_{i_j+1}, \dots, y_{i_{j+1}-1} \in \{z_2, \dots, z_k\}$. From the restriction on w that w contains only $t_{\bar{z}_1}$ and t_{z_k} among elements in the form t_y, $y \in \mathrm{Edge}Y_d$, it follows that $r_{i_j+1} = r_{i_j+2} = \cdots = r_{i_{j+1}-1} = 1$; that is, $A_j = \phi(r_{i_j})$. It is enough to prove that the word $\phi'(w)$ is reduced and not equal to the word $\{1\}$. Suppose that this word is not reduced. Then it contains a subword

$$\phi(y_{i_j})A_j\phi(y_{i_{j+1}})$$

with $A_j \in G_{t(\phi(y_{i_j}))}$ and $\phi(y_{i_{j+1}}) = \overline{\phi(y_{i_j})}$.

Case 1. $y_{i_j} \neq z_1$ nor $\bar{z}_1$. In this case, we have $\phi(y_{i_j}) = y_{i_j}$, $\phi(y_{i_{j+1}}) = y_{i_{j+1}} = \bar{y}_{i_j}$, and either $i_{j+1} = i_j + 1$ or $y_{i_j+1} \in \{z_2, \dots, z_k\}$ holds. Suppose that $i_{j+1} = i_j + 1$ holds. Since $\phi : G_{t(y_{i_j})} \to G_{t(y_{i_j})}$ is an isomorphism and

$$\phi(y_{i_j})\phi(r_{i_j})\phi(y_{i_j+1})$$

is reducible,

$$y_{i_j}r_{i_j}y_{i_j+1}$$

is also reducible, contradicting the assumption. Now we may assume $i_{j+1} \neq i_j + 1$. It is easy to see that $t(y_{i_j}) = t(z_k)$ and $y_{i_j+1} = \bar{z}_k$ hold. Since $Y - |z_1| - |z_k|$ is not connected and w contains only $z_i, \bar{z}_i$ among $\mathrm{Edge}(Y) - \mathrm{Edge}(Y_r)$, $\{y_{i_{j+1}}\cdots y_{i_{j+1}-1}\}$ is a walk from $t(z_k)$ to $t(z_k)$ consists of only $z_i, \bar{z}_i$ for $2 \leq i \leq k$. The assumption on w and irreducibility of w imply that this walk has no back-tracking. Thus, this walk is empty, contradicting the fact that $y_{i_j+1} = \bar{z}_k$.

Case 2. $y_{i_j} = z_1$ or $\bar{z}_1$. If $y_{i_j} = z_1$, then similarly to Case 1, $\{y_{i_j+1}\cdots y_{i_{j+1}-1}\}$ must be a path from $t(z_1)$ to $t(z_1)$. The corresponding part of w must be

$$z_1\cdots z_k r_k \bar{z}_k \cdots \bar{z}_1$$

for some $r_k \notin \mathrm{Image}(G_{z_k})$. By passage by ϕ, we have a reduced word

$$er_k\bar{e},$$

contradicting the assumption. If $y_{i_j} = \bar{z}_1$, then we have

$$y_{i_j}r_{i_j}y_{i_j+1} = \bar{z}_1 r_{i_j} z_1,$$

whose image by ϕ is also reduced.

Consequently, both cases are impossible and every y_i must coincide with one of z_j for some $2 \leq j \leq k$. This implies that $P = t(z_k)$, and similarly to Case 1, the irreducibility and the assumption on w imply that $w = 1$ holds. (q.e.d.)

Lemma 6.9. *The kernel of* ϕ

$$\mathrm{Ker}\{\phi : \pi_1(G, Y, P) \to \pi_1(G', Y', P)\}$$

is generated as a normal subgroup by the following four types of elements. Recall that we decomposed a word of $\pi_1(G, Y, P)$ *and defined four types in Definition 6.2, and defined a function* s *in Lemma 6.7. Let* U *be a path from* P *to* $o(e_1)$ *and* V *be a path from* P *to* $t(e_2)$*, which contain no edge in* $\mathrm{Edge}(Y_d) - \{z_1, \bar{z}_1, \dots, z_k, \bar{z}_k\}$*. The generators are:*

(1) $UB(t_{\bar{e}_1})^{s(B)}U^{-1}$ *for* B *of type 1,*
(2) $VB(t_{e_2})^{-s(B)}V^{-1}$ *for* B *of type 2,*
(3) $U(t_{\bar{e}_1})^{s(B)}B\bar{z}_k\bar{z}_{k-1}\cdots\bar{z}_1U^{-1}$ *for* B *of type 3,*
(4) $VB(t_{\bar{e}_1})^{s(B)}z_1z_2\cdots z_kV^{-1}$ *for* B *of type 4.*

Proof. It is easy to check that these elements belong to $\mathrm{Ker}(\phi)$ by using Lemma 6.7. Let x be an element of $\mathrm{Ker}(\phi)$. Take a reduced word w representing x, and decompose it into

$$w = A_1B_1A_2B_2\cdots A_sB_sA_{s+1}$$

as in Definition 6.2. Suppose that B_1 is of type 3 for example. Other cases can be handled similarly. In this case,

$$w = A_1(t_{\bar{e}_1})^{-s(B_1)}U^{-1}U(t_{\bar{e}_1})^{s(B_1)}B_1\bar{z}_k\bar{z}_{k-1}\cdots\bar{z}_1U^{-1}Uz_1z_2\cdots z_kA_2\cdots B_sA_{s+1}$$

holds. Thus, it is enough to prove that

$$w_0 = A_1(t_{\bar{e}_1})^{-s(B)}z_1z_2\cdots z_kA_2\cdots B_sA_{s+1}$$

can be generated by the above four types of elements. Applying this operation on $B_2, \dots, B_s$ in this order, we obtain w' satisfying the assumption of Lemma 6.8. Since $\phi(w') = 1$, we have $w' = 1$ by Lemma 6.8. (q.e.d.)

Lemma 6.10. *For any* $y \in \mathrm{Edge}(Y)$*, the edge twist* $D_y \in \mathrm{Aut}(\pi_1(G, Y, P))$ *stabilizes* $\mathrm{Ker}(\phi)$.

Proof. It is enough to check that the image by D_y of each generator of $\mathrm{Ker}(\phi)$ belongs to $\mathrm{Ker}(\phi)$ again. This is straightforward; for example, let w be an element of $\mathrm{Ker}(\phi)$ of type 3 in Lemma 6.9. If y does not occur in $B\bar{z}_k\bar{z}_{k-1}\cdots\bar{z}_1$, then we have

$$\begin{aligned}&\phi \circ D_y(U(t_{\bar{e}_1})^{s(B)}B\bar{z}_k\bar{z}_{k-1}\cdots\bar{z}_1U^{-1})\\ &\quad= \phi(D_y(U))\phi((t_{\bar{e}_1})^{s(B)}B\bar{z}_k\bar{z}_{k-1}\cdots\bar{z}_1)\phi(D_y(U^{-1})) = \phi(D_y(UU^{-1})) = 1.\end{aligned}$$

Thus, we may assume that y occurs in $B\bar{z}_k\bar{z}_{k-1}\cdots\bar{z}_1$, and it follows that y does not belong to Edge(Y_r). If y is not any of $z_i, \bar{z}_i$, then we have

$$\phi(D_y(v)) = \phi(v)$$

for any word v since $\phi(t_y) = \phi(t_{\bar{y}}) = 1$, and consequently, $\phi(D_y(w)) = \phi(w) = 1$ holds.

The rest case is $y = z_i$ or $\bar{z}_i$ for some i. In this case, observe that the number of occurrence of z_i in $B\bar{z}_k\bar{z}_{k-1}\cdots\bar{z}_1$ is the same with that of $\bar{z}_i$ for each i. Since $D_y(z_i) = z_i t_{z_i}$ and $D_y(\bar{z}_i) = \bar{z}_i t_{\bar{z}_i} = t_{z_i}^{-1}\bar{z}_i$, it is easy to see that

$$\phi \circ D_y(B\bar{z}_k\bar{z}_{k-1}\cdots\bar{z}_1) = \phi(B\bar{z}_k\bar{z}_{k-1}\cdots\bar{z}_1)$$

holds; and consequently, we have $\phi \circ D_y(w) = \phi(w) = 1$. The other types can be checked similarly. (q.e.d.)

Now we can define

$$\varphi : I_Y \to \mathrm{Out}(\pi_1(G', Y', P))$$

by

$$\varphi(D_y)\cdot\phi(x) = \phi(D_y \cdot x),$$

since D_y stabilizes Ker(ϕ).

(6.5.3) Explicit description of φ.

Lemma 6.11. *The homomorphism φ maps an edge twist to an edge twist or identity as follows:*

$$\begin{array}{lcll} D_y & \mapsto & D_y & \text{if } y \in \mathrm{Edge}Y_r, \\ D_y & \mapsto & \mathrm{id} & \text{if } y \in \mathrm{Edge}Y_d - \{z_1, \bar{z}_1, \ldots, z_k, \bar{z}_k\}, \\ D_{z_i} & \mapsto & D_e & \text{for } 1 \le i \le n. \end{array}$$

Proof. Let w be a word in $\pi_1(G, Y, P)$. Let D'_y denote the RHS of the above table. It is enough to show that

$$\phi(D_y \cdot w) = D'_y \cdot \phi(w)$$

for each case. If $y \in$ Edge(Y_r), there is no problem.

If $y \in$ Edge$(Y_d) - \{z_1, \bar{z}_1, \ldots, z_k, \bar{z}_k\}$, it is obvious that $\phi(D_y \cdot w) = \phi(w)$ holds since $D_y \cdot y = yt_y$, $D_y \cdot \bar{y} = \bar{y}t_{\bar{y}} = t_y^{-1}\bar{y}$, and $\phi(t_y) = 1$. Suppose that $y = z_i$ or $\bar{z}_i$ for some i. Decompose w into $w = A_1B_1A_2B_2\cdots A_sB_sA_{s+1}$ as shown in Definition 6.2. Since $D_y(A_i) = A_i$ and $D'_y(\phi(A_i)) = \phi(A_i)$, it is enough to show that

$$\phi(D_y \cdot B) = D'_y \cdot \phi(B) = D_e \cdot \phi(B)$$

for B of any one of four types in Definition 6.2. This can easily be checked; for example, suppose that $y = z_i$ and B is of type 3. Then, it is easy to see that

$$\phi(D_y \cdot B) = e(t_e)^{s(B)+1}$$

and that

$$D'_y(\phi(B)) = D_e(e(t_e)^{s(B)}) = e(t_e)^{s(B)+1}.$$

Other cases follow similarly. (q.e.d.)

Lemma 6.12. *The homomorphism φ preserves the filtration.*

Proof. Let α be an element of $I_Y^{(m)}$. Then,

$$\varphi(\alpha)(\phi(x)) = \phi(\alpha(x))$$

and

$$\begin{aligned}(\varphi(\alpha)(\phi(x)))\phi(x)^{-1} = \phi(\alpha(x)x^{-1}) &\in \phi(\pi_1(G, Y, P)(m+1)) \\ &\subset \pi_1(G', Y', P)(m+1)\end{aligned}$$

hold for all x. Therefore we have $\varphi(\alpha) \in I_{Y'}^{(m)}$. (q.e.d.)

We have proved all statements in Proposition 6.1.

REFERENCES

[A]. Asada, M., *Two properties of the filtration of the outer automorphism groups of certain groups*, Preprint.

[BLM]. Birman, J. Lubotsky, A., McCarthy, J., *Abelian and solvable subgroups of the mapping class group*, Duke Math. J. **50** (1983), 1107–1120.

[Br]. Brylinski, Jean-Luc., *Propriétés de ramification a l'infini du groupe modulaire de Teichmüller*, Ann. Scient.Éc.Norm.Sup. 4e série **t.12** (1979), 295-333.

[DM]. Deligne, P. and Mumford, D., *The irreducibility of the space of curves of given genus*, Publ. Math. I.H.E.S. **36** (1969), 75–110.

[K]. Kaneko, M., *Certain automorphism groups of pro-l fundamental groups of punctured Riemann surfaces.*, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **36** (1989), 363-372.

[Kn]. Knusten, F.F., *The projectivity of the moduli space of stable curves II.*, Math. Scad. **52** (1983), 161l-199.

[N]. Namikawa, Y., *On the canonical holomorphic map from the moduli space of stable curves to the Igusa monoidal transform*, Nagoya Math. J. **52** (1973), 197–259.

[O]. Oda, T., *A note on the ramification of the fundamental group of an algebraic curve II*, To appear in J. of Number Theory.

[S]. Serre, J.-P., *"Arbres, amalgames, SL_2"*, Astérisque **46** (1977), 1–189.

[ZVC]. Zieschang, H., Vogt, E. and Coldewey, H.-D., *"Surfaces and Planar Discontinuous Groups"*, Lec. Notes in Math. 835, Springer, 1980.

Tokyo Denki Univ., Tokyo,
RIMS, Kyoto Univ., Kyoto,
RIMS, Kyoto Univ., Kyoto

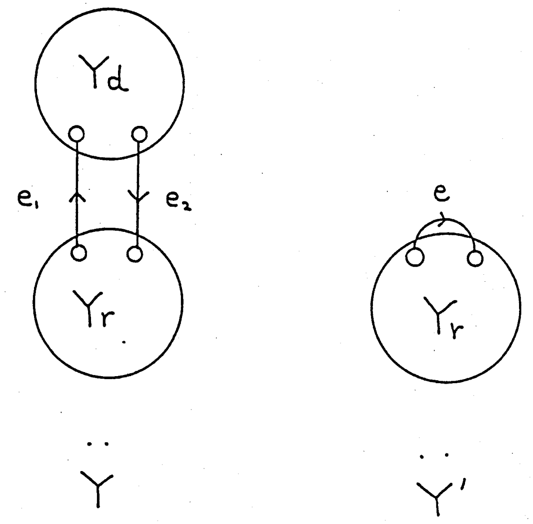

Figure 1.

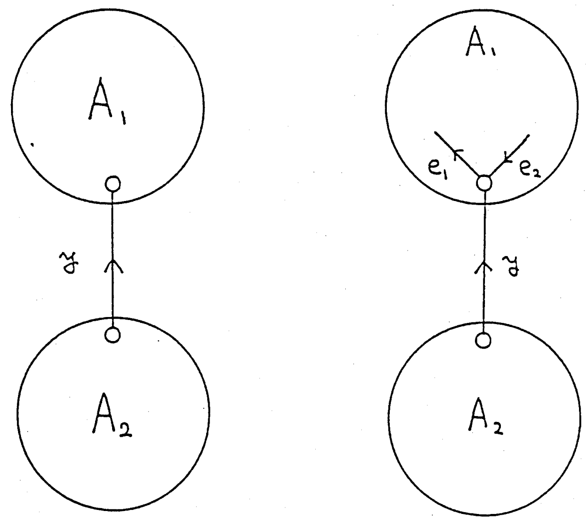

Figure 2.

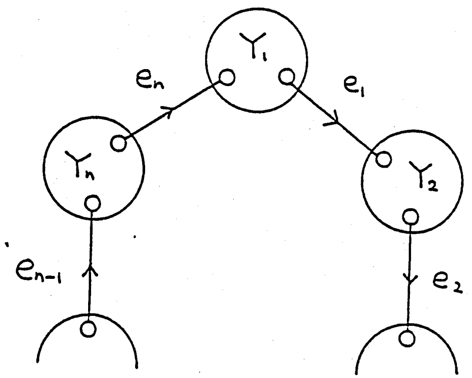

Figure 4.

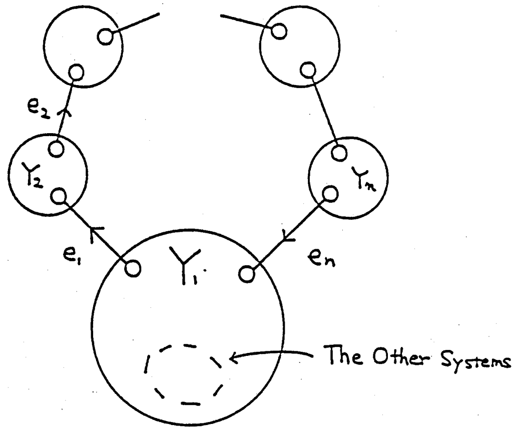

Figure 5.

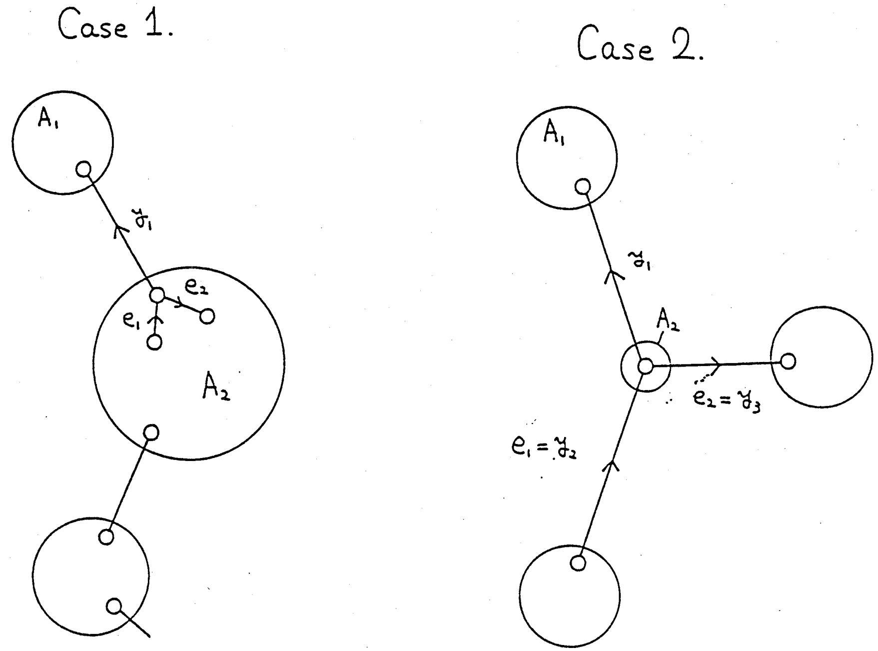

Figure 3.

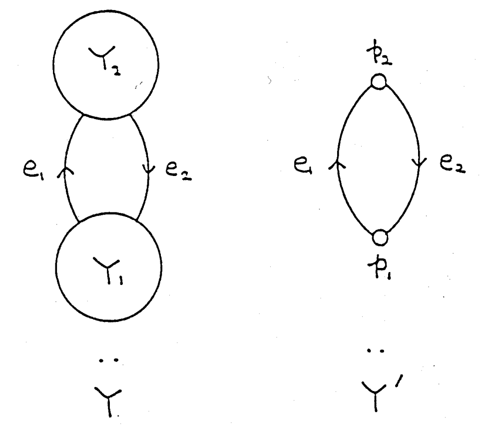

Figure 6.

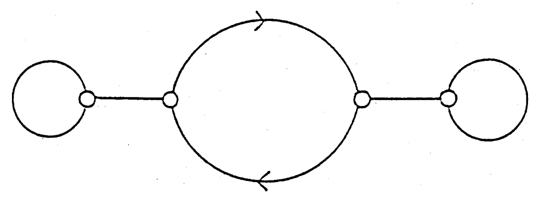

Figure 7.

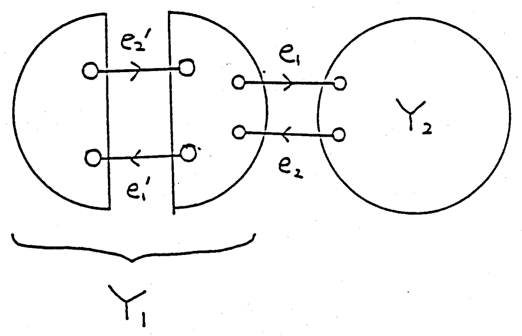

Figure 8.

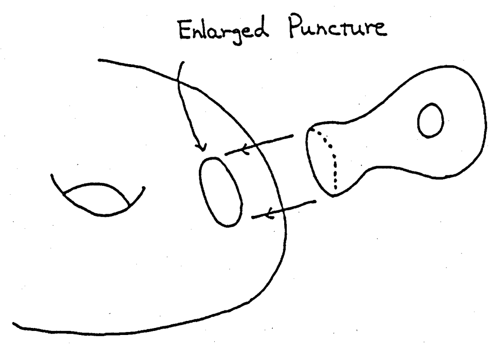

Figure 9.