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Parabolic Variational Inequality for the Cahn-Hilliard Equation with Constraint

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1. Introduction

In this paper we study the Cahn-Hilliard equation with constraint by means of subdifferential operator techniques. Such a state constraint problem was resently proposed by Blowey-Elliott [1] as a model of diffusive phase separation. The questions of the existence, uniqueness and asymptotic behaviour of solutions, treated in [1] for the special case of the deep quench limit, are considered in our paper without such a restriction.

The standard Cahn-Hilliard equation is a model of diffusive phase separation in isothermal binary systems, and in terms of the concentration u of one of the components it has the form

$$u_t + \nu\Delta^2 u - \Delta f(u) = 0 \quad \text{in} \quad Q_T = (0,T)\times\Omega. \tag{1.1}$$

Here Ω is a bounded domain in $\mathbf{R}^N, N \geq 1$, with a smooth boundary $\Gamma = \partial\Omega$, ν is a positive constant related to the surface tension, $f(u)$ corresponds to the volumetric part of the chemical potential difference between components and is given by

$$f(u) = F'(u), \tag{1.2}$$

where $F(u)$ is a homogeneous (volumetric) free energy parametrized by temperature θ, with the characteristic double-well form for θ below the critical temperature θ_c. Usually the free energy is approximated by polynomials $F:\ \mathbf{R}\to\mathbf{R}$, e.g. in the simplest case by quartic polynomial

$$F(u) = F_o(\theta) + \alpha_2(\theta-\theta_c)u^2 + \alpha_4 u^4 \tag{1.3}$$

with constants $\alpha_2, \alpha_4 > 0$ and a given function $F_o(\theta)$ of temperature. To preserve an explicit physical sense, the state variable u often is subject to some constraints, e.g. in the case of concentration natural limitation is

$$0 \leq u \leq 1. \tag{1.4}$$

Then the free energy $F(u)$ can be assumed in the form of the so-called regular solution model

$$F(u) = F_o(\theta) + \alpha_o\theta[u\log u + (1-u)\log(1-u)] + \alpha_1(\theta-\theta_c)u(u-1) \tag{1.5}$$

with a function $F_o(\theta)$ and positive constants α_o, α_1. The corresponding form of the chemical potential $f(u)$ is shown in Fig. 1. Moreover, as the deep quench limit of (1.5), i.e. as the

(b)

$$tX(t,v(t)) + \int_0^t \tau|v'(\tau)|^2_{V^*}d\tau \le \int_0^t \{\tau|\alpha'(\tau)| + X(\tau,v(\tau))\}d\tau \cdot \exp(\int_0^t |\alpha'(\tau)|d\tau)$$
$$\text{for all } t > 0,$$

and

$$X(t,v(t)) + \int_s^t |v'|^2_{V^*}d\tau \le \{X(s,v(s)) + \int_s^t |\alpha'(\tau)|d\tau\} \cdot \exp(\int_s^t |\alpha'(\tau)|d\tau) \qquad (2.1)$$
$$\text{for all } 0 < s < t.$$

In particular, if $v_o \in D$, then (2.1) holds for $0 = s < t$, too.

The third theorem is concerned with the large time behaviour of the solution $v(t)$ of (VI).

Theorem 2.3. *In addition to the assumptions $(\varphi 1)-(\varphi 3)$ and (p) suppose that $\alpha' \in L^1(\mathbf{R}_+)$, and*

$(\varphi 4)$ φ^t converges to a proper l.s.c. convex function φ^∞ on H in the sense of Mosco [11] as $t \to \infty$, i.e.
(M1) for any $z \in D(\varphi^\infty)$ there exists a function $w : \mathbf{R}_+ \to H$ such that $w(t) \to z$ in H and $\varphi^t(w(t)) \to \varphi^\infty(z)$ as $t \to \infty$;
(M2) if $w : \mathbf{R}_+ \to H$ and $w(t) \to z$ weakly in H as $t \to \infty$, then $\liminf_{t\to\infty} \varphi^t(w(t)) \ge \varphi^\infty(z)$.

Let v be the solution of (VI) on $\mathbf{R}_+$ associated with initial datum $v_o \in D_*$, and denote by $\omega(v_o)$ the ω-limit set of $v(t)$ in H as $t \to \infty$, i.e. $\omega(v_o) := \{z \in H;\ v(t_n) \to z$ in H for some t_n with $t_n \to \infty\}$. Then $\omega(v_o) \ne \emptyset$ and

$$\partial\varphi^\infty(v_\infty) + p(v_\infty) \ni 0 \qquad \text{for all } v_\infty \in \omega(v_o).$$

Finally we give a result on the continuous dependence of solutions of (VI) upon the data v_o, $\{\varphi^t\}$ and $p(\cdot)$.

Theorem 2.4. *Let $\{\varphi^t_n\}$ be a sequence of families of proper l.s.c. convex functions on H such that conditions $(\varphi 1) - (\varphi 3)$ are satisfied for common positive constants C_o, C_1 and a common function $\alpha \in W^{1,1}_{loc}(\mathbf{R}_+)$. Also, let p_n be a sequence of Lipschitz continuous operators in H such that condition (p) is satisfied for a common Lipschitz constant $L_o > 0$ and a non-negative C^1-function P_n on H. Suppose that for each $t \le 0$, φ^t_n converges to φ^t on H in the sense of Mosco as $n \to \infty$, i.e.*

(m1) for any $z \in D$, there exists $\{z_n\} \subset H$ such that $z_n \in D_n$ $(= D(\varphi^t_n))$, $z_n \to z$ in H and $\varphi^t_n(z_n) \to \varphi^t(z)$ as $n \to \infty$;

(m2) if $z_n \in H$ and $z_n \to z$ weakly in H as $n \to \infty$, then $\liminf_{n\to\infty} \varphi^t_n(z_n) \ge \varphi^t(z)$.

Furthermore suppose that for each $z \in H$,

$$p_n(z) \to p(z) \quad \text{in } H, \qquad P_n(z) \to P(z) \quad \text{as } n \to \infty.$$

The cases (1.3),(1.5) and (1.6) of free energies can be written in the form (1.7) with appropriate functions $\hat{\beta}$ and $\hat{g}$, and these special cases have been studied by Blowey-Elliott [1] and Elliott-Luckhaus [5].

2. Abstract results

We shall study evolution system (1.8)-(1.10) in an abstract framework.

Let H and V be (real) Hilbert spaces such that V is densely and compactly embedded in H. V^* will be the dual of V. Then, identifying H with its dual, we have

$$V \subset H \subset V^*$$

with dense and compact injections. Further, let J^* be the duality mapping from V^* onto V, and for $t \in \mathbf{R}_+ = [0,\infty)$, let $\varphi^t(\cdot)$ be a proper, l.s.c., non-negative and convex function on H. We shall consider the following problem (VI):

$$\begin{cases} J^*(v'(t)) + \partial\varphi^t(v(t)) + p(v(t)) \ni 0 & \text{in } H,\ t > 0, \\ v(0) = v_o, \end{cases}$$

where $v' = (\frac{d}{dt})v$, $\partial\varphi^t$ is the subdifferential of φ^t in H; $p(\cdot)\ :\ H \to H$ is a Lipschitz continuous operator and v_o a given initial datum.

When it is necessary to indicate the data φ^t, p and v_o explicitly, (VI) is denoted by (VI;φ^t, p, v_o).

Throughout this paper we use the following notations:

$(\cdot,\cdot)$: the inner product in H;

$\langle\cdot,\cdot\rangle$: the duality pairing between V^* and V;

$|\cdot|_W$: the norm in W for any normed space W;

J: the duality mapping from V onto V^*, hence $J^* = J^{-1}$.

We use some basic notions and results about monotone operators and subdifferentials of convex functions; for details we refer to Brézis [2] and Lions [10].

We shall discuss (VI)=(VI;φ^t, p, v_o) under the following additional hypotheses:

(φ1) The effective domain $D(\varphi^t)$ $(= \{z \in H; \varphi^t(z) < \infty\})$ of φ^t is independent of $t \in \mathbf{R}_+$, $D := D(\varphi^t) \subset V$ and

$$\varphi^t(z) \geq C_o|z|_V^2 \quad \text{for all } z \in V \text{ and all } t \in \mathbf{R}_+,$$

where C_o is a positive constant.

(φ2) $(z_1^* - z_2^*, z_1 - z_2) \geq C_1|z_1 - z_2|_V^2$ for all $z_i \in D$, $z_i^* \in \partial\varphi^t(z_i), i = 1,2$, and all $t \in \mathbf{R}_+$, where C_1 is a positive constant.

(φ3) There is a function $\alpha \in W_{loc}^{1,1}(\mathbf{R}_+)$ such that

$$\varphi^t(z) - \varphi^s(z) \leq |\alpha(t) - \alpha(s)|(1 + \varphi^s(z))$$

for all $z \in D$ and $s,\ t \in \mathbf{R}_+$ with $s \leq t$.

(p) p is a Lipschitz continuous operator in H and there is a non-negative C^1-function $P : H \to \mathbf{R}$ whose gradient coincides with p, i.e. $p = \nabla P$; hence

$$\frac{d}{dt}P(w(t)) = (p(w(t)), w'(t)) \quad \text{for } a.e.\ t \in \mathbf{R}, \text{ if } w \in W^{1,2}_{loc}\mathbf{R}_+; H).$$

We now introduce a notion of the solution in a weak sense to problem (VI).

Definition 2.1. (i) Let $0 < T < \infty$. Then a function $v : [0,T] \to H$ is called a solution of (VI) on $[0,T]$, if $v \in L^2(0,T;V) \cap C([0,T];V^*)$, $v' \in L^2_{loc}((0,T];V^*)$, $v(0) = v_o, \varphi^{(\cdot)}(v) \in L^1(0,T)$ and

$$-J^*(v'(t)) - p(v(t)) \in \partial\varphi^t(v(t)) \qquad \text{for } a.e.\ t \in [0,T].$$

(ii) A function $v : \mathbf{R}_+ \to H$ is called a solution of (VI) on $\mathbf{R}_+$, if the restriction of v to $[0,T]$ is a solution of (VI) on $[0,T]$ for every finite $T > 0$.

Our results for (VI) are given as follows.

Theorem 2.1. *Assume that* $(\varphi 1)-(\varphi 3)$ *and* (p) *are satisfied. Let* T *be any positive number. Then the following two statements (a) and (b) hold:*

(a) If v_o *is given in the closure* D_* *of* D *in* V^**, then (VI) has one and only one solution* v *on* $[0,T]$ *such that*

$$t^{\frac{1}{2}}v' \in L^2(0,T;V^*), \qquad \sup_{0<t\leq T} t\varphi^t(v(t)) < \infty.$$

(b) If $v_o \in D$*, then the solution* v *of (VI) on* $[0,T]$ *satisfies that*

$$v' \in L^2(0,T;V^*), \qquad \sup_{0\leq t\leq T} \varphi^t(v(t)) < \infty;$$

hence $v \in C([0,T];H)$.

The second theorem is concerned with the energy inequality for (VI).

Theorem 2.2. *Assume that* $(\varphi 1)-(\varphi 3)$ *and (p) hold. Let* v *be the solution of (VI) on* $\mathbf{R}_+$ *associated with initial datum* $v_o \in D_*$*. Define*

$$X(t,z) = \varphi^t(z) + P(z) \qquad \text{for } z \in D \text{ and } t \in \mathbf{R}_+.$$

Then: (a)

$$\sup_{0\leq\tau\leq t}|v(\tau)|^2_{V^*} + \int_0^t \varphi^\tau(v(\tau))d\tau \leq M_o\{|v_o|^2_{V^*} + \int_0^t \varphi^\tau(z)d\tau + (|z|^2_H + 1)\}e^{M_o t}$$

$$\text{for all } z \in D \text{ and } t > 0,$$

where M_o *is a positive constant dependent only on* C_o *in* $(\varphi 1)$*, the Lipschitz constant* L_p *of* $p(\cdot)$ *and the value* $|p(0)|_H$.

limit of (1.5) as $\theta \to 0$, the non-smooth free energy

$$F(u) = \begin{cases} F_o(\theta) + \alpha_1\theta_c u(1-u) & \text{if } 0 \leq u \leq 1, \\ \infty & \text{otherwise} \end{cases} \tag{1.6}$$

is obtained (see Fig. 2); the constraint (1.4) is included in formula (1.6). This type of free energy (1.6) was introduced by Oono-Puri [12], and the corresponding Cahn-Hilliard equation was numerically studied by them; subsequently this model was analized theoretically, too, by Blowey-Elliott [1].

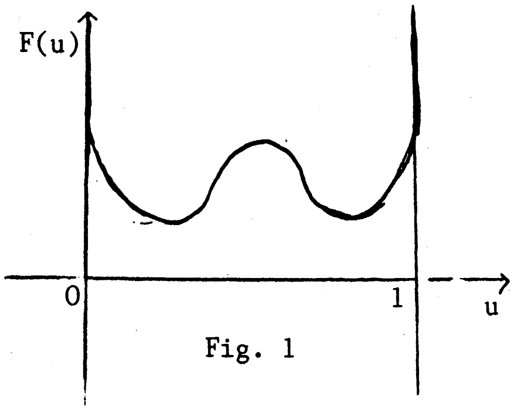

Fig. 1

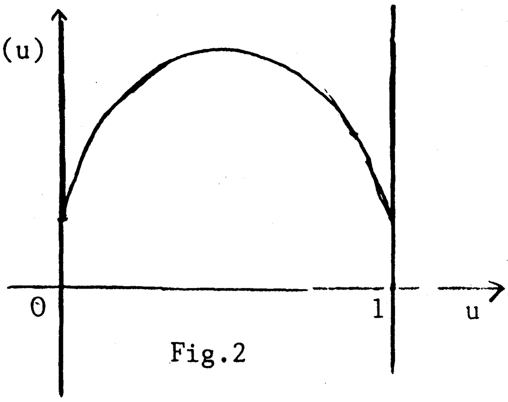

Fig.2

For generality we propose in this paper the representation of (possibly non-smooth) free energy in the form

$$F(u) = \hat{\beta}(u) + \hat{g}(u), \tag{1.7}$$

where $\hat{\beta}$ is a proper, l.s.c. and convex function on $\mathbf{R}$ and $\hat{g}$ is a non-negative function of C^1-class on $\mathbf{R}$ with Lipschitz continuous derivative $g = \hat{g}'$ on $\mathbf{R}$. In such a non-smooth case of free energy functionals, the formula (1.2), giving the volumetric part $f(u)$ of the chemical potential difference, does not make sense any longer. Therefore, following the idea in [1], we introduce a generalized notion of chemical potential which is represented in terms of the multivalued function

$$F(u) = \{\xi + g(u);\ \xi \in \beta(u)\},$$

where β is the subdifferential of $\hat{\beta}$ in $\mathbf{R}$. Then the Cahn-Hilliard equation (1.1) is extended to the general form

$$u_t + \nu\Delta^2 u - \Delta(\xi + g(u)) = 0, \qquad \xi \in \beta(u) \qquad \text{in } Q_T. \tag{1.8}$$

Equation (1.8) is to be satisfied together with boundary conditions

$$\frac{\partial u}{\partial n} = 0, \qquad \frac{\partial}{\partial n}(\nu\Delta u + \xi + g(u)) = 0 \qquad \text{on } \Sigma_T := (0,T) \times \gamma \tag{1.9}$$

and initial condition

$$u(0,\cdot) = u_o \qquad \text{in } \Omega, \tag{1.10}$$

where u_o is a given initial datum, and $\frac{\partial}{\partial n}$ denotes the outward normal derivative on Γ.

Let $\{v_{on}\}$ be a sequence in V^* such that $v_{on} \in D_{n*}$ (=the closure of D_n in V^*), $v_o \in D_*$ and $v_{on} \to v_o$ in V^* as $n \to \infty$. Then the solution v_n of $(VI)_n := (VI; \varphi_n^t, p_n, v_{on})$ converges to the solution v of $(VI) := (VI; \varphi^t, p, v_o)$ as $n \to \infty$ in the following sense: for every finite $T > 0$ and every $0 < \delta < T$,

$$v_n \to v \qquad \text{in } C([0,T]; V^*),$$

$$t^{\frac{1}{2}} v_n' \to t^{\frac{1}{2}} v' \qquad \text{weakly in } L^2(0,T; V^*),$$

$$v_n \to v \qquad \text{in } C([\delta,T]; H) \text{ and weakly}^* \text{ in } L^\infty(\delta,T; V),$$

as $n \to \infty$.

3. Sketch of the proofs

We sketch the proofs of the main theorems.

(1) (Uniqueness) Let v_i, $i = 1, 2$, be two solutions of (VI) on $[0,T]$ and put $v := v_1 - v_2$. Multiply the difference of two equations, which v_1 and v_2 satisfy, by v, and then use the inequality

$$|z|_H^2 \le \varepsilon |z|_V^2 + C(\varepsilon)|z|_{V^*}^2 \qquad \text{for all } z \in V,$$

where ε is an arbitrary positive number and $C(\varepsilon)$ is a suitable positive constant dependent only on ε. Then we have an inequality of the form

$$\frac{1}{2}\frac{d}{dt}|v(t)|_{V^*}^2 + k_1 |v(t)|_V^2 \le k_2 |v(t)|_{V^*}^2 \qquad \text{for } a.e.\ t \in [0,T],$$

where k_1 and k_2 are some positive constants. Therefore, Gronwall's lemma implies that $v = 0$.

(2) (Approximate problems) Let $v_o \in D$ and μ be any parameter in $(0,1]$. Consider the following approximate problem $(VI)_\mu$ for (VI):

$$\begin{cases} (J^* + \mu I)(v_\mu'(t)) + \partial\varphi^t(v_\mu(t)) + p(v_\mu(t)) \ni 0 & \text{in } H, \quad 0 < t < T, \\ v_\mu(0) = v_o. \end{cases}$$

By making use of the results in [9] this problem $(VI)_\mu$ has one only one solution $v_\mu \in W^{1,2}(0,T; H) \cap L^\infty(0,T; V)$. Also, multiplying the equation of $(VI)_\mu$ by v_μ, v_μ' and tv_μ', we have similar estimates as those in Theorem 2.2.

(3) (Existence and estimates for (VI)) In the case when $v_o \in D$, by the standard monotonicity and compactness methods we can prove that the solution v_μ tends to the solution v of (VI) as $\mu \to 0$ in the sense that

$$v_\mu \to v \qquad \text{in } C([0,T]; H) \text{ and weakly}^* \text{in } L^\infty(0,T; V),$$

$$v_\mu' \to v' \qquad \text{weakly in } L^2(0,T; V^*),$$

$$\mu v_\mu' \to 0 \qquad \text{in } L^2(0,T; H).$$

Moreover we have the estimates in Theorem 2.2 for v. In the case when $v_o \in D_*$, it is enough to approximate v_o by a sequence $\{v_{on}\} \subset D$ and to see the convergence of the solution v_n associated with initial datum v_{on}.

(4) (Proof of Theorem 2.3) From the energy estimates which were obtained in Theorem 2.2, it follows that $v' \in L^2(1,\infty;V^*)$ and $v \in L^\infty(1,\infty;V)$; hence Theorem 2.3 holds.

(5) (Proof of Theorem 2.4) Under the assumptions of Theorem 2.4, we see from the energy estimates for v_n that $\{v_n\}$ is bounded in $C([0,T];H) \cap L^2(0,T;V) \cap L^\infty_{loc}((0,T];V) \cap W^{1,2}_{loc}((0,T];V^*)$. Hence by the usual monotonicity and compactness argument we have the assertions of Theorem 2.4.

4. Application to the Cahn-Hilliard equation with constraint

We denote by (CHC) the Cahn-Hilliard equation with constraint (1.8)-(1.10). Here we suppose that

(A1) $g : \mathbf{R} \to \mathbf{R}$ is a Lipschitz continuous function with a non-negative primitive $\hat{g}$ on $\mathbf{R}$.

(A2) β is a maximal monotone graph in $\mathbf{R}\times\mathbf{R}$ such that $0 \in R(\beta)$ and $int.D(\beta) \neq \emptyset$; we may assume that there is a non-negative proper l.s.c. convex function on $\mathbf{R}$ such that its subdifferential $\partial\hat{\beta}$ coincides with β in $\mathbf{R}$.

(A3) $u_o \in L^2(\Omega)$, $u_o(x) \in \overline{D(\beta)}$ for $a.e.\ x \in \Omega$.

Definition 4.1. Let $0 < T < \infty$. Then $u : [0,T] \to H$ is called a (weak) solution of (CHC) on $[0,T]$, if u satisfies the following properties (w1)-(w3):

(w1) $u \in L^2(0,T;H^1(\Omega)) \cap C([0,T];(H^1(\Omega))^*) \cap L^2_{loc}((0,T];H^2(\Omega)) \cap L^\infty_{loc}((0,T];H^1(\Omega)) \cap W^{1,2}_{loc}((0,T];(H^1(\Omega))^*)$ and $\hat{\beta}(u) \in L^1(Q_T)$;

(w2) $u(0,\cdot) = u_o$ $a.e.$ in Σ_T;

(w3) there is a function $\xi : [0,T] \to L^2(\Omega)$ such that

$$\xi \in L^2_{loc}((0,T];L^2(\Omega)), \qquad \xi \in \beta(u) \qquad a.e. \text{ in } Q_T$$

and

$$\frac{d}{dt}(u(t),\eta) + \nu(\Delta u(t),\Delta\eta) - (\xi(t) + g(u(t)),\Delta\eta) = 0$$

for all $\eta \in H^2(\Omega)$ with $\frac{\partial\eta}{\partial n}$ $a.e.$ on Γ, and for $a.e.\ t \in [0,T]$.

Applying Theorems 2.1-2.4 to (CHC) we have:

Theorem 4.1. *Assume that (A1)-(A3) hold and*

$$m := \frac{1}{|\Omega|}\int_\Omega u_o dx \in int.D(\beta).$$

Then for every finite $T > 0$ problem (CHC) has one and only one solution u on $[0,T]$, and the following statements (a) and (b) hold:

(a) $u \in L^\infty(\delta,\infty;H^1(\Omega))$, $u'(\delta,\infty;(H^1(\Omega))^*)$ for every $\delta > 0$, and hence the ω-limit set $\omega(u_o) := \{z \in L^2(\Omega); u(t_n) \to z$ in $L^2(\Omega)$ for some t_n with $t_n \to \infty\}$ is non-empty;

(b) $\omega(u_o) \subset H^2(\Omega)$, and any $u_\infty \in \omega(u_o)$ with some $\mu_\infty \in \mathbf{R}$ and $\xi_\infty \in L^2(\Omega)$ solves the following stationary problem

$$-\nu\Delta u_\infty + \xi_\infty + g(u_\infty) = \mu_\infty \qquad in\ \Omega, \qquad \xi_\infty \in \beta(u_\infty) \qquad a.e. \in \Omega,$$

$$\frac{\partial u_\infty}{\partial n} = 0 \qquad a.e.\ on\ \Gamma, \qquad \frac{1}{|\Omega|}\int_\Omega u_\infty dx = m.$$

Now, let us reformulate (CHC) as an evolution problem of the form (VI) in the space

$$H := \{z \in L^2(\Omega);; \int_\Omega zdx = 0\} \quad \text{with } |z|_H = |z|_{L^2(\Omega)};$$

put also

$$V := H \cap H^1(\Omega) \quad \text{with } |z|_V = |\nabla z|_{L^2(\Omega)}.$$

For this purpose we consider the data $\varphi^t = \varphi$, $p(\cdot)$ and v_o as follows:.

$$\varphi(z) := \begin{cases} \frac{\nu}{2}|\nabla z|^2_{L^2(\Omega)} + \int_\Omega \hat\beta(z+m)dx & \text{if } z \in V, \\ \infty & \text{otherwise,} \end{cases}$$

where $m = \frac{1}{|\Omega|}\int_\Omega u_o dx$;

$$p(z) := \pi(g(z+m)), \quad P(z) := \int_\Omega \hat g(z+m)dx, \quad z \in H;$$

$$v_o := u_o - m.$$

By virtue of the following lemma, problems (CHC) and (VI) associated with the data defined above are equivalent.

Lemma 4.1. *Let $\ell \in L^2(\Omega)$. Then $\pi(\ell) \in \partial\varphi(z)$ if and only if $z_m = z + m$ satisfies that there are $\mu_m \in \mathbf{R}$ and $\xi_m \in L^2(\Omega)$ such that*

$$-\nu\Delta z_m + \xi_m = \ell + \mu_m \qquad in\ L^2(\Omega), \qquad \xi_m \in \beta(z_m) \qquad a.e.\ in\ \Omega,$$

$$\frac{\partial z_m}{\partial n} = 0 \qquad a.e.\ on\ \Gamma, \qquad \frac{1}{|\Omega|}\int_\Omega z_m dx = m;$$

hence $z_m \in H^2(\Omega)$. Moreover, μ_m can be chosen so that

$$|\mu_m| \le M(1 + |\ell|_{L^2(\Omega)}),$$

where $M > 0$ is a certain constant dependent only upon β and m, and z_m satisfies that

$$\nu|\Delta z_m|_{L^2(\Omega)} \le |\ell|_{L^2(\Omega)} + |\mu_m||\Omega|^{\frac{1}{2}}.$$

By Theorem 2.1 problem (VI) has one and only one solution v. Moreover we see from the above lemma that the function $u := v + m$ is the unique solution of (CHC), and from Theorems 2.2 and 2.3 that (a) and (b) hold.

When the state constraint $\xi \in \beta(u)$ is not imposed, the system (1.8)-(1.10) becomes the standard Cahn-Hilliard problem. For such a problem various existence, uniqueness and asymptotic results have been establised; see e.g. Elliott [3], Elliott-Zheng [6] and Zheng [15]. For related results in abstract setting we refer to Temam [13] and von Wahl [14]. For the Cahn-Hilliard models with non-smooth free energy functionals we refer to Elliott-Mikelic [4]. The structure of stationary solutions corresponding to the Cahn-Hilliard equation was studied by Gurtin-Matano [7]; their analysis covers also some cases of free energy $F(u)$ with infinite walls.

Finally we give examples of β and the corresponding Cahn-Hilliard equations.

Example 4.1. (i) (Logarithmic form) For constants $\alpha_o > 0$ and $\theta > 0$, θ being a parameter,

$$\beta(u) := \beta^{\theta}(u) = \begin{cases} \{\alpha_o \theta \log \frac{u}{1-u}\} & \text{for } 0 < u < 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Gien any Lipschitz continuous function $\bar{g}$ on $[0,1]$, we extend it to a Lipschitz continuous function g, with support in $[-1,2]$, on the whole line $\mathbf{R}$.

(ii) (The limit of β^{θ} as $\theta \to 0$)

$$\beta(u) := \beta^{0}(u) = \begin{cases} [0,\infty) & \text{if } u = 1, \\ \{0\} & \text{if } 0 < u < 1, \\ (-\infty, 0] & \text{if } u = 0, \\ \emptyset & \text{otherwise,} \end{cases}$$

and g is the same as in (i).

Example 4.2. Denote by $(\mathrm{CHC})_\theta$ and $(\mathrm{CHC})_0$ the Cahn-Hilliard equations (CHC) associated with $\beta = \beta^{\theta}$ and $\beta = \beta^{0}$, respectibely. Then, by the theorems proved above, $(\mathrm{CHC})_\theta$ and $(\mathrm{CHC})_0$ have the unique solutions u^{θ} and u^{0}, respectively, and moreover $u^{\theta} \to u^{0}$ as $\theta \to 0$ in the similar sense as Theorem 2.4.

References

[1] J. F. Blowey and C. M. Elliott, The Cahn-Hilliard gradient theory for phase separation with non-smooth free energy, Part I: Mathematical analysis, European J. Appl. Math. 2(1991), 233-280.

[2] H. Brézis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, 1973.

[3] C. M. Elliott, The Cahn-Hilliard model for the kinetics of phase separation, in *Mathematical Models for Phase Change Problems* J. F. Rodrigues ed., ISNM 88, Birkhäuser, Basel, 1989, pp.35-73.

[4] C. M. Elliott and A. Mikelic, Existence for the Cahn-Hilliard phase separation model with a non-differentiable energy, Ann. Mat. pura appl. 158(1991), 181-203.

[5] C. M. Elliott and S. Luckhaus, A generalized diffusion equation for phase separation of a multi-component mixture with interfacial free energy, preprint.

[6] C. M. Elliott and S. Zheng, On the Cahn-Hilliard equation, Arch. Rat. Mech. Anal. 96(1986), 339-357.

[7] M. E. Gurtin and H. Matano, On the structure of equilibrium phase transitions within the gradient theory of fluids, Quart. Appl. Math. 156(1988), 301-317.

[8] N. Kenmochi, M. Niezgódka and I. Pawlow, Subdifferential operator approach to the Cahn-Hilliard equation with constraint, preprint.

[9] N. Kenmochi and I. Pawlow, A class of nonlinear elliptic-parabolic equations with time-dependent constraints, Nonlinear Anal. TMA 10(1986), 1181-1202.

[10] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod Gauthier- Villars, Paris, 1969.

[11] U. Mosco, Convergence of convex sets and of solutions of variational inequalities, Avdances Math. 3(1969), 510-585.

[12] Y. Oono and S, Puri, Study of the phase separation dynamics by use of call dynamical systems, I. Modelling Phys. Rev. (A) 38(1988), 434-453.

[13] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*,Springer Verlag, Berlin, 1988.

[14] W. von Wahl, On the Cahn-Hilliard equation $u' + \Delta^2 u - \Delta f(u) = 0$, Delft Progress Report 10(1985), 291-310.

[15] S. Zheng, Asymptotic behaviour of the solution to the Cahn- Hilliard equation, Applic. Anal. 23(1986), 165-184.

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