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Nontrivial Solutions of Semilinear Elliptic Equations with Continuous or Discontinuous Nonlinearities

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1 Introduction.

We begin this paper by considering the existence of nontrivial solutions of the boundary value problem of the form

$$-\Delta u = g(u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \tag{1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbf{R}^n$ and g is a real-valued continuous function on $\mathbf{R}$ such that $g(0) = 0$.

Let $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ denote the eigenvalues of the self-adjoint realization in $L^2(\Omega)$ of $-\Delta$ with the Dirichlet boundary condition. Many authors have studied the existence of nontrivial solutions of the problem (1) when $g(t)/t$ crosses finitely many eigenvalues of $-\Delta$ as t varies from $-\infty$ to $+\infty$. Amann and Zehnder [2] proved by generalized Morse theory that (1) has at least one nontrivial solution if $g \in C^2(R, R)$ satisfies

$$\sup_{t\in R} |g'(t)| < \infty$$

and

$$\lambda_{k-1} \leq g'(0) < \lambda_k \leq \lambda_m < a_* \leq a^* < \lambda_{m+1} \quad \text{for some } m, k \geq 1$$

where

$$a_* = \liminf_{|t|\to\infty} \frac{g(t)}{t} \quad \text{and} \quad a^* = \limsup_{|t|\to\infty} \frac{g(t)}{t}.$$

On the other hand, using Leray-Schauder deree, Hirano [6] established the existence of one nontrivial solution of (1) under

$$\lambda_{k-1} < b_* \leq b^* < \lambda_k \leq \lambda_m < a_* \leq a^* < \lambda_{m+1} \quad \text{for some } k, m \geq 1,$$

where a_* and a^* are as above,

$$b_* = \liminf_{|t|\to 0} \frac{g(t)}{t} \quad \text{and} \quad b^* = \limsup_{|t|\to 0} \frac{g(t)}{t},$$

without any assumptions of differentiability of g. Hirano's result cannot be applied in the case of resonance at 0. We obtain the existence of one nontrivial solution of (1) under weaker conditions of g near 0 which contain the resonance case at 0 (Theorem 1). Moreover, there are no results for g with $b_* > a^*$ in [6]. We deal with such a function g in Theorem 2.

It is seen in §3, that the assertions of Theorem 1 and Theorem 2 remain valid in the case that g is a piecewise continuous function on any bounded closed interval of R (may be discontinuous at 0), that is,

$$-\Delta u \in [\underline{g}(u), \overline{g}(u)] \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \tag{2}$$

where

$$\underline{g}(t) = \liminf_{s\to t} g(s) \quad \text{and} \quad \overline{g}(t) = \limsup_{s\to t} g(s).$$

2 The case that g is continuous.

Our purpose in this section is to prove the following two theorems.

Theorem 1. Let $g : R \to R$ be a continuous function with $g(0) = 0$. If g satisfies the following condition

$$b^* < \lambda_m < a_* \leq \overline{a} < \lambda_{m+1} \tag{3}$$

for some $m \geq 1$, where

$$a_* = \liminf_{|t|\to\infty} \frac{g(t)}{t}, \quad \overline{a} = \sup_{t\neq 0} \frac{g(t)}{t} \quad \text{and} \quad b^* = \limsup_{|t|\to 0} \frac{g(t)}{t},$$

then the equation (1) has at least one nontrivial solution in $H^2(\Omega) \cap H_0^1(\Omega)$.

Theorem 2. Let $g : R \to R$ be a continuous function with $g(0) = 0$. If g satisfies that

$$\lambda_{k-1} < \underline{a} \leq a^* < \lambda_k < b_* \tag{4}$$

for some $k \geq 1$, where

$$\underline{a} = \inf_{t\neq 0} \frac{g(t)}{t}, \quad a^* = \limsup_{|t|\to\infty} \frac{g(t)}{t} \quad \text{and} \quad b_* = \liminf_{|t|\to 0} \frac{g(t)}{t},$$

then there exists at least one nontrivial solution of (1) in $H^2(\Omega) \cap H_0^1(\Omega)$.

In the following, we write H, H^{-1} and L^2 instead of $H_0^1(\Omega), H^{-1}(\Omega)$ and $L^2(\Omega)$, respectively. We denote by $\|\cdot\|, \|\cdot\|_*$ and $|\cdot|$ the norms of H, H^{-1} and L^2, respectively. The notation $|\cdot|$ is often used for the absolute value of a real number without notice if there is no possibility of their confusion. The pairing between H and H^{-1} is denoted by $\langle\cdot,\cdot\rangle$. We take $k \in Z^+$ with $b^* < \lambda_k \leq \lambda_m$ if g satisfies the condition (3), and $m \in Z^+$ with $\lambda_k \leq \lambda_m < b_*$ if g satisfies the condition (4). Let H_1, H_2 and H_3 be closed subspaces of H spanned by the eigenfunctions corresponding to the eigenvalues $\{\lambda_{m+1}, \lambda_{m+2}, \cdots\}, \{\lambda_k, \cdots, \lambda_m\}$ and $\{\lambda_1, \lambda_2, \cdots, \lambda_{k-1}\}$, respectively (We consider $\lambda_0 = 0$ and $H_3 = \{0\}$ if $k = 1$.).

For $i = 1, 2, 3$, P_i means the projection from H onto H_i. Define a real valued function f on H by

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega \int_0^{u(x)} g(t)dtdx \quad \text{for } u \in H. \tag{5}$$

Then we have

$$\langle f'(u), v\rangle = \langle -\Delta u - g(u), v\rangle \quad \text{for any } u, v \in H,$$

and hence weak solutions of (1) coincide with critical points of f.

We need the following two lemmas in order to prove our theorems.

Lemma 1. If g satisfies the conditions (3) or (4), then the Palais-Smale condition holds for the function f defined by (5), that is, for any sequence $\{u_n\}$ in H such that $\{f(u_n)\}$ is bounded and $\|f'(u_n)\|_* \to 0$, there exists a convergent subsequence of $\{u_n\}$.

Proof. Let $\{u_n\}$ in H satisfy that $\{f(u_n)\}$ is bounded and $\|f'(u_n)\|_* = \| -\Delta u - g(u)\|_* \to 0$. For each u_n, we put $v_n = P_1 u_n$, $w_n = P_2 u_n$ and $z_n = P_3 u_n$. Then

$$\begin{aligned} &\langle -\Delta u_n - g(u_n), v_n - (w_n + z_n)\rangle \\ =\ & \|v_n\|^2 - \|w_n + z_n\|^2 - \int_\Omega g(u_n)(v_n - (w_n + z_n))dx. \end{aligned}$$

Suppose that g satisfies the condition (3). Then there exist positive numbers α with $\lambda_m < \alpha < a_*$ and ρ such that $\alpha \leq \frac{g(t)}{t} \leq \bar{a}$ for all $t \in R$ with $|t| \geq \rho$. From the continuity of g, for some constant K, we have $|g(t)| \leq K$ for all t with $|t| < \rho$. If $|u_n(x)| \geq \rho$, then

$$\alpha \leq \frac{g(u_n(x))}{v_n(x) + w_n(x) + z_n(x)} \leq \bar{a}. \tag{6}$$

If $|u_n(x)| < \rho$, then

$$|v_n(x)|^2 - |w_n(x) + z_n(x)|^2 \geq -\rho(|v_n(x)| + |w_n(x) + z_n(x)|).$$

We set

$$
\begin{aligned}
A &= \{x \in \Omega : |v_n(x)| > |w_n(x) + z_n(x)|\}, \\
A_1 &= \{x \in A : |u_n(x)| \geq \rho\} \quad \text{and } A_2 = \{x \in A : |u_n(x)| < \rho\}.
\end{aligned}
$$

By the second inequality in (6), we have

$$
\begin{aligned}
&\int_A g(u_n)(v_n - (w_n + z_n))dx \\
&\leq \int_{A_1} \overline{a}(|v_n|^2 - |w_n + z_n|^2)dx + \int_{A_2} K(|v_n| + |w_n + z_n|)dx \\
&\leq \int_A (\overline{a}|v_n|^2 - \alpha|w_n + z_n|^2)dx + \int_{A_2} (\overline{a}\rho + K)(|v_n| + |w_n + z_n|)dx.
\end{aligned}
$$

Putting

$$
\begin{aligned}
B &= \{x \in \Omega : |v_n(x)| \leq |w_n(x) + z_n(x)|\}, \\
B_1 &= \{x \in B : |u_n(x)| \geq \rho\} \quad \text{and } B_2 = \{x \in B : |u_n(x)| < \rho\},
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\int_B g(u_n)(v_n - (w_n + z_n))dx \\
&\leq \int_B (\overline{a}|v_n|^2 - \alpha|w_n + z_n|^2)dx + \int_{B_2} (\overline{a}\rho + K)(|v_n| + |w_n + z_n|)dx
\end{aligned}
$$

from the first inequality in (6). Therfore we have

$$
\begin{aligned}
&\int_\Omega g(u_n)(v_n - (w_n + z_n))dx \\
&\leq \overline{a}|v_n|^2 - \alpha|w_n + z_n|^2 + 2|\Omega|^{1/2}(\overline{a}\rho + K)|u_n|.
\end{aligned}
$$

Thus it holds that

$$
\begin{aligned}
&\langle -\Delta u_n - g(u_n), v_n - (w_n + z_n)\rangle \\
&\geq (1 - \frac{\overline{a}}{\lambda_{m+1}})\|v_n\|^2 + (\frac{\alpha}{\lambda_m} - 1)\|w_n + z_n\|^2 - 2|\Omega|^{1/2}(\overline{a}\rho + K)|u_n| \\
&\geq \omega_1\|u_n\|^2 - \omega_2\|u_n\|
\end{aligned}
$$

for some $\omega_1, \omega_2 > 0$. The assumption $\| -\Delta u_n - g(u_n)\|_* \to 0$ and this inequality imply the boundedness of $\{u_n\}$ in H and hence the existence of a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ which converges weakly to some u in H. Then we have

$$\langle -\Delta u_{n_j} - g(u_{n_j}), u_{n_j} - u\rangle \to 0.$$

Since H is compactly embedded into L^2, $\{u_{n_j}\}$ strongly converges to u in L^2 and $\langle g(u_{n_j}), u_{n_j} - u\rangle \to 0$, so $\langle -\Delta u_{n_j}, u_{n_j} - u\rangle \to 0$. Since $\{-\Delta u_{n_j}\}$ weakly converges to $-\Delta u$ in H^{-1}, we have

$$\lim_{j\to\infty} \|u_{n_j}\|^2 = \lim_{j\to\infty}\langle -\Delta u_{n_j}, u_{n_j} - u\rangle + \lim_{j\to\infty}\langle -\Delta u_{n_j}, u\rangle = \|u\|^2.$$

Thus we obtain the strong convergence of $\{u_{n_j}\}$ in H. The proof is similar in the case that g satisfies the condition (4).

Lemma 2. Under the assumption (3), there exist positive constants c_i ($i = 1,2,3,4$), ε_j ($j = 1,2$) and K such that

i) if $\|P_1 u\| \geq c_1, \|P_2 u\| \leq c_2$ and $\|P_3 u\| \leq c_3$, then $f(u) \geq \varepsilon_1$;

ii) if $\|P_2 u\| \leq c_4$ and $\|P_3 u\| \leq K\|P_2 u\|$, then $f(u) \geq \varepsilon_2 \|P_2 u\|^2$.

Proof. For simplicity, we set $v = P_1 u, w = P_2 u$ and $z = P_3 u$. By $\overline{a} < \lambda_{m+1}$, we have

$$\begin{aligned} f(u) &\geq \frac{1}{2}\|v+w+z\|^2 - \frac{1}{2}\overline{a}|v+w+z|^2 \\ &\geq \frac{1}{2}\{(1-\frac{\overline{a}}{\lambda_{m+1}})\|v\|^2 - (\frac{\overline{a}}{\lambda_k} - 1)\|w\|^2 - (\frac{\overline{a}}{\lambda_1} - 1)\|z\|^2\}, \end{aligned}$$

so there exist positive constants c_i ($i = 1,2,3$) and ε_1 for which the statement i) holds. From $b^* < \lambda_k$, we obtain positive constants δ and α with $\alpha < \lambda_k$ such that $\dfrac{g(t)}{t} \leq \alpha$ for all t with $|t| \le \delta$. In the case that $|v(x)+w(x)+z(x)| \le \delta$, we have

$$\begin{aligned}
&\frac{1}{2}(\lambda_{m+1}|v|^2+\lambda_k|w|^2+\lambda_1|z|^2)-\int_0^{v+w+z} g(t)dt \\
\ge\ & \frac{1}{2}(\lambda_{m+1}-\alpha)|v|^2+\frac{1}{2}(\lambda_k-\alpha)|w|^2+\frac{1}{2}(\lambda_1-\alpha)|z|^2-\alpha(vw+wz+zv) \\
\ge\ & \frac{1}{2}(\lambda_k-\alpha)|w|^2+\frac{1}{2}(\lambda_1-\alpha)|z|^2-\alpha(vw+wz+zv).
\end{aligned}$$

Now, we choose $d>0$ such that

$$(\lambda_{m+1}-\alpha)p^2+2(\lambda_{m+1}-\alpha)pq+(\overline{a}-\alpha)q^2 \le (\lambda_{m+1}-\overline{a})\delta^2$$

for all $p,q \ge 0$ with $p+q \le d$. Moreover we can take $c>0$ such that

$$\sup_{x\in\Omega}(|P_2u(x)|+|P_3u(x)|) \le d$$

if $\|P_2u+P_3u\| \le c$. Let $\|w+z\| \le c$. In the case that $|v(x)+w(x)+z(x)| > \delta$, we have

$$|\int_0^{v+w+z} g(t)dt| \le \frac{1}{2}\overline{a}(v+w+z)^2-\frac{1}{2}(\overline{a}-\alpha)\delta^2$$

and hence

$$\begin{aligned}
&\frac{1}{2}(\lambda_{m+1}|v|^2+\lambda_k|w|^2+\lambda_1|z|^2)-\int_0^{v+w+z} g(t)dt \\
&\ge \frac{1}{2}(\lambda_{m+1}-\overline{a})\{|v|+\frac{\alpha-\overline{a}}{\lambda_{m+1}-\overline{a}}(|w|+|z|)\}^2 \\
&\quad -\frac{\overline{a}-\alpha}{2(\lambda_{m+1}-\overline{a})}\{\lambda_{m+1}-\alpha)|w|^2+2(\lambda_{m+1}-\alpha)|w||z|+(\overline{a}-\alpha)|z|^2-(\lambda_{m+1}-\overline{a})\delta^2\} \\
&\quad +\frac{1}{2}(\lambda_k-\alpha)|w|^2+\frac{1}{2}(\lambda_1-\overline{a})|z|^2-\alpha(vw+wz+zv) \\
&\ge \frac{1}{2}(\lambda_k-\alpha)|w|^2+\frac{1}{2}(\lambda_1-\overline{a})|z|^2-\alpha(vw+wz+zv).
\end{aligned}$$

It follows that

$$\begin{aligned}
f(u)\ &\ge\ \int_\Omega\{\frac{1}{2}(\lambda_{m+1}|v|^2+\lambda_k|w|^2+\lambda_1|z|^2)-\int_0^{v+w+z} g(t)dt\}dx \\
&\ge\ \frac{1}{2}(\lambda_k-\alpha)|w|^2+\frac{1}{2}(\lambda_1-\overline{a})|z|^2 \\
&\ge\ \frac{1}{2}\{\frac{\lambda_k-\alpha}{\lambda_m}\|w\|^2-\frac{\overline{a}-\lambda_1}{\lambda_1}\|z\|^2\}
\end{aligned}$$

if $\|w+z\| \le c$. Taking K, c_4 and ε_2 such that

$$0 < K < \sqrt{\frac{\lambda_1(\lambda_k - \alpha)}{\lambda_m(\overline{a} - \lambda_1)}}, \quad 0 < (1+K)c_4 \le c$$

and

$$0 < \varepsilon_2 < \frac{\lambda_k - \alpha}{2\lambda_m}\left(1 - K^2 \frac{\lambda_m(\overline{a} - \lambda_1)}{\lambda_1(\lambda_k - \alpha)}\right),$$

the statement ii) holds.

We are now ready to prove Theorem 1.

Proof of Theorem 1. By $\lambda_m < a_* \le \overline{a} < \lambda_{m+1}$, there exists $r > 0$ such that $f(w+z) < \inf_{v \in H_1} f(v)$ for all $w \in H_2$ and $z \in H_3$ with $\|w+z\| \ge r$. We define

$$\begin{aligned}\Gamma^* = \{A \subset H : A &\text{ is a compact set such that } \sigma(A) \ni 0 \text{ for any} \\ &\text{continuous mapping } \sigma : A \to H_2 \oplus H_3 \text{ satisfying} \\ &\sigma(u) = u \text{ for all } u \in A \cap S\} \ (\neq \emptyset),\end{aligned}$$

where

$$S = \{w + z : w \in H_2, z \in H_3 \text{ and } \|w+z\| = r\}$$

and

$$c^* = \inf_{A \in \Gamma^*} \max_A f \quad (\ge \inf_{v \in H_1} f(v)\ ).$$

It is easily seen that if $A \in \Gamma^*$ and $\eta : A \to H$ is a continuous mapping such that $\eta(u) = u$ for all $u \in A \cap S$, then $\eta(A) \in \Gamma^*$. Since f satisfies the Palais-Smale condition by Lemma 1, c^* is a critical value of f by a method similar to Rabinowitz's saddle point theorem ([9] and [7]). Assume that 0 is the only critical point of f. Let c_i $(i = 1,2,3,4), \varepsilon_j$ $(j = 1,2)$ and K be positive numbers in Lemma 2. We set

$$U = \{u \in H; \|P_1 u\| < a, \|P_2 u\| < b \text{ and } \|P_3 u\| < \frac{c}{2}\}$$

and

$$V = \{u \in H : \|P_1 u\| < a, \|P_2 u\| < b \text{ and } \|P_3 u\| < c\},$$

where

$$a = c_1, b = \min\{c_2, c_4\} \text{ and } c = \min\{c_3, Kb\}.$$

We may suppose that $r > \sqrt{b^2 + c^2}$ with no loss of generality. Putting $\gamma = \min\{\varepsilon_1, \varepsilon_2 b^2\}$, it follows that $f \geq \gamma$ on $\{u \in H : \|P_1 u\| \geq a, \|P_2 u\| \leq b \text{ and } \|P_3 u\| \leq c\} \cup \{u \in H : \|P_1 u\| \leq a, \|P_2 u\| = b \text{ and } \|P_3 u\| \leq c\}$. From $c^* = 0$, for $0 < \varepsilon < \gamma$, there exists $A \in \Gamma^*$ with $\max_A f < \varepsilon$. Now, we define $T : H \to H$ by

$$T(u) = \begin{cases} u & \text{if } u \notin V \\ \varphi(\|P_3 u\|)(P_1 + P_2)u + P_3 u & \text{if } u \in V, \end{cases}$$

where $\varphi : [0, +\infty) \to [0, 1]$ is defined by

$$\varphi(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{c}{2} \\ \frac{2}{c}t - 1 & \text{if } \frac{c}{2} < t \leq c \\ 1 & \text{if } c < t. \end{cases}$$

Then, T is continuous on $\{u \in H : \|P_1 u\| = a, \|P_2 u\| \leq b \text{ and } \|P_3 u\| \leq c\}^c \cap \{u \in H : \|P_1 u\| \leq a, \|P_2 u\| = b \text{ and } \|P_3 u\| \leq c\}^c$. By $\dim H_2 \neq 0$, we can choose $w_0 \in H_2$ with $0 < \|w_0\| < \dfrac{b}{2}$. Define $\tilde{T} : T(A) \to H$ by

$$\tilde{T}(u) = \begin{cases} u & \text{if } \|P_3 u\| \geq \frac{c}{2} \\ P_1 u + Q((P_2 + P_3)u) & \text{if } \|P_3 u\| < \frac{c}{2}, \end{cases}$$

where $Q((P_2 + P_3)u)$ means the intersection of the half-line $\{t(P_2 + P_3)u + (1 - t)w_0 : t \geq 0\}$ and the relative boundary of $\{w + z : w \in H_2, z \in H_3, \|w\| < b \text{ and } \|z\| < \dfrac{c}{2}\}$ in $H_2 \oplus H_3$. Putting $\sigma = (P_2 + P_3) \circ \tilde{T} \circ T$, σ is a continuous mapping from A into $H_2 \oplus H_3$ such that $\sigma(u) = u$ for all $u \in A \cap S$. Since $f \geq \gamma > \varepsilon$ on $\{u \in H : \|P_1 u\| \geq a, \|P_2 u\| \leq b \text{ and } \|P_3 u\| \leq c\}$, we have $\sigma(A) \not\ni 0$. This is contrary to $A \in \Gamma^*$. This completes the proof.

Next we prove Theorem 2.

Proof of Theorem 2. From $\lambda_{k-1} < \underline{a} \leq a^* < \lambda_k$, we take $r > 0$ largely enough such that $f(z) < \inf_{v \in H_1, w \in H_2} f(v+w)$ for all $z \in H_3$ with $\|z\| \geq r$. We set $B = \{z \in H_3 : \|z\| \leq r\}$ and $S = \{z \in H_3 : \|z\| = r\}$. Define

$$\Gamma = \{g : g \text{ is a continuous mapping from } B \text{ into } H \text{ such that } g(z) = z \text{ for all } z \in S\} \quad (\neq \emptyset)$$

and

$$c = \inf_{g \in \Gamma} \max_{z \in B} f(g(z)) \quad (\geq \inf_{v \in H_1, w \in H_2} f(v+w)\ ).$$

Similarly to the proof of Theorem 1, c is a critical value of f. Now, suppose that f does not have any nonzero critical points in H. From $\underline{a} > \lambda_{k-1}$, if follows that

$$f(z) \leq \frac{1}{2}\|z\|^2 - \frac{1}{2}\lambda_{k-1}|z|^2 \leq 0 \quad \text{for all } z \in H_3.$$

By $b_* > \lambda_m$, there exists $\delta > 0$ such that $\dfrac{g(t)}{t} \geq \lambda_m$ for all t with $|t| \leq \delta$. Then, we obtain $c_1 > 0$ such that $\sup_{x \in \Omega} |w(x) + z(x)| \leq \delta$ if $w \in H_2, z \in H_3$ and $\|w + z\| \leq c_1$. Therefore we have

$$f(w+z) \leq \frac{1}{2}\|w+z\|^2 - \frac{1}{2}\lambda_m|w+z|^2 \leq 0$$

for all $w \in H_2$ and $z \in H_3$ with $\|w+z\| \leq c_1$. We may assume $c_1 < r$ without loss of generality. Choosing $c_2 > 0$ arbitrarily, we put

$$U = \{u \in H : \|P_1 u\| < c_2 \text{ and } \|(P_2 + P_3)u\| < \frac{c_1}{2}\}.$$

Since $\dim H_2 \neq 0$, by an argument similar to the proof of Theorem 1, we can construct a continuous mapping $g : B \to H$ such that $g(z) = z$ for all $z \in S$, $g(B) \cap U = \emptyset$ and

$f(g(z)) \leq 0$ for all $z \in B$. From the well-known deformation lemma, for sufficiently small $\varepsilon_0 > 0$, there exist a continuous mapping $\eta : H \to H$ and a positive number $\varepsilon < \varepsilon_0$ satisfying the following conditions

i) $\eta(u) = u$ if $u \notin f^{-1}([-\varepsilon_0, \varepsilon_0])$;

ii) $\eta(f^{-1}((-\infty, \varepsilon]) \setminus U) \subset f^{-1}((-\infty - \varepsilon])$.

Putting $\tilde{g} = \eta \circ g$, it is clear that $\tilde{g} \in \Gamma$. On the other hand, $\max_{z \in B} f(\tilde{g}(z)) \leq -\varepsilon$ since $g(B) \cap U = \emptyset$. This is contrary to $c = 0$. This completes the proof.

3 The case that g is discontinuous.

In this section, we consider the existence of one nontrivial solution of the equation (2). Let $g : R \to R$ be a piecewise continuous function on any bounded closed interval (may be discontinuous at 0) with $0 \in [\underline{g}(0), \overline{g}(0)]$. Then, it is easily seen that the functional f defined by (5) is locally Lipschitz continuous if g satisfies the conditions (3) or (4). Then, we cannot apply the usual critical point theory for differentiable functionals since f may be nondifferentiable. In order to solve the problem (2), Chang [4] made use of the generalized gradients for locally Lipschitz continuous functionals introduced by Clarke [5]. In fact, it was shown that

$$\partial f(u) \subset -\Delta u - [\underline{g}(u), \overline{g}(u)] \qquad \text{for each } u \in H,$$

where $\partial f(u)$ means the generalized gradient of f at u.

Further, he proved in [4] that the deformation lemma holds in this case. On the other hand, Mizoguchi [8] obtained the existence of one nontrivial solution of (2) under the same conditions as Theorem 1 in [6].

We remark that g is automatically continuous at 0 in [8]. According to the proofs of Thoerem 1 and Theorem 2, we see that the equation (2) has at least one nontrivial solution if the conditions (3) or (4) are assumed.

Theorem 3. Let $g : R \to R$ be a piecewise continuous function on any bounded closed interval with $0 \in [\underline{g}(0), \overline{g}(0)]$. If g satisfies the condition (3), then the equation (2) has at least one nontrivial solution in $H^2(\Omega) \cap H_0^1(\Omega)$.

Theorem 4. Let $g : R \to R$ be a piecewise continuous function on any bounded closed interval with $0 \in [\underline{g}(0), \overline{g}(0)]$. If g satisfies the condition (4), then there exists at least one nontrivial solution of (2) in $H^2(\Omega) \cap H_0^1(\Omega)$.

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