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INFINITESIMAL DEFORMATION OF PRINCIPAL BUNDLES, DETERMINANT BUNDLES AND AFFINE LIE ALGEBRAS

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In the present notes we shall show that affine Lie algebras appear as infinitesimal automorphism groups of the determinant bundles of a family of associated vector bundles of the versal family of principal bundles over a Riemann surface R. More natural but shophistcated approach can be found in [BS] and [T]. In the following we fix a Riemann surface R.

§1 Infinitesimal deformations of principal bundles.

The arguments in this section are valid for all complex manifolds. Let G be a simply connected complex simple algebraic group realized as a closed subgroup of $GL(N, \mathbf{C})$ for a sufficiently large integer N. Let $\pi : P \to R$ be a holomorphic principal G-bundle. Let $R = \cup_{\lambda\in\Lambda} U_\lambda$ be an open covering of the Riemann surface such that the principal bundle $\pi : P \to R$ is trivialized on each U_λ. Then the principal bundle can be determined by transition functions $\{g_{\lambda\mu}\}$ with $g_{\lambda\mu} \in \Gamma(U_{\lambda\mu}, \mathcal{G})$ where $\mathcal{G}$ is the sheaf of germs of holomorphic sections of R to G. The transition functions $g_{\lambda\mu}$ satisfy the relation

$$g_{\lambda\mu} g_{\mu\nu} = g_{\lambda\nu} \quad \text{on } U_{\lambda\mu\nu}.$$

Let ϵ be the dual number, that is $\epsilon \equiv z \mod (z^2)$ in $\mathbf{C}[z]/(z^2)$. To change the structure of the principal bundle $\pi : P \to R$ infinitesimally put

$$\hat{g}_{\lambda\mu} := g_{\lambda\mu}(I + \epsilon h_{\lambda\mu})$$

where I is the identity matrix and $h_{\lambda\mu} \in \Gamma(U_{\lambda\mu}, \underline{\mathfrak{g}})$. Here, $\mathfrak{g}$ is the Lie algebra of the Lie group G realized as a Lie subalgebra of the $N \times N$ matrix algebra $M(N, \mathbf{C})$ and $\underline{\mathfrak{g}}$ is the sheaf of germs of holomorphic sections of R to $\mathfrak{g}$. These new transition functions satisfy the compatibility condition

$$\hat{g}_{\lambda\mu}\hat{g}_{\mu\nu} = \hat{g}_{\lambda\nu} \quad \text{on } U_{\lambda\mu\nu}.$$

The condition can be rewritten in the form

$$h_{\lambda\nu} = g_{\mu\nu}^{-1} h_{\lambda\mu} g_{\mu\nu} + h_{\mu\nu} \tag{1}$$

on $U_{\lambda\nu\mu}$. Let $\underline{ad}(P)$ be the associated vector bundle (adjoint bundle) $P \underset{G}{\times} \mathfrak{g}$ associated with the adjoint representation of G. Then, the condition (1) means that a Chech cocycle $\{h_{\lambda\mu}\}$ defines an element in $H^1(R, \underline{ad}(P))$.

Theorem 1. *There is a one to one correspondence between the set of infinitesimal deformations of the principal bundle* $\pi : P \to R$ *and* $H^1(R, \underline{ad}(P))$.

§2 Principal G-bundles with trivializations.

Let us choose a point Q of the Riemann surface R and a local coordinate ξ of R with center Q. In the following we fix the data $(R; Q; \xi)$. We let $(P; \eta^{(k)})$ be a holomorphic principal G-bundle with k-th infinitesimal trivialization at the point Q:

$$\eta^{(k)} : \mathcal{O}_R(P) \otimes \mathcal{O}_{R,Q}/\mathfrak{m}_Q^{k+1} \simeq G(\mathbf{C}[\xi]/(\xi^{k+1})).$$

For $k \to +\infty$ we have a formal trivialization at Q:

$$\hat{\eta} : \mathcal{O}_R(P) \otimes \widehat{\mathcal{O}}_{R,Q} \simeq G(\mathbf{C}[[\xi]]).$$

Theorem 1 can be generalized in the following form.

Theorem 2. *For each positive integer k here is a one to one correspondence between the set of infinitesimal deformations of the data* $(P; \eta^{(k)})$ *and by* $H^1(R, \underline{ad}(P)(-(k+1)Q)$.

Let $\mathfrak{M}_R^{(k)}(G)$ be the coarse moduli scheme of stable pairs $(P; \eta^{(k)})$. At a point $\mathfrak{X} = (P; \eta^{(k)})$ of $\mathfrak{M}_R^{(k)}(G)$ we have a canonical isomorphismof the tangent space at $\mathfrak{X}$ to the first cohomology group:

$$T_{\mathfrak{X}}\mathfrak{M}_R^{(k)}(G) \simeq H^1(R, \underline{ad}(P)(-(k+1)Q)).$$

Let us consider an exact sequence

$$0 \to \underline{ad}(P)(-(k+1)Q) \to \underline{ad}(P)((m-(k+1))Q) \to \bigoplus_{\ell=-m+k+1}^{k} \mathfrak{g} \otimes \xi^{\ell} \to 0.$$

If $m \gg 0$, we have

$$H^1(R, \underline{ad}(P)((m-(k+1))Q)) = 0.$$

Hence, there is an isomorphism

$$\bigoplus_{\ell=-m+k+1}^{k} \mathfrak{g} \otimes \xi^{\ell} / H^0(R, \underline{ad}(P)((m-(k+1))Q)) \simeq H^1(R, \underline{ad}(P)(-(k+1)Q)).$$

Taking $m \to \infty$, we have

(2) $$\mathfrak{g} \otimes \left(\mathbf{C}[\xi, \xi^{-1}]/(\xi^{k+1})\right) H^0(R, \underline{ad}(P)(*Q) \simeq H^1(R, \underline{ad}(P)(-(k+1)Q)).$$

Note that for every principal G-bundle P over R, there is a positive integer ℓ such that $(P; \eta^{(k)})$, $k \geq \ell$ is always stable. Therefore, if we take $k \to \infty$, the corse moduli scheme $\mathfrak{M}_R(G)$ of pairs$(P; \hat{\eta})$ of principal G-bundle with formal trivialization at the point Q contains all the pair $(P; \hat{\eta})$ of principal G-bundle with formal trivialization at Q. Moreover, the coarse moduli scheme is fine and there is a universal family $\varpi : \mathcal{P} \to R \times \mathfrak{M}_R(G)$ of principal G-bundles with formal trivialization.

Now by virtue of (2), the tangent space of $\mathfrak{M}_R(G)$ at a point $\hat{\mathfrak{X}} = (P; \hat{\eta})$ is given by

$$\mathfrak{g} \otimes \mathbf{C}((\xi))/H^0(R, \underline{ad}(P)(*Q)).$$

This means that the affine Lie algebra $\mathfrak{g} \otimes \mathbf{C}((\xi))$ without centre operates on $\mathfrak{M}_R(G)$ infinitesimally and the action is infinitesimally homogeneous.

§3 Determinant bundles.

Let V be a G-module and $\rho : G \to \mathrm{Aut}(V)$ be the corresponding representation. Let $\hat{\varpi} : \mathcal{P} \times_G V \to R \times \mathfrak{M}_R(G)$ be the associated family of vector bundles with the universal family $\varpi : \mathcal{V} = \mathcal{P} \to R \times \mathfrak{M}_R(G)$ of principal G-bundles with formal trivializations. For each principal G-bundle on R put

$$V(P) := P \underset{G}{\times} V.$$

For the second projection $q : R \times \mathfrak{M}_R(G) \to \mathfrak{M}_R(G)$ we let $\det \mathbb{R}q_*\mathcal{V}$ be the determinant bundle of the family of vector bundles $\mathcal{V}$. For a point $\hat{\mathfrak{X}} = (P; \hat{\eta}) \in \mathfrak{M}_R(G)$ the fibre of the determinant bundle $\det \mathbb{R}q_*\mathcal{V}$ at $\hat{\mathfrak{X}}$ is given by

$$\left(\overset{max}{\wedge} H^0(R, V(P))\right) \otimes \left(\overset{max}{\wedge} H^1(R, V(P))\right)^{-1}.$$

The determinant bundle can be easily described by using the universal Grassmann manifold (UGM) due to Sato and the fermion Fock space. (See, for example [KNTY].) At a point $\hat{\mathfrak{X}} = (P; \hat{\eta})$, by taking the Laurent expansion at the point Q, we have a natural inclusion

$$t : H^1(R, V(P)(*Q)) \hookrightarrow V \underset{\mathbf{C}}{\otimes} \mathbf{C}((\xi)).$$

This embedding determines a point of $UGM(V)$ and gives an embedding

$$\tau : \mathfrak{M}_R(G) \hookrightarrow UGM(V).$$

Now $UGM(V)$ can be embedded into $\mathbf{P}(\mathcal{F})$ by the Plücker embedding where we may regard $\mathcal{F}$ to be a fermion Fock space. Thus we have a projective embedding

$$\hat{\tau} : \mathfrak{M}_R(G) \hookrightarrow \mathbf{P}(\mathcal{F}).$$

Then, the pull-back of the dual of hyperplane bundle of $\mathbf{P}(\mathcal{F})$ to $\mathfrak{M}_R(G)$ is nothing but the determinant bundle $\det \mathbb{R}q_*\mathcal{V}$.

The projective embedding can be described in the following way. Let us choose and fix a basis $\{e_1, e_2, \dots, e_n\}$ of the vector space V. Put

$$V_j = \langle e_1, e_2, \dots, e_j \rangle_{\mathbf{C}}$$
$$H_{jk} := t^{-1}(V_j \otimes \mathbf{C}((\xi)).$$

Then, $\{H_{jk}\}$ is a increasing filtration and we choose a normalized basis $\{h_1, h_2, \dots\}$ of $H^0(R, V(P)(*Q))$ by lexicographic ordering with respect to the filtration with normalization at thecoefficient of the first leading term. Then, the infinite exterior product

$$h_1 \wedge h_2 \wedge \cdots \quad .$$

gives the point $\hat{\tau}(\hat{\mathfrak{X}})$ in $\mathbf{P}(\mathcal{F})$.

Now let us consider the action of $\otimes \mathbf{C}((\xi))$ on $\det \mathbb{R}q_*\mathcal{V}$ which is the lift of the one on $\mathfrak{M}_R(G)$. For an element A of $\mathfrak{g} \otimes \mathbf{C}((\xi))$ this is very easy, since A acts on $\mathfrak{M}_R(G)$ as infinitesimal change of formal trivializations. That is, $A(h_j)$ is well-defined and the infinite product

$$A(h_1) \wedge A(h_2) \wedge \cdots$$

is also well-defined. This gives the desired action. Let us define the action of an element A of $\mathfrak{g} \otimes \mathbf{C}((\xi))$. Let $R = \cup_{\lambda \in \Lambda} U_\lambda$ be a small open covering of R such that a principal G-bundle is given by transition functions $\{g_{\lambda\mu}\}$. The section h_j is given by V-valued holomorphic functions f_λ on U_λ's with

$$f_\lambda = \rho(g_{\lambda\mu}) f_\mu.$$

We define the action of A on h_j in such a way that

$$A(f_\lambda) = f_\lambda + \epsilon \eta_\lambda$$

for each λ. By the isomorphism (2), the element A defines an element $\{h_{\lambda\mu}\} \in H^1(R, \underline{ad}(P)(-(k+1)Q)$ for a suitable k. Then, we need to have

$$A(f_\lambda) = \rho(g_{\lambda\mu} + \epsilon_{\lambda\mu}) A(f_\mu).$$

This is equivalent to saying that

$$\eta_{\lambda\mu} = \rho(h_{\lambda\mu}) f_\mu + \rho(g_{\lambda\mu}) \eta_\mu. \tag{3}$$

Since $\rho(h_{\lambda\mu}) f_\mu$ defines an element of $H^1(R, V(P)(mQ))$ for a certain integer m and we have

$$H^1(R, V(P)(*Q)) = 0$$

we can always find $\{\eta_\lambda\} \in \Gamma(U_\lambda, V(P)(*Q))$ which satisfy (3). $\{\eta_\lambda\}$ is uniquely determined up to the addition of an element in $H^0(R, V(P)(*Q))$. Therefore, we may choose $\{\eta_\lambda\}$ in such a way that

$$\eta_\lambda \in \Gamma(U_\lambda, V(P)(\ell Q))$$

with

$$\ell \gg \text{order of pole of } h_j \text{ at } Q.$$

Then, the infinite wedge product

$$A(h_1) \wedge A(h_2) \wedge \cdots$$

is well-defined. Since the above argument does not determine $A(h_j)$ uniquely, the action of A does not necessarily defines the action of $\mathfrak{g} \otimes \mathbf{C}((\xi))$ on the determinant bundle $\det \mathbb{R}q_*\mathcal{V}$. There is a canonical way to define the action of $\mathfrak{g} \otimes \mathbf{C}((\xi))$ on the determinant bundle $\det \mathbb{R}q_*\mathcal{V}$ by using the second quantization (or renormalization) of operators acting on the fermion Fock space $\mathcal{F}$. (See [KNTY].) The process shows that we need to take a central extension

$$\mathfrak{g} \otimes \mathbf{C}((\xi)) \oplus \mathbf{C} \cdot c$$

of the Lie algebra $\mathfrak{g} \otimes \mathbf{C}((\xi))$ to lift the operation of $\mathfrak{g} \otimes \mathbf{C}((\xi))$ on $\mathfrak{M}_R(G)$ to $\det \mathbb{R}q_*\mathcal{V}$.

References

[BS] A. A. Beilinson and V. V. Schechtman, *Determinant bundles and Virasoro algebras*, Commun. Math. Phys. **118** (1988), 651 – 701.

[KNTY] N. Kawamoto, Y. Namikawa, A. Tsuchiya and Y. Yamada, *Geometric realization of conformal field theory*, Commun. Math. Phys. **116** (1988), 247 – 308.

[N] Y. Namikawa, *A conformal field theory on Riemann surfaces realized as quantized moduli theory of Riemann surfaces*, Proceedings of Symposia in Pure Math. **49** (1989), 413 – 443.

[S]] C.S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Astérisque **96** (1982).

[T] Y. Tsuchimoto, *On the coordinate-free description of the conformal blocks*, to appear in J. Math. Kyoto Univ. (1992).

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