

Title	Bounding the Number of Columns (1, $k-2$, 1) in the Intersection Array of a Distance-Regular Graph
Author(s)	Higashitani, Naomi
Citation	数理解析研究所講究録 (1991), 768: 69-80
Issue Date	1991-11
URL	http://hdl.handle.net/2433/82329
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

Bounding the Number of Columns (1, k-2, 1)
in the Intersection Array of a Distance-Regular Graph

東谷直美 (Naomi Higashitani)

Department of Mathematics,
Osaka Kyoiku University

1. Introduction.

Let $\Gamma = (X, R)$ be an undirected connected finite graph without loops and multiple edges, where X and R are the vertex and edge sets. For a vertex x , $\Gamma_i(x)$ denotes the set of vertices having distance i from x . Γ is said to be *distance-regular* if the numbers (which are called *intersection numbers*)

$$c_i = | \Gamma_{i-1}(x) \cap \Gamma_1(y) |$$

$$a_i = | \Gamma_i(x) \cap \Gamma_1(y) |$$

$$b_i = | \Gamma_{i+1}(x) \cap \Gamma_1(y) |$$

are independent of the choices of $x \in X$ and $y \in \Gamma_1(x)$. In what follows, we always assume that Γ is a distance-regular graph. The valency and diameter are denoted by k and d :

$$k = | \Gamma_1(x) |,$$

$$d = \text{Max} \{ i \mid \Gamma_i(x) \neq \emptyset \}.$$

The intersection array is denoted by

$$\left[\begin{array}{cccccccc} * & c_1 & c_2 & \cdots & c_i & \cdots & c_{d-1} & c_d \\ 0 & a_1 & a_2 & \cdots & a_i & \cdots & a_{d-1} & a_d \\ k & b_1 & b_2 & \cdots & b_i & \cdots & b_{d-1} & * \end{array} \right].$$

Let

$$\ell = \ell(c, a, b) = \# \{ i \mid (c_i, a_i, b_i) = (c, a, b) \}.$$

Then by the relation [1, page 195]

$$1 = c_1 \leq c_2 \leq \dots \leq c_d, \quad k = b_0 \geq b_1 \geq \dots \geq b_{d-1},$$

these ℓ columns appear in the intersection array consecutively.

In [2], it is conjectured that $\ell(c, a, b)$ is bounded by a function of the valency k and it is shown that if $c = b$,

$$\ell(c, a, b) \leq 10k2^k.$$

We shall improve the above bound when $c = b = 1$.

Theorem.

$$\text{If } k \geq 5, \quad \ell(1, k-2, 1) < 46\sqrt{k-3}.$$

Notice that distance-regular graphs of valency 3 or 4 are classified by [3], [4]. Our result is useful for the classification of distance-regular graphs with small valencies, for example $k = 5, 6$.

2. Preliminaries.

In what follows, let the intersection array be

$$\left[\begin{array}{cccccccccccc} * & 1 & \dots & 1 & c_{\alpha+1} & \dots & c_s & c & \dots & c & c_{s+l+1} & \dots & c_{d-1} & c_d \\ 0 & a_1 & \dots & a_1 & a_{\alpha+1} & \dots & a_s & a & \dots & a & a_{s+l+1} & \dots & a_{d-1} & a_d \\ k & \underline{b_1} & \dots & \underline{b_1} & b_{\alpha+1} & \dots & b_s & \underline{b} & \dots & \underline{b} & b_{s+l+1} & \dots & b_{d-1} & * \end{array} \right].$$

The i -th adjacency matrix of Γ is denoted by A_i ($0 \leq i \leq d$) and

we set $A = A_1$.

Proposition 1. ([2, Proposition 1])

Let $l = l(c, a, b)$. Then there exists an eigenvalue θ of A such that

$$k - b - c + 2\sqrt{bc} \cos \frac{\nu + 2}{l} \pi < \theta < k - b - c + 2\sqrt{bc} \cos \frac{\nu}{l} \pi$$

for each $\nu = 1, 2, \dots, l-3$.

There exists a polynomial $v_i(x)$ of degree i such that $v_i(A) = A_i$, and we have $v_i(k) = k_i$. See [1].

Proposition 2. ([2, Proposition 2])

$v_s(x)$ has roots all less than $k - b_s - c_s + 2\sqrt{b_s c_s}$.

Proposition 3. ([5])

Let $\alpha = l(c_j, a_j, b_j)$ and $\theta \neq \pm k$ an eigenvalue of A .

If $a_j = 0$, then the multiplicity $m(\theta)$ of θ in A satisfies

$$m(\theta) \geq k(k-1)^{r-1}$$

with $r = [(\alpha + 1)/2]$, the integer part of $(\alpha + 1)/2$.

Proposition 4. ([6])

Let $\alpha = l(c_j, a_j, b_j)$ and $\alpha' \leq l(c_u, a_u, b_u)$. Suppose

$c_{u+\alpha'} = 1$ and $a_j \neq a_u$. Then the following hold

$$(1) \quad \alpha' \leq \alpha$$

(2) $\alpha' \leq \alpha - 1$ if $\alpha \geq 3$

(3) $\alpha' \leq \alpha - 2$ if $\alpha \geq 5$.

The results in Proposition 4 are also obtained by A.V.Ivanov (personal communication).

Proposition 5. ([7])

Let $\alpha = \ell(1, 0, k-1)$ and $\alpha_j \leq \ell(1, 1, k-2)$.

Suppose $c_{\alpha+\alpha_j+1} = 1$, $k \geq 4$ and $\alpha \geq 1$. Then $\alpha_j \leq 2$.

Lemma 6. If $\ell(1, k-2, 1) \geq 2$, then $a_1 = 0$.

Proof. Suppose $a_1 \neq 0$. Then for an edge $\{y, z\}$ with $y \in \Gamma_{s+1}(x)$, $z \in \Gamma_{s+2}(x)$, there exists a triangle yzw , and so we have $b_{s+1} > 1$ or $c_{s+2} > 1$, which is a contradiction. \square

3. Proof of Theorem.

Let Γ be a distance regular graph with valency k and diameter d . Let $\ell = \ell(1, k-2, 1) > 0$, and

$$\begin{aligned} (c_{s+1}, a_{s+1}, b_{s+1}) &= \cdots = (c_{s+\ell}, a_{s+\ell}, b_{s+\ell}) \\ &= (1, k-2, 1). \end{aligned}$$

Let $\alpha = \ell(c_1, a_1, b_1)$ and $r = [(\alpha+1)/2]$.

By the relation $c_i \leq c_{i+1}$, $b_i \geq b_{i+1}$, we have the following intersection array :

$$\begin{bmatrix} * & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & c_{s+\ell+1} & \cdots & c_{d-1} & c_d \\ 0 & a_1 & \cdots & a_1 & a_{\alpha+1} & \cdots & a_s & k-2 & \cdots & k-2 & a_{s+\ell+1} & \cdots & a_{d-1} & a_d \\ k & \underbrace{b_1}_{\alpha} & \cdots & \underbrace{b_1}_{\alpha+1} & b_{\alpha+1} & \cdots & b_s & \underbrace{1}_{\ell} & \cdots & \underbrace{1}_{\ell} & 1 & \cdots & 1 & * \end{bmatrix}.$$

Firstly we need two lemmas to estimate the number of vertices n and the number s above.

Lemma 7. Let n be the number of vertices. Then

$$n < k_s \cdot \left((k-1)/(k-2) + (\ell+1)(k-1) \right).$$

Proof. Since $n = k_0 + k_1 + \cdots + k_d$, we evaluate k_i 's using the property,

$$b_i k_i = c_{i+1} k_{i+1}.$$

For $i \leq s$, since $c_i = 1$ and $k = b_0 \geq b_1 \geq \cdots \geq b_s > 1$, we have

$$k_{i-1} = k_i / b_{i-1} \leq k_i / b_s \leq k_{i+1} / b_s^2 \leq k_s / b_s^{s-(i-1)}.$$

Hence

$$\begin{aligned} k_0 + k_1 + \cdots + k_s &\leq k_s \left(\left(1 / b_s \right)^s + \left(1 / b_s \right)^{s-1} + \cdots + 1 \right) \\ &= k_s \cdot \frac{b_s}{b_s - 1} \left(1 - \left(1 / b_s \right)^{s+1} \right) \\ &< \frac{b_s}{b_s - 1} \cdot k_s. \end{aligned}$$

Obviously, it holds that

$$k_{s+1} = k_{s+2} = \cdots = k_{s+\ell} = b_s k_s.$$

For $i \geq s + \ell + 1$, since $b_i = 1$ and $1 < c_{s+\ell+1} \leq \cdots \leq c_d$, we have

$$k_{s+l+j} \leq k_{s+l} / (c_{s+l+1})^j.$$

Hence

$$\begin{aligned} & k_{s+l+1} + \cdots + k_d \\ & \leq k_{s+l} \left(1 / c_{s+l+1} + (1 / c_{s+l+1})^2 \right. \\ & \quad \left. + \cdots + (1 / c_{s+l+1})^{d-(s+l)} \right) \\ & = k_{s+l} \cdot \frac{1}{c_{s+l+1} - 1} \left(1 - (1 / c_{s+l+1})^{d-(s+l)} \right) \\ & < \frac{1}{c_{s+l+1} - 1} \cdot k_{s+l} \\ & \leq k_{s+l} = b_s k_s. \end{aligned}$$

Therefore we get

$$\begin{aligned} n & < k_s \left(b_s / (b_s - 1) + (\ell + 1) b_s \right) \\ & \leq k_s \left(\frac{k - 1}{(k - 1) - 1} + (\ell + 1)(k - 1) \right). \end{aligned}$$

Note that $(\ell + 1)x + \frac{x}{x - 1}$ is increasing if $x \geq 2$. \square

Lemma 8. *Suppose $\ell \geq 5$ and $k \geq 5$. Then $s \leq \alpha(k - 3)$.*

Proof. Let $\alpha' = \ell - 1$ in Proposition 4 and $u = s + 1$.

Since $\alpha' \geq 4$, $\alpha \geq 5$. Let

$$\alpha_i = \ell (1, i, k - i - 1).$$

Then $\alpha_0 = \alpha$ and $\alpha_{k-2} = \ell$. Note that $a_1 = 0$ by Lemma 6.

By Proposition 5, $\alpha_1 \leq 2$.

By Proposition 4.(3), $\alpha_i \leq \alpha - 2$ $i = 2, \dots, k - 3$.

Therefore we get

$$\begin{aligned}
 s &\leq \alpha + \alpha_1 + \alpha_2 + \cdots + \alpha_{k-3} \\
 &\leq \alpha + 2 + (k-4)(\alpha-2) \\
 &= \alpha(k-3) - 2(k-5) \\
 &\leq \alpha(k-3) \quad \text{as } k \geq 5. \quad \square
 \end{aligned}$$

Now we start the proof of Theorem.

By Proposition 1, there exists an eigenvalue θ of A such that

$$k - b - c + 2\sqrt{bc} \cos(3\pi/\ell) < \theta < k - b - c + 2\sqrt{bc} \cos(\pi/\ell).$$

Since $b = c = 1$, it holds that

$$\begin{aligned}
 k - 2 + 2 \cos(3\pi/\ell) &\geq k - 2 + 2 \left(1 - \frac{1}{2} (3\pi/\ell)^2 \right) \\
 &= k - (3\pi/\ell)^2.
 \end{aligned}$$

Thus we have an eigenvalue θ of A such that

$$k - \delta < \theta < k,$$

with $\delta = (3\pi/\ell)^2$.

By Proposition 2, $v_s(x)$ is positive for

$x \geq k - b_s - c_s + 2\sqrt{b_s c_s}$, while

$$\begin{aligned}
 k - b_s - c_s + 2\sqrt{b_s c_s} &= k - \left(\sqrt{b_s} - \sqrt{c_s} \right)^2 \\
 &= k - \left(\sqrt{b_s} - 1 \right)^2 \\
 &\leq k - \left(\sqrt{2} - 1 \right)^2.
 \end{aligned}$$

Hence we get

$$v_s(x) > 0 \quad \text{for } x \geq k - \left(\sqrt{2} - 1 \right)^2. \quad \cdots(1)$$

Now assume that $\ell > 46\sqrt{k-3}$. Since $\ell \geq 3\pi(\sqrt{2} + 1)$,
 $\theta > k - (\sqrt{2} - 1)^2$ and so $v_s(\theta) > 0$.

Let $m(\theta)$ be the multiplicity of θ in A .
 By Biggs' formula [1, page 72], it holds that

$$m(\theta) = n / \sum_{i=0}^n \{ v_i(\theta)^2 / k_i \}. \quad \dots (2)$$

Since $\ell \geq 3\pi(\sqrt{2} + 1) > 2$, $a_1 = 0$ by Lemma 6.

Hence we can apply the Terwilliger bound (Proposition 3)

$$m(\theta) \geq k(k-1)^{r-1}. \quad \dots (3)$$

We shall find an upper bound of $m(\theta)$, and by comparing it with
 (3), we shall prove that ℓ can not exceed $46\sqrt{k-3}$.

Applying Lemma 7 to (2), we have

$$\begin{aligned} m(\theta) &\leq n \cdot k_s / v_s(\theta)^2 \\ &\leq \left(k_s / v_s(\theta) \right)^2 \cdot \left((k-1)/(k-2) + (\ell+1)(k-1) \right). \end{aligned}$$

Let $\lambda_1, \lambda_2, \dots, \lambda_s$ be the roots $v_s(x)$.

By (1), we may assume

$$\theta > k - (\sqrt{2} - 1)^2 > \lambda_1 > \lambda_2 > \dots > \lambda_s.$$

Since $(k - \lambda_i)/(k - \theta) = 1 + (k - \theta)/(\theta - \lambda_i)$ increases

with λ_i , it follows that

$$\begin{aligned} \frac{k - \lambda_i}{\theta - \lambda_i} &< \frac{k - \{ k - (\sqrt{2} - 1)^2 \}}{\theta - \{ k - (\sqrt{2} - 1)^2 \}} \\ &= \frac{1}{1 + (\theta - k)(\sqrt{2} + 1)^2} \end{aligned}$$

$$< \frac{1}{1 - \delta(\sqrt{2} + 1)^2} \quad \text{as } k - \delta < \theta.$$

So we have

$$\frac{k_s}{v_s(\theta)} = \frac{v_s(k)}{v_s(\theta)} = \prod_{i=1}^s \frac{k - \lambda_i}{\theta - \lambda_i} < \frac{1}{\{1 - \delta(\sqrt{2} + 1)^2\}^s},$$

$$m(\theta) < \frac{(k-1)/(k-2) + (\ell+1)(k-1)}{\{1 - \delta(\sqrt{2} + 1)^2\}^{2s}} \quad \dots (4)$$

Since $r = [(\alpha + 1)/2]$, $\alpha \leq 2r$ and it follows from Lemma 8 that

$$s \leq \alpha(k-3) \leq 2r(k-3). \quad \dots (5)$$

By (3), (4) and (5), we have

$$k(k-1)^{r-2} < \frac{1/(k-2) + (\ell+1)}{\{1 - \delta(\sqrt{2} + 1)^2\}^{4r(k-3)}}, \quad \dots (6)$$

while $0 < 1 - \delta(\sqrt{2} + 1)^2 < 1$ by $\ell \geq 3\pi(\sqrt{2} + 1)^2$ and

$$\delta = (3\pi / \ell)^2.$$

Since $\ell + 1 \leq \alpha \leq s$ by Proposition 4.

[the right hand side of (6)]

$$\leq \frac{1/(k-2) + s}{\{1 - \delta(\sqrt{2} + 1)^2\}^{4r(k-3)}}$$

$$\leq \frac{s+1}{\{1 - \delta(\sqrt{2} + 1)^2\}^{4r(k-3)}}$$

$$\leq \frac{2r(k-3) + 1}{\{1 - \delta(\sqrt{2} + 1)^2\}^{4r(k-3)}}$$

$$< \frac{2rk}{\{1 - \delta(\sqrt{2} + 1)^2\}^{4r(k-3)}}.$$

Hence

$$\log k + (r-2)\log(k-1)$$

$$\langle \log 2 + \log r + \log k - 4r(k-3) \log \{ 1 - \delta(\sqrt{2} + 1)^2 \}. \quad \dots (7)$$

We want to show that (7) does not hold for large ℓ .

Namely we shall show the opposite inequality for $\ell \geq 46\sqrt{k-3}$:

$$(r-2) \log(k-1) \geq \log 2 + \log r - 4r(k-3) \log \{ 1 - \delta(\sqrt{2} + 1)^2 \}. \quad \dots (8)$$

By $r = \lceil (\alpha + 1)/2 \rceil$ and Proposition 4.(3),

$$r \geq (\ell + 1)/2 \geq \frac{46\sqrt{k-3} + 1}{2} \geq \frac{46\sqrt{2} + 1}{2}.$$

[the left hand side of (8)] - [the right hand side of (8)]

$$= (r-2) \log(k-1) - \log 2 - \log r + 4r(k-3) \log \{ 1 - (\frac{3\pi}{\ell})^2 \cdot (\sqrt{2} + 1)^2 \}$$

$$\geq (r-2) \log(k-1) - \log 2 - \log r$$

$$+ 4r(k-3) \log \left[1 - \left(\frac{3\pi}{46\sqrt{k-3}} \right)^2 \cdot (\sqrt{2} + 1)^2 \right]$$

$$> (r-2) \log(k-1) - \log 2 - \log r$$

$$+ 4r(k-3) \cdot \left[- \frac{\left(\frac{3\pi}{46\sqrt{k-3}} \right)^2 \cdot (\sqrt{2} + 1)^2}{1 - \left(\frac{3\pi}{46\sqrt{k-3}} \right)^2 \cdot (\sqrt{2} + 1)^2} \right]$$

$$\left(\text{by } \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \text{ (for } 0 < x < 1 \text{)} \right)$$

$$> -x - x^2 - x^3 - \dots$$

$$= - \frac{x}{1-x} \Bigg)$$

$$= (r-2) \log(k-1) - \log 2 - \log r$$

$$+ 4r(k-3) \cdot \left[- \frac{\left(\frac{3\pi}{46} \right)^2 (\sqrt{2} + 1)^2 / (k-3)}{1 - \left(\frac{3\pi}{46} \right)^2 (\sqrt{2} + 1)^2 / (k-3)} \right]$$

$$> (r-2) \log(k-1) - \log 2 - \log r$$

$$\begin{aligned}
& + 4r(k-3) \cdot \left[- \frac{0.2447/(k-3)}{1-0.2447/(k-3)} \right] \\
& \left(\text{by } (3\pi/46)^2 \cdot (\sqrt{2}+1)^2 \approx 0.244668445 \right) \\
& = (r-2) \log(k-1) - \log 2 - \log r \\
& \quad - 4r \times 0.2447 \times \frac{k-3}{(k-3)-0.2447} \\
& \geq (r-2) \log 4 - \log 2 - \log r - 4r \times 0.2447 \times \frac{2}{2-0.2447} \\
& \quad (\text{by } k \geq 5) \\
& \geq \left(\frac{46\sqrt{2}+1}{2} - 2 \right) \log 4 - \log 2 - \log \left(\frac{46\sqrt{2}+1}{2} \right) \\
& \quad - 4 \times 0.2447 \times \frac{46\sqrt{2}+1}{2} \times \frac{2}{2-0.2447} \\
& \quad \left(\text{as it is increasing with } r, \text{ where } r \geq \frac{46\sqrt{2}+1}{2} \right) \\
& \approx 43.01243307 - 0.69314718 - 3.497322742 - 36.83329505 \\
& > 43.01 - 0.70 - 3.50 - 36.84 \\
& = 1.97 > 0.
\end{aligned}$$

This proves the theorem. \square

Acknowledgement

The author would like to thank Prof. T. Ito and Dr. H. Suzuki for valuable suggestions and comments.

References

- [1] E. Bannai and T. Ito : Algebraic Combinatorics I, Benjamin/Cummings Lecture Note Series (1984).

- [2] E. Bannai and T. Ito : On Distance-Regular Graphs with Fixed Valency, *Graphs and Combinatorics*, 3 (1987), 95-109.
- [3] N. L. Biggs, A. G. Boshier and J. Shawe-Taylor : Cubic distance-regular graphs, *J. London Math. Soc.*, (2) 33 (1986), 385-394.
- [4] E. Bannai, T. Ito and K. Nomura : In preparation.
- [5] P. Terwilliger : Eigenvalue Multiplicities of highly symmetric graphs, *Discrete Math.*, 41 (1982), 295-302.
- [6] H. Suzuki : On a Distance Regular Graph with $b_e = 1$, Preprint.
- [7] A. Hiraki : An Improvement of the Boshier-Nomura Bound, Preprint.