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ON THE SUBSCHEMES OF THE JOHNSON SCHEME $J(v, d)$

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Abstract.

For two association schemes χ and χ' , defined on the same set, we call χ' a subscheme of χ if each relation of χ' is a union of some relations of χ . In this paper we prove that the Johnson scheme $J(v, d)$ has no non-trivial subscheme if $v > 2d + \frac{3}{2} + \sqrt{(d - \frac{5}{2})^2 + 6}$. This slightly improves the earlier result of Muzichuk that the conclusion holds if $v \geq 3d + 4$.

1. Introduction

Let $\chi = (X, \{R_i\}_{0 \leq i \leq d})$ and $\chi' = (X, \{R'_i\}_{0 \leq i \leq d})$ be two association schemes defined on the same set X . We say that χ' is a subscheme of χ if each relation R'_i is a union of some relations R_i 's.

The purpose of this paper is to prove that the Johnson scheme $J(v, d)$ have no non-trivial subscheme if $v > 2d + \frac{3}{2} + \sqrt{(d - \frac{5}{2})^2 + 6}$. For a subscheme χ' of χ , we have a partition $\tau = \{T_0 = \{0\}, T_1, \dots, T_d\}$ of the index set $\{0, 1, \dots, d\}$ such that

$$R'_i = \bigcup_{\alpha \in T_i} R_\alpha \quad \left(\text{using the adjacency matrices, } A'_i = \sum_{\alpha \in T_i} A_\alpha \right).$$

Clearly $\chi_0 = (X, \{R_0, \bigcup_{i>0} R_i\})$ and χ are subschemes of χ . We call these subschemes trivial subschemes. Clearly if the association scheme χ is commutative then it's subscheme χ' is commutative, and if χ is symmetric then χ' is symmetric, too.

Throughout the whole paper we shall consider only symmetric association schemes. If we look at the Bose-Mesner algebra \mathfrak{a} and \mathfrak{a}' , it immediately holds that χ' is a subscheme of χ if and only if $\mathfrak{a}' \subset \mathfrak{a}$. Then we also have the partition $\pi = \{\pi_0 = \{0\}, \pi_1, \dots, \pi_d\}$ of the index set $\{0, 1, \dots, d\}$ such that

$$E'_j = \sum_{\beta \in \pi_j} E_\beta.$$

Now we calculate the entries of the first and second eigenmatrices of χ' . (For the notation of association scheme, see [2],[4])

$$|X|E'_j = \sum_{0 \leq i \leq d'} q'_j(i)A'_i. \quad \dots (1,1)$$

$$\begin{aligned} \text{The LHS of (1,1)} &= |X| \sum_{\beta \in \pi_j} E_\beta \\ &= \sum_{0 \leq \alpha \leq d} \left(\sum_{\beta \in \pi_j} q_\beta(\alpha) \right) A_\alpha. \end{aligned}$$

$$\text{The RHS of (1,1)} = \sum_{0 \leq i \leq d'} \sum_{\alpha \in T_i} q'_j(i)A_\alpha.$$

Therefore comparing the both sides, we get $q'_j(i) = \sum_{\beta \in \pi_j} q_\beta(\alpha)$ ($\alpha \in T_i$).

Dually we get $p'_i(j) = \sum_{\alpha \in T_i} p_\alpha(\beta)$ ($\beta \in \pi_j$).

With the above notation, we get the following lemma.

Lemma 1. Let χ' be a subscheme of χ , and τ, π be the partitions of χ' .

The indices i, j are glued in τ (namely R_i and R_j are in a same relation R' of χ') (dually in π) if and only if for each $0 \leq k \leq d'$

$$\sum_{\beta \in \pi_k} q_\beta(i) = \sum_{\beta \in \pi_k} q_\beta(j) \quad \left(\text{dually } \sum_{\alpha \in T_k} p_\alpha(i) = \sum_{\alpha \in T_k} p_\alpha(j) \right).$$

(cf. [1],[4])

Using this lemma, we get the following corollary.

Corollary. Let $\chi = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme and

$\chi' = (X, \{R'_i\}_{0 \leq i \leq d})$ be a non-trivial subscheme of χ .

If there exist $0 < i_0, j_0 \leq d$ such that

$q_j(i_0) > q_j(i)$ ($0 < i \neq i_0, 0 < j \neq j_0$) , then

$R_{i_0} \in \{R'_i\}_{0 \leq i \leq d}$.

Proof Since χ' is non-trivial, there exists a part π ($\neq \{0\}$) of the partition π which does not contain j_0 . Then

$$\sum_{j \in \pi} q_j(i_0) > \sum_{j \in \pi} q_j(i) \quad (0 < i \neq i_0) .$$

With Lemma 1 , we have $R_{i_0} \in \{R'_i\}_{0 \leq i \leq d}$. (Q.E.D.)

We now state our main result. Before that, we mention a brief history of this problem (according to Muzichuk [4]) . Let $J(v, d)$ be the Johnson scheme of class d (see [2],[3], for the details about the Johnson scheme) . As for the study of subscheme of $J(v, d)$ or on the enumeration of subschemes of the Johnson scheme, first L.A.Kaluznin M.H.Klin proved that there exists a function $f(d)$ such that Johnson scheme $J(v, d)$ does not have non-trivial subscheme for $v \geq f(d)$. M.E.Muzichuk proved that the same conclusion is true if $v \geq 3d+4$. The purpose of this paper is to prove the following Theorem.

Theorem A The Johnson scheme $J(v, d)$ has no non-trivial subscheme if $v > 2d + \frac{3}{2} + \sqrt{(d - \frac{5}{2})^2 + 6}$.

This slightly improves the result of Muzichuk [4], because our condition becomes $v \geq 3d$ for $d \geq 6$.

The author thanks Professor Eiichi Bannai for suggesting this problem. The author also thanks Mr. Go Seki for checking the calculation in this paper. Also the idea of the proof of the present paper is based

on that of Muzichuk [4] . The auther thanks M.E.Muzichuk for making the preprint of [4] available to the auther before publication.

2. Proof of the Theorem A

For the Johnson scheme $J(v,d)$, we have

$$p_i(j) = \sum_{0 \leq v \leq j} (-1)^v \binom{j}{v} \binom{d-j}{i-v} \binom{v-d-j}{i-v} , \quad k_i = \binom{d}{i} \binom{v-d}{i} . \quad (\text{ see [2],[3],[4]})$$

In order to prove Theorem A , we need the following lemma.

Lemma 2. For $J(v,d)$ with $v > 2d + \frac{3}{2} + \sqrt{(d - \frac{5}{2})^2 + 6}$, we have

$$q_j(1) > |q_j(i)| \quad (1 < i \leq d, 0 < j < d) .$$

First we show that Thorem A is easily obtained for Lemma 2 .

Proof of Theorem A from Lemma 2.

Let $\chi' = (X, \{R_i\}_{0 \leq i \leq d})$ be a non-trivial subscheme of $J(v,d)$.

Then by Corollary, we have $R_1 \in \{R'_i\}_{0 \leq i \leq d}$. i.e. we have $\{1\} \in \tau$.

If the indices i, j are glued in π , with Lemma 1 , we have

$$p_1(i) = p_1(j) . \text{ Since}$$

$$p_1(i) = i^2 - (v+1)i + d(v-d) , \quad p_1(j) = j^2 - (v+1)j + d(v-d) , \text{ we have}$$

$$i = j . \quad (\text{ Q.E.D.})$$

Now we prove Lemma 2 with the inductive formula of the Johnson scheme:

$$p_i^{v,d}(j) = \begin{cases} p_i^{v-2,d-1}(j-1) - p_{i-1}^{v-2,d-1}(j-1) & (0 < i < d) \\ -p_{d-1}^{v-2,d-1}(j-1) & (i = d) . \end{cases}$$

From now on, if necessary, we let $p_i^{v,d}(j)$ and $k_i(v,d)$ be the entries of the first eigenmatrix P and the valencies of $J(v,d)$ respectively.

Concerning Lemma 2 , we check at the special value of $j = d-1$ by the following Proposition.

Proposition. For $J(v,d)$ with $v > 2d + \frac{3}{2} + \sqrt{(d - \frac{5}{2})^2 + 6}$, we have

$$q_{d-1}(1) > |q_{d-1}(i)| \quad (1 < i \leq d) .$$

Proof Since $q_j(i)/m_j = p_i(j)/k_i$, we only have to show that

$$|p_i(d-1)|/k_i < p_1(d-1)/k_1 \quad (1 < i \leq d) .$$

With the direct calculation, we have

$$p_i(d-1)/k_i = (-1)^{i-1} \frac{(v-2d+2)i-d}{d \binom{v-d}{i}} . \text{ Then}$$

$$\begin{aligned} & d(v-d-i) \binom{v-d}{i} (|p_i(d-1)|/k_i - |p_{i+1}(d-1)|/k_{i+1}) \\ &= -2(v-2d+2)i^2 + (v^2 - 3dv + 2d^2 + 4d - 4)i - (d+1)v + d^2 + 3d - 2 \quad \dots (*) \end{aligned}$$

This reaches it's minimum when $i=1$ or $d-1$, i.e.

$$(*) \geq \min\{v^2 - (4d+3)v + 3d^2 + 11d - 10, (d-1)(v-2d+1)\{v - \frac{3d+\frac{2}{d-1}}{\frac{2}{d-1}}\}\}$$

Since $v^2 - (4d+3)v + 3d^2 + 11d - 10 > 0$ is equal to $v > 2d + \frac{3}{2} + \sqrt{(d-\frac{5}{2})^2 + 6}$,

we have

1) If $v \geq 3d + \frac{2}{d-1}$ (i.e. $v \geq 3d+1$), then we have $(*) \geq 0$. Then

$$p_1(d-1)/k_1 > |p_2(d-1)|/k_2 \geq \dots \geq |p_d(d-1)|/k_d .$$

2) In the case of $v=3d$ ($d \geq 6$), we have

$$\begin{aligned} |p_i(d-1)|/k_i &\leq \max\{p_1(d-1)/k_1, |p_d(d-1)|/k_d\} \\ &= \max\{1/d^2, (d+1)/\binom{2d}{d}\} . \text{ Since} \end{aligned}$$

$$\frac{d+1}{\binom{2d}{d}} d^2 = \frac{d^2(d+1)!}{2d(2d-1)\cdots(d+1)}$$

$$< \frac{5 \times 4 \times 3}{(2d-1)(2d-2)} \leq \frac{6}{11} , \text{ we have}$$

$$|p_i(d-1)|/k_i < p_1(d-1)/k_1 \quad (i > 1) . \quad (\text{Q.E.D.})$$

Proof of Lemma 2

By Proposition, we only have to show that

$$|p_i(j)|/k_i < p_1(j)/k_1 \quad (1 < i \leq d, 0 < j < d-1) .$$

1) $d=3$

With the direct calculation, we have

$$p_1(1)/k_1 = \frac{2v-9}{3(v-3)}, \quad p_2(1)/k_2 = \frac{v-9}{3(v-3)}, \quad p_3(1)/k_3 = \frac{-3}{v-3} .$$

Since $v > 10$, we have

$$|p_i(1)|/k_i < |p_1(1)|/k_1 \quad (i = 2, 3).$$

2) $d > 3$

We consider the following two cases, Case 2.1; $1 < i < d$, $0 < j < d-1$ and Case 2.2; $i = d$, $0 < j < d-1$, separately.

Case 2.1 $1 < i < d$, $0 < j < d-1$

With the inductive formula, we have

$$p_i^{v,d}(j) = p_i^{v-2,d-1}(j-1) - p_{i-1}^{v-2,d-1}(j-1).$$

Since $v-2 > 2(d-1) + \frac{3}{2} + \sqrt{\{(d-1) - \frac{5}{2}\}^2 + 6}$, we have

$$|p_i^{v-2,d-1}(j-1)| \leq p_1^{v-2,d-1}(j-1) \frac{k_i(v-2,d-1)}{k_1(v-2,d-1)},$$

$$|p_{i-1}^{v-2,d-1}(j-1)| \leq p_1^{v-2,d-1}(j-1) \frac{k_{i-1}(v-2,d-1)}{k_1(v-2,d-1)}, \text{ and}$$

$$|p_i^{v,d}(j)|/k_i(v,d) \leq \frac{k_i(v-2,d-1) + k_{i-1}(v-2,d-1)}{k_1(v-2,d-1)k_i(v,d)} p_1^{v-2,d-1}(j-1).$$

We show that

$$\frac{k_i(v-2,d-1) + k_{i-1}(v-2,d-1)}{k_1(v-2,d-1)k_i(v,d)} p_1^{v-2,d-1}(j-1) k_1(v,d) / p_1^{v,d}(j) < 1 \quad (*)$$

With the direct calculation, we have

$$\text{The LHS of } (*) = \frac{2i^2 - vi + d(v-d)}{(d-1)(v-d-1)} \left(1 + \frac{1}{j^2 - (v+1)j + d(v-d)}\right).$$

This reaches its maximum when $i=2$, $j=d-2$. Therefore

$$\text{The LHS of } (*) \leq \left(1 - \frac{v-7}{(d-1)(v-d-1)}\right) \left(1 + \frac{1}{2v-5d+6}\right).$$

Since $\frac{1}{2v-5d+6} \leq \frac{1}{d+6} < \frac{v-7}{(d-1)(v-d-1)}$, we have the LHS of $(*) < 1$.

Case 2.2 $i = d$, $0 < j < d-1$

Since $p_d^{v,d}(j) = -p_{d-1}^{v-2,d-1}(j-1)$, we have

$$|p_d^{v,d}(j)|/k_d(v,d) \leq \frac{k_{d-1}(v-2,d-1)}{k_1(v-2,d-1)k_d(v,d)} p_1^{v-2,d-1}(j-1).$$

Now we get

$$\begin{aligned}
& \frac{k_{d-1}(v-2, d-1)}{k_1(v-2, d-1)k_d(v, d)p_1^{v-2, d-1}(j-1)k_1(v, d)/p_1^{v, d}(j)} \\
&= \frac{d^2}{(d-1)(v-d-1)} \left(1 + \frac{1}{j^2 - (v+1)j + d(v-d)}\right) \\
&\leq \left(1 - \frac{v-7}{(d-1)(v-d-1)}\right) \left(1 + \frac{1}{2v-5d+6}\right) < 1 .
\end{aligned}$$

This completes the proof of Lemma 2, hence of Theorem A.

Added in proof

Recently we improve the Theorem A in this paper as follows;

The Johnson scheme $J(v, d)$ has no non-trivial subscheme

if $v > 2d + \frac{3}{2} + \sqrt{(d - \frac{7}{2})^2 + 6}$.

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