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Propagation of singularities of solutions for a degenerate hyperbolic equation

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In this note we give a short review on the Gevrey wave front sets and give a properties of the propagation of singularities for solutions of the Cauchy problem for a degenerate hyperbolic operator.

§1. Gevrey wave front sets. Let κ satisfy $\kappa > 1$. As in [5] we introduce a class of ultradistributions as

$$\mathcal{D}_{L^2}^{\{\kappa\}'} = \operatorname*{proj\,lim}_{\varepsilon \downarrow 0} \mathcal{D}_{L^2,\varepsilon}^{\{\kappa\}'} .$$

Here, for $\varepsilon > 0$, $\mathcal{D}_{L^2,\varepsilon}^{\{\kappa\}'}$ is the dual space of the Hilbert space $\mathcal{D}_{L^2,\varepsilon}^{\{\kappa\}} \equiv \{u(x) \in L^2 \ ; \ \exp(\varepsilon\langle\xi\rangle^{1/\kappa})\hat{u}(\xi) \in L^2\}$ and $F[u](\xi) \equiv \hat{u}(\xi)$ denotes the Fourier transform of $u(x)$. For the symbol classes of pseudo-differential operators we employ

Definition 1.1. i) Let $0 \leqq \delta \leqq \rho \leqq 1$ and $\delta < 1$. We say that a symbol $p(x,\xi)$ belongs to a class $S^m_{\rho,\delta,G(\kappa)}$ if

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leqq CM^{-|\alpha+\beta|}(\alpha!^\kappa + \alpha!^{\kappa\rho}\langle\xi\rangle^{(1-\rho)|\alpha|})$$
$$\times(\beta!^\kappa + \beta!^{\kappa(1-\delta)}\langle\xi\rangle^{\delta|\beta|})\langle\xi\rangle^{m-|\alpha|}$$
$$\text{for all } x \text{ and } \xi$$

hold for constants C and M independent of α and β. We also denote $S^m_{G(\kappa)} = S^m_{1,0,G(\kappa)}$.

ii) We say that a symbol $p(x,\xi)$ $(\in S^{-\infty})$ belongs to a class $\mathcal{R}_{G(\kappa)}$ if for any α there exists a constant C_α such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leqq C_\alpha M^{-|\beta|}\beta!^\kappa \exp(-\varepsilon\langle\xi\rangle^{1/\kappa})$$

hold with a positive constant ε independent of α and β.

The following lemma follows by the similar way to th proof of Theorem 1 in [13].

Lemma 1.2. i) The pseudo-differential operator $p(X,D_x)$ with a symbol $p(x,\xi)$ in $S^m_{\rho,\delta,G(\kappa)}$ maps $\mathcal{D}^{\{\kappa\}'}_{L^2}$ to $\mathcal{D}^{\{\kappa\}'}_{L^2}$.

ii) The pseudo-differential operator $p(X,D_x)$ with a symbol $p(x,\xi)$ in $\mathcal{R}_{G(\kappa)}$ maps $\mathcal{D}^{\{\kappa\}'}_{L^2}$ to $\bigcup_{\varepsilon>0}\mathcal{D}^{\{\kappa\}}_{L^2,\varepsilon}$.

From Lemma 1.2 we can call pseudo-differential operators $p(X,D_x)$ with symbols in $\mathcal{R}_{G(\kappa)}$ regularizers.

Next, we introduce Gevrey wave front sets $WF_{G(\kappa_1)}(u)$ of ultradistributions $u(x)$ as follows.

Definition 1.3. Let κ_1 satisfy $\kappa_1 \geqq \kappa$ and let (x_0,ξ_0) be a point in $T^*(R^n)\setminus\{0\}$. Then, (x_0,ξ_0) does not belong to a Gevrey wave front set $WF_{G(\kappa_1)}(u)$ of u $(\in \mathcal{D}^{\{\kappa\}'}_{L^2})$ if and only if there exists a symbol $a(x,\xi)$ in $S^0_{G(\kappa)}$ such that the following i) and ii) hold:

i) For a conic neighborhood Γ of (x_0,ξ_0) an inequality

$$|a(x,\xi)| \geqq C \qquad \text{for } (x,\xi)\in\Gamma$$

holds for a positive constant C.

ii) The function $a(X,D_x)u$ belongs to the Gevrey class $\gamma^{(\kappa_1)}$ of order κ_1, that is, $f(x) \equiv a(X,D_x)u$ satisfies

$$|\partial_x^\alpha f(x)| \leqq CM^{-|\alpha|}\alpha!^{\kappa_1} \qquad \text{for any } x\in R^n_x.$$

For a distribution $u(x)$ $(\in \mathcal{E}')$ with compact support we have proved in [13]

Lemma 1.4 (cf. Theorem 3 of [13]). Let $u \in \mathcal{E}'$ and $\kappa_1 > 1$. Then, the point $(x_o, \xi_o) \in T^*(R^n) \setminus \{0\}$ does not belong to $WF_{G(\kappa_1)}(u)$ if and only if

(*) There exist a function $\chi(x)$ in $\gamma^{(\kappa_1)}(R_x^n)$ with $\chi(x_o) \neq 0$ and a conic neighborhood Γ_ξ of ξ_o such that

$$|F[\chi u](\xi)| \leqq C \exp(-\varepsilon \langle \xi \rangle^{1/\kappa_1}) \qquad \text{for } \xi \in \Gamma_\xi$$

holds for a positive constants C and ε.

The (*) is the original definition of the Gevrey wave front sets for distiributions given by Hörmander [3]. So, our definition is an extension of the original one. We also note that from our definition we can treat Gevrey wave front sets by the almost same way as the treatment of wave front sets for the C^∞ case (cf. [2], [7].)

2. A degenerate hyperbolic operator. Consider a degenerate hyperbolic operator

$$\text{(2.1)} \qquad L = D_t^2 - t^{2\ell} g(x)^{2\ell'} \sum_{j,j'=1}^{n} a_{j,j'}(t,x) D_{x_j} D_{x_{j'}} + t^k g(x)^{k'} \sum_{j=1}^{n} a_j(t,x) D_{x_j} + c(t,x) \quad \text{on } [0, T] \times R_x^n .$$

We first assume

(A-1) $\ell - 1 \geqq k \geqq 0$, $\ell' \geqq k' \geqq 1$ and $\ell' \geqq 2$.

Define

(2.2) $\sigma = \max((\ell - k - 1)/(2\ell - k)\ ,\ (\ell' - k')/(2\ell' - k'))$.

Then, σ satisfies $\sigma < 1/2$. Now, we assume

(A-2) $\kappa \geqq 2$ and $\kappa\sigma < 1$ with σ in (2.2).

(A-3) The function $g(x)$ belongs to a Gevrey class of order κ with a uniform estimate:

(2.3) $|D_x^\alpha g(x)| \leqq CM^{-|\alpha|}\alpha!^\kappa$ for all $x \in R^n$.

(A-4) The coefficients $a_{j,j'}(t,x)$, $a_j(t,x)$ and $c(t,x)$ are analytic in t and of a Gevrey class of order κ in x with a uniform estimate (2.3).

(A-5) $a_{j,j'}(t,x)$ are real-valued and there exists a positive constant C such that

$$\sum_{j,j'} a_{j,j'}(t,x)\xi_j\xi_{j'} \geqq C|\xi|^2 \quad \text{for all} \quad (t,x) \in [0,T]\times R_x^n.$$

Denote the characteristic roots of L by

(2.4) $\lambda_\pm(t,x,\xi) = \pm\, t^\ell g(x)^{\ell'}\{\sum_{j,j'} a_{j,j'}(t,x)\xi_j\xi_{j'}\}^{\frac{1}{2}}$

and let $\phi_\pm(t,s;x,\xi)$ be the phase functions corresponding to the modified characteristic roots

(2.5) $\tilde{\lambda}_\pm(t,x,\xi) = \lambda(t,x,\xi)(1 - \chi(\xi))$

of $\lambda_\pm(t,x,\xi)$, that is, $\phi_\pm(t,s;x,\xi)$ are solutions of

$$\begin{cases} \dfrac{d\phi_\pm}{dt} = \tilde{\lambda}_\pm(t,x,\nabla_x\phi_\pm), \\ \phi_\pm|_{t=s} = x\cdot\xi \ . \end{cases}$$

Here, in (2.5), $\chi(\xi)$ is the function in $\gamma^{(\kappa)}$ satisfying

$$\chi(\xi) = 1 \quad \text{for} \quad |\xi| \leqq 1/2, \qquad \chi(\xi) = 0 \quad \text{for} \quad |\xi| \geqq 1.$$

Then, we have

Theorem 1 (cf. Shinkai-Taniguchi [12]). Assume (A-1)-(A-5). Set $\rho = 1 - (1 - \sigma)/\ell'$. Then, for a small T_0 ($\leqq T$) the fundamental solution of the Cauchy problem

(2.6) $$\begin{cases} Lu = 0 & \text{on } [s, T_0], \\ u(s)=0, & \partial_t u(s) = u_0 \end{cases}$$

with $s \in [0, T_0)$ can be constructed in the form

(2.7) $$E(t,s) = \sum_{\pm} I_{\phi_\pm}(t,s) E_\pm(t,s) + E_0(t,s) + E_\infty(t,s).$$

Here, $I_{\phi_\pm}(t,s)$ are Fourier integral operators with phase functions $\phi_\pm(t,s;x,\xi)$ and the symbol 1; and $E_j(t,s)$, $j = 0, \pm, \infty$, are pseudo-differential operators with symbols $e_j(t,s;x,\xi)$ satisfying

(2.8) $$|\partial_\xi^\alpha \partial_x^\beta e_\pm(t,s;x,\xi)| \leqq CM^{-|\alpha+\beta|}((\alpha+\beta)!^\kappa + (\alpha+\beta)!^{\kappa\rho}\langle\xi\rangle^{(1-\rho)|\alpha+\beta|}) \times \langle\xi\rangle^{-|\alpha|}\exp(C_1\langle\xi\rangle^{\sigma'})$$

and

(2.9) $$|\partial_\xi^\alpha \partial_x^\beta e_0(t,s;x,\xi)| \leqq CM^{-|\alpha+\beta|}((\alpha+\beta)!^\kappa + (\alpha+\beta)!^{\kappa\rho}\langle\xi\rangle^{(1-\rho)|\alpha+\beta|}) \times \langle\xi\rangle^{-|\alpha|}\exp(C_1\langle\xi\rangle^{\sigma'} - \varepsilon_1 t^{\ell+1}|g(x)|^{\ell'}\langle\xi\rangle^{1-\sigma})$$

for a positive constant ε_1 and the constant σ' satisfying

(2.10) $$\sigma < \sigma' < 1/\kappa, \qquad \sigma' \geqq (1 + (\ell'-1)\sigma)/(\ell'\kappa - \ell' + 1).$$

Moreover, for any multi-index α there exists a constant C_α such that

(2.11) $$|\partial_\xi^\alpha \partial_x^\beta e_\infty(t,s;x,\xi)| \leqq C_\alpha M^{-|\beta|}\beta!^\kappa \exp(-\varepsilon_2\langle\xi\rangle)^{1/\kappa})$$

holds for a positive constsant ε_2.

We note that the factor $\exp(C_1\langle\xi\rangle^{\sigma'})$ appearing in (2.8)-(2.9) makes us to consider the Cauchy problem (2.6) of (2.1) in the class of ultradistributions and the Gevrey wave front sets.

Next, we consider the propagation of Gevrey wave front sets for the solution of (2.6). For the characteristic roots $\lambda_\pm(t, x,\xi)$ of (2.4), let $\{q^\pm,p^\pm\}(t,s;y,\eta)$ be the solution of

$$\begin{cases} \dfrac{dq^\pm}{dt} = -\nabla_\xi \lambda_\pm(t,q^\pm,p^\pm), \quad \dfrac{dp^\pm}{dt} = \nabla_x \lambda_\pm(t,q^\pm,p^\pm) \\ \qquad\qquad (s \leqq t \leqq T_0), \\ \{q^\pm,p^\pm\}|_{t=s} = (y,\eta) \end{cases}$$

and, for $s \leqq 0 \leqq t$, let $\{\tilde{q}^\pm,\tilde{p}^\pm\}(t,s;y,\eta)$ be the solution of

$$\begin{cases} \dfrac{d\tilde{q}^\pm}{dt} = -\nabla_\xi \lambda_\pm(t,\tilde{q}^\pm,\tilde{p}^\pm), \quad \dfrac{d\tilde{p}^\pm}{dt} = \nabla_x \lambda_\pm(t,\tilde{q}^\pm,\tilde{p}^\pm) \\ \qquad\qquad (0 \leqq t \leqq T_0), \\ \{\tilde{q}^\pm,\tilde{p}^\pm\}|_{t=0} = \{q^\mp,p^\mp\}(0,s;y,\eta). \end{cases}$$

Then, combining Theorem 1 and the discussion in [9] we obtain

Theorem 2. Consider a Cauchy problem (2.6) with $s < 0$. Then we have, when $t > 0$, for a solution $u(t)$ of (2.6)

(2.12) $$WF_{G(\kappa)}(u(t)) \subset \Gamma_+(t) \cup \Gamma_-(t) \cup \tilde{\Gamma}_+(t) \cup \tilde{\Gamma}_-(t) \cup \Gamma_0(t),$$

where

$$\Gamma_\pm(t) = \{(q^\pm(t,s;y,\eta)\,,\, p^\pm(t,s;y,\eta))\ ;\ (y,\eta) \in WF_{G(\kappa)}(u_0),\ |\eta| >> 1\},$$

$$\tilde{\Gamma}_\pm(t) = \{(\tilde{q}^\pm(t,s;y,\eta)\,,\, \tilde{p}^\pm(t,s;y,\eta))\ ;\ (y,\eta) \in WF_{G(\kappa)}(u_0),\ |\eta| >> 1\}$$

and

$$\Gamma_0(t) = \{(y,\eta)\ ;\ (y,\eta) \in WF_{G(\kappa)}(u_0),\ \ g(y) = 0\}.$$

This theorem corresponds to the branching property for the C^∞ case, that is, for the Cauchy problem of the operator

(2.13) $$D_t^2 - t^{2\ell} D_x^2 + at^k D_x$$

with $k = \ell - 1$ (see [1], [14] and [9]). For the case $k < \ell - 1$ in (2.13), setting $\kappa = (2\ell - k)/(\ell - k - 1)$, Shinkai gave $WF_{G(\kappa)}(u(t))$ exactly by using the exact form of the fundamental solution for the operator (2.13) (see [10] and [11]). Finally, we note that in (A-1) we assumed $\ell' \geqq 2$ and $k' \geqq 1$. But, when we assume $\ell' = 1$ or $k' = 0$ we have $\sigma = 1/2$ and the problem for the operator (2.1) is reduced to the case $1 < \kappa < 2$, which is the case of free lower order problem. In that case, $1 < \kappa < 2$, the problem (2.6) for (2.1) is always $\gamma^{(\kappa)}$-well-posed and the propagation of singularities (2.12) for solutions of (2.6) is obtained in [8]. So, we assumed $\ell' \geqq 2$ and $k' \geqq 1$.

§3. A higher order degenerate hyperbolic operator. Following [4] we consider a higher order degenerate hyperbolic operator

$$(3.1) \qquad L = D_t^m + \sum_{\substack{|\alpha|+j=m \\ j<m}} t^{|\alpha|\ell} g(x)^{|\alpha|\ell'} a_{\alpha,j}(t,x) D_x^\alpha D_t^j + \sum_{|\alpha|+j<m} t^{\nu(\alpha,j)} g(x)^{\mu(\alpha,j)} a_{\alpha,j}(t,x) D_x^\alpha D_t^j$$

with integers ℓ, ℓ' $\nu(\alpha,j)$ and $\mu(\alpha,j)$. For the operator L of (3.1) we define an irregularity θ (cf. [6]) as

$$\theta = \max_{|\alpha|+j<m} \left\{ \frac{m-j-\{\nu(\alpha,j)+m-j\}/(\ell+1)}{m-j-|\alpha|} , \frac{m-j-\mu(\alpha,j)/\ell'}{m-j-|\alpha|} , 1 \right\}$$

and set

$$(3.2) \qquad \sigma = (\theta - 1)/\theta .$$

We assume

(A-1)' $\ell' \geqq 2$ and $\theta < 2$.

Then, σ satisfies $\sigma < 1/2$. We assume, furthermore, (A-2) and

(A-3) in Section 2 and

(A-4)' The coefficients $a_{\alpha,j}(t,x)$ of (3.1) are analytic in t and of a Gevrey class of order κ in x with a uniform estimate (2.3).

(A-5)' The equation

$$\lambda^m + \sum_{\substack{|\alpha|+j=m \\ j<m}} a_{\alpha,j}(t,x)\xi^\alpha\lambda^j = 0$$

has distinct roots $\lambda_j^0(t,x,\xi)$, $j = 1,\cdots,m$, and they satisfy an uniform estimate:

$$|\lambda_j^0(t,x,\xi) - \lambda_{j'}^0(t,x,\xi)| \geqq C|\xi| \qquad \text{for all} \quad (t,x,\xi) \in [0,T]\times R^{2n}_{x,\xi}$$

with a positive constant C.

From (A-5)' the characteristic roots of (3.1) are

$$\lambda_j(t,x,\xi) = t^\ell g(x)^{\ell'}\lambda_j^0(t,x,\xi), \qquad j = 1,\cdots,m.$$

Let $\phi_j(t,s;x,\xi)$, $j=1,\cdots,m$, be the phase functions corresponding to the characteristic roots $\lambda_j(t,x,\xi)$ and denote by $I_{\phi_j}(t,s)$ the Fourier integral operators with phase functions $\phi_j(t,s;x,\xi)$ and with the symbol 1. Moreover, let $\{q^j,p^j\}(t,s;y,\eta)$ be the bicharacteristics of $\lambda_j(t,x,\xi)$ and we denote, for $s<0<t$, the (broken) bicharacteristics as $\{\tilde{q}^{j,j'},\tilde{p}^{j,j'}\}(t,s;y,\eta) = \{q^j,p^j\}(t,0;\{q^{j'},p^{j'}\}(0,s;y,\eta))$. Then, by the same method of proving Theorems 1 and 2 we obtain the following two theorems.

Theorem 3. We assume (A-1)', (A-2), (A-3), (A-4)' and (A-5)'. Let s satisfy $0 \leqq s \leqq T_0$ for a small T_0 ($\leqq T$). Then, the fundamental solution $E(t,s)$ of the Cauchy problem

$$(3.3)\quad \begin{cases} Lu = 0 & \text{on } [s, T_0], \\ \partial_t^j u(s) = 0 \ (j=0,\cdots,m-2), & \partial_t^{m-1}u(s) = u_0 \end{cases}$$

for the operator L of (3.1) can be constructed in the form

$$E(t,s) = \sum_{j=1}^{m} I_{\phi_j}(t,s)E_j(t,s) + E_0(t,s) + E_\infty(t,s)$$

with pseudo-differential operators $E_j(t,s)$, $j = 0,1,\cdots,m,\infty$, whose symbols $e_j(t,s;x,\xi) = \sigma(E_j(t,s))$ satisfy

$$|\partial_\xi^\alpha \partial_x^\beta e_j(t,s;x,\xi)| \leq CM^{-|\alpha+\beta|}((\alpha+\beta)!^\kappa + (\alpha+\beta)!^{\kappa\rho}\langle\xi\rangle^{(1-\rho)|\alpha+\beta|}) \times\langle\xi\rangle^{-|\alpha|}\exp(C_1\langle\xi\rangle^{\sigma'}) \qquad (j = 1,\cdots,m)$$

and (2.9) and (2.11). Here, σ' is a constant satisfying (2.10).

Theorem 4. Consider a Cauchy problem (3.3) with $s < 0$. Then, we have, when $t > 0$, for a solution $u(t)$ of (3.3)

$$WF_{G(\kappa)}(u(t)) \subset \bigcup_{j=1}^{m} \Gamma_j(t) \cup \bigcup_{j,j'=1}^{m} \tilde{\Gamma}_{j,j'}(t) \cup \Gamma_0(t) ,$$

where

$$\Gamma_j(t) = \{(q^j(t,s;y,\eta), p^j(t,s;y,\eta)) \; ; \; (y,\eta) \in WF_{G(\kappa)}(u_0), \; |\eta| \gg 1 \},$$

$$\tilde{\Gamma}_{j,j'}(t) = \{(\tilde{q}^{j,j'}(t,s;y,\eta), \tilde{p}^{j,j'}(t,s;y,\eta)) \; ; \; (y,\eta) \in WF_{G(\kappa)}(u_0), \; |\eta| \gg 1 \}$$

and

$$\Gamma_0(t) = \{(y,\eta) \; ; \; (y,\eta) \in WF_{G(\kappa)}(u_0), \; g(y) = 0 \}.$$

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