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Regularity of solutions for some elliptic equations with nonlinear boundary conditions

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1 Introduction

In this paper, we consider the following heat equation with nonlinear boundary condition:

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times (0,T), \\ -\dfrac{\partial u}{\partial n} = \beta(u) & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

The peculiarity of the equation lies in its nonlinear boundary condition. These nonlinear flux condition on the boundary often comes from the so-called Stefan-Boltzmann's radiation law, which says that the heat energy radiation from the surface of the body J is given by $J = \sigma(T^4 - T_s^4)$, where $\sigma > 0$ is a physical constant, T is the surface temperature and T_s is outside temperature. This nonlinear flux condition from Stefan-Boltzmann's law implies that $\beta(u)$ is monotone increasing function. In this case, the solvability and the uniqueness for this parabolic equation is completely covered by the abstract theory by H.Brézis [1].

However, if we consider the case where the heat flux radiated from the surface is reflected by its surrounding materials, then we must consider also the absorption effect. For such a case $\beta(u)$ could not be a monotone increasing function.

In fact, such a kind of non-monotone radiation-absorption model are already proposed from the view point of engineering (see e.g. [5]).

In this note we are concerned with such a non-monotone radiation-absorption model. In order to analyse the basic nature of nonlinear boundary conditions, here we consider the following elliptic equation with the nonlinear boundary condition:

$$\begin{cases} -\Delta u + bu = f(x) & \text{in } \Omega, \\ -\dfrac{\partial u}{\partial n} = \beta(u) - g(u) & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $b \geq 0$ and $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary $\partial\Omega$. We assume the following conditions:

(β1) $\beta(0) = 0$, $\beta(u)$ is continuous and monotone increasing function.

(β2) $\lim_{u\to\infty} \dfrac{\beta(u)}{u} = \infty$.

(g1) $g(0) = 0$, $g(u)$ is locally Lipschitz continuous function on $\mathbb{R}$.

(g2) There exist $k \in (0,1)$, $C_1 > 0$ such that

$$|g'(u)| \leq k\beta'(u) + C_1 \quad \text{a.e. } u \in \mathbb{R}. \tag{2}$$

Here (g2) is the crucial condition in our later arguments which implies that $g(u)$ can be regarded as the small perturbation for the leading term $\beta(u)$.

2 Main result

We set

$$D(j) = \left\{ u \in H^1(\Omega) : \int_{\partial\Omega} j(u)dS < \infty \right\}, \quad j(u) = \int_0^u \beta(s)\, ds.$$

The effective domain $D(j)$ of $j(\cdot)$ gives the natural domain where the associated functional for our equation can be well defined. Our first main result can be stated as follows.

Theorem 2.1. *Assume* (β1), (β2), (g1) *and* (g2). *Then for any $f \in L^2(\Omega)$ there exists a solution $u \in H^2(\Omega) \cap D(j)$ of* (1) *satisfying*

$$\|u\|^2_{H^2(\Omega)} + \|j(u)\|_{L^1(\partial\Omega)} \leq C(1 + \|f\|^2_{L^2(\Omega)}), \tag{3}$$

where C is a positive constant.

Remark 2.1. *When $g(u) \equiv 0$, it is well known that for any $f \in L^2(\Omega)$ there exists an unique solution $u \in H^2(\Omega)$ of* (1) *(see e.g. H. Brézis* [1]*).*
However for our case the uniqueness does not hold in general. In fact, if we take $\beta(u) = |u|^{q-2}u$ $(q > 2)$, $g(u) = \alpha u$, $f \equiv 0$ and $\alpha > 0$ large enough, then the uniqueness does not hold.

Remark 2.2. *Theorem* 2.1 *assures only the existence of solution satisfying the elliptic estimates* (3), *but does not give any information about elliptic estimates for any given weak solutions of* (1). *However if we impose the additional condition:*

($\beta 3$) $\exists C_2 > 0$ *such that* $u\beta(u) \leq C_2 j(u)$ *for all* $u \in \mathbb{R}$,

we can show that for any weak solution $u \in H^1(\Omega) \cap D(j)$ *of* (1) *should belong to* $H^2(\Omega)$ *and satisfies* (3).

Next we consider the following nonlinear elliptic equations with the nonlinear boundary condition:

$$\begin{cases} -\Delta u + bu = |u|^{p-1}u & \text{in } \Omega, \\ -\frac{\partial u}{\partial n} = |u|^{q-1}u - g(u) & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $b > 0$ and $1 < q < p < 2^* - 1 = \frac{N+2}{N-2}$.
Here, instead of (g2), we need to assume a little bit stronger condition (g2)'.

(g2)' For any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|g'(u)| \leq \epsilon |u|^{q-1} + C_\epsilon \quad \text{a.e. } \mathbb{R}. \tag{5}$$

We also need the following additional assumption.

(g3) $\lim_{u \to 0} g(u)/u = 0$.

Then our main results for (4) can be stated as follows.

Theorem 2.2. *We assume* (g1), (g2)' *and* (g3). *Then there exists a nontrivial solution* $u \in H^2(\Omega) \cap L^\infty(\Omega) \cap D(j)$ *of* (4).

Theorem 2.3. *Assume all the assumptions in Theorem* 2.2 *and let* $g(u)$ *be a odd function. Then there exist infinitely many solutions* $\{u_k\}_{k=1}$ *of* (4) *in* $H^2(\Omega) \cap L^\infty(\Omega) \cap D(j)$ *satisfying*

$$\lim_{k \to \infty} I(u_k) = \infty,$$

where

$$I(u) = \int_\Omega \frac{1}{2}(|\nabla u|^2 + bu^2)dx + \int_{\partial\Omega} (j(u) - G(u))dS - \int_\Omega F(u)dx,$$

and $j(u) = \frac{1}{q+1}|u|^{q+1}$, $G(u) = \int_0^s g(s)ds$, $F(u) = \frac{1}{p+1}|u|^{p+1}$.

3 Proofs of Theorems

3.1 Proof of Theorem 2.1

step1: Approximation problem

We rely on the variational approach. Our functional $I(\cdot)$ associated with (1) is given by

$$I(u) = \int_\Omega \frac{1}{2}\left(|\nabla u|^2 + bu^2\right) dx + \int_{\partial\Omega} (j(u) - G(u))\, dS - \int_\Omega f(x)u\, dx,$$

where $j(u) = \int_0^u \beta(s)ds$, $G(u) = \int_0^u g(s)ds$. But this functional may not be defined on $H^1(\Omega)$ in general, since the term $j(u)$ and $G(u)$ may not be integrable for all $u \in H^1(\Omega)$. To avoid this difficulty, we introduce the following approximations $\beta_n(\cdot)$ and $g_n(\cdot)$ for $\beta(\cdot)$ and $g(\cdot)$ respectively.

$$\beta_n(u) = \begin{cases} \beta(n) + (u-n) & u > n, \\ \beta(u) & |u| \le n, \\ \beta(-n) + (u+n) & u < -n, \end{cases} \qquad g_n(u) = \begin{cases} g(n) + (u-n) & u > n, \\ g(u) & |u| \le n, \\ g(-n) + (u+n) & u < -n. \end{cases}$$

Then approximation problem associated with these approximations is given by

$$\begin{cases} -\Delta u + bu = f(x) & \text{in } \Omega, \\ -\dfrac{\partial u}{\partial n} = \beta_n(u) - g_n(u) & \text{on } \partial\Omega. \end{cases} \tag{6}$$

By the trace embedding theorem $(H^1(\Omega) \subset L^2(\partial\Omega))$, we can well define on $H^1(\Omega)$ the associated functional I_n for the approximation problem (6).

$$I_n(u) = \int_\Omega \frac{1}{2}\left(|\nabla u|^2 + bu^2\right) dx + \int_{\partial\Omega} (j_n(u) - G_n(u))\, dS - \int_\Omega f(x)u\, dx.$$

We can easily find that there exists a minimizer u_n of I_n. In fact, by assumption (β1) and (g2) we can easily check I_n is bounded below and I_n is coercive on $H^1(\Omega)$. Thus there exists a global minimizer u_n of I_n, and u_n gives a solution of the approximation problem (6).

step2: A priori estimates

Multiplying (6) by u_n, integrating over Ω and using assumption (g2), we get

$$\|u_n\|^2_{H^1(\Omega)} + \|j_n(u_n)\|_{L^1(\partial\Omega)} \le C\left(1 + \|f\|^2_{L^2(\Omega)}\right), \tag{7}$$

where C is independent of n.
The following H^2-estimate is a key lemma.

Lemma 3.1. *There exists a positive constant C independent of n such that*

$$\|u_n\|_{H^2(\Omega)} \le C\left(\|u_n\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}\right). \tag{8}$$

Proof. The interior estimate can be done by the standard arguments, since it is not affected by the (nonlinear) boundary condition. As for the estimates near the boundary, we need to work by using local charts as in [1].
Let $x_0 \in \partial\Omega$ and U is a neighborhood of x_0, and let $H : Q_+ \to \Omega \cap U$ be a standard transformation mapping with $Q_+ = \{y = (y', y_N); |y'| < 1,\ 0 < y_N < 1\}$ and $Q_0 = \{y = (y', 0); |y'| < 1\}$. We define $\tilde{u}_n = u_n \circ H$, $\tilde{f} = f \circ H$. In the new coordinate, $\tilde{u}_n \in H^1(Q_+)$ satisfies

$$\sum_{i,j=1}^{N} \int_{Q_+} a_{ij}(y) \frac{\partial \tilde{u}_n}{\partial y_i} \frac{\partial \phi}{\partial y_j} J(y) dy + \int_{Q_+} b \tilde{u}_n \phi J(y) dy$$
$$+ \int_{Q_0} (\beta_n(\tilde{u}_n) - g_n(\tilde{u}_n)) \phi \sigma(y') dy' = \int_{Q_+} \tilde{f} \phi J(y) dy, \tag{9}$$

for any $\phi \in \{\phi \in C^1(\overline{Q_+}); \mathrm{supp}\ \phi \subset Q_+ \cup Q_0\}$, where $J(y)$ is the absolute value of Jacobian, $\sigma(y')$ is the surface element, and $a_{ij}(y)$ is a coefficient satisfying the uniformly elliptic condition.
We test (9) by the following function ϕ given by,

$$\phi = D_{-h}(\theta^2 D_h \tilde{u}_n) \frac{1}{\sigma(y')}$$

where $D_h \tilde{u} = \frac{1}{|h|}(\tau_h \tilde{u} - \tilde{u})$, $\tau_h \tilde{u}(y) = \tilde{u}(y + h)$, h is a vector orthogonal to y_N and θ is a smooth function composing the partition of unity.
Let $\tilde{v}_n = \theta \tilde{u}_n$. Since $a_{ij}(y)$ satisfies the uniformly elliptic condition, each term in (9) is estimated as

$$\begin{aligned}
\text{(the first term)} &\geq a_0 \|D_h \nabla \tilde{v}_n\|_{L^2}^2 - C \|D_h \nabla \tilde{v}_n\|_{L^2} \|\tilde{u}_n\|_{H^1} - C \|\tilde{u}_n\|_{H^1}^2, \\
\text{(the second term)} &\leq C \|\tilde{u}_n\|_{H^1}^2, \\
\text{(the fourth term)} &\leq C \|\tilde{f}\|_{L^2} (\|D_h \nabla \tilde{v}_n\|_{L^2} + \|\tilde{u}_n\|_{H^1}).
\end{aligned}$$

The following estimate for the third term is crucial.

$$\begin{aligned}
\text{(the third term)} &= \int_{Q_0} D_h(\beta_n(\tilde{u}_n) - g_n(\tilde{u}_n)) \theta^2 D_h \tilde{u}_n \, dy', \\
&\geq \int_{Q_0} (D_h \tilde{u}_n)^2 \theta^2 \int_0^1 \Big(\beta_n'(s\tau_h \tilde{u}_n + (1-s)\tilde{u}_n) \\
&\qquad - g_n'(s\tau_h \tilde{u}_n + (1-s)\tilde{u}_n) \Big) ds dy', \\
&\geq \int_{Q_0} (D_h \tilde{u}_n)^2 \theta^2 \int_0^1 ((1-k)\beta_n' - C_1) ds dy', \\
&\geq -C \left(\int_{Q_0} (D_h \tilde{v}_n)^2 dy' + \int_{Q_0} \tilde{u}_n^2 \, dy' \right), \\
&\geq -\epsilon \|D_h \tilde{v}_n\|_{H^1(Q_+)}^2 - C(\epsilon) \|D_h \tilde{v}_n\|_{L^2(Q_+)}^2 - C \|\tilde{u}_n\|_{H^1(Q_+)}^2.
\end{aligned}$$

In the first inequality, we used the following Lemma 3.2 and the last inequality is deduced from the interpolation lemma and the trace lemma.

Lemma 3.2. *Let f be a monotone increasing function. Then*

$$\int_a^b f'(s) \leq f(b) - f(a).$$

Consequently combing these estimates, we get

$$\left\| \frac{\partial^2 \tilde{v}_n}{\partial y_i \partial y_j} \right\|_{L^2(Q_+)} \leq C \left(\|\tilde{u}_n\|_{H^1(Q_+)} + \|\tilde{f}\|_{L_2(Q_+)} \right),$$

for $(i,j) \neq (N,N)$.
To obtain the estimate for $\frac{\partial^2 \tilde{v}_n}{\partial y_N^2}$, going back to (9) and choosing $\phi = \frac{\theta \psi}{a_{NN} J}$, we obtain

$$\begin{aligned} {}_{\mathscr{D}'(Q_+)}\langle -\frac{\partial^2 \tilde{v}_n}{\partial y_N^2}, \psi \rangle_{\mathscr{D}(Q_+)} &= \int_{Q_+} \frac{\partial \tilde{v}_n}{\partial y_N} \frac{\partial \psi}{\partial y_N} dy \\ \leq C &\left(\sum_{(i,j)\neq(N,N)} \left\| \frac{\partial^2 \tilde{v}_n}{\partial y_i \partial y_j} \right\|_{L^2(Q_+)} + \|\tilde{u}_n\|_{H^1(Q_+)} + \|\tilde{f}\|_{L_2(Q_+)} \right) \|\psi\|_{L^2(Q_+)}, \end{aligned}$$

for any $\psi \in C_c^\infty(Q_+)$.
Thus H^2-estimate for $\tilde{v}_n$ is derived. This estimate leads to the estimate for u_n and (8) is assured.

step3: Convergence to the original problem

By (7) and (8), $\{u_n\}_{n\in\mathbf{N}}$ is bounded in $H^2(\Omega)$. Then there exists $u \in H^2(\Omega) \cap D(j)$ and subsequences $\{u_n\}_{n\in\mathbf{N}}$ such that

$$\begin{aligned} u_n &\rightharpoonup u && \text{weakly in } H^2(\Omega), \\ u_n(x) &\to u(x) && \text{a.e. } \partial\Omega, \\ \frac{\partial u_n}{\partial n}(x) &\to \frac{\partial u}{\partial n}(x) && \text{a.e. } \partial\Omega. \end{aligned}$$

Hence by Lebesgue's dominant convergence theorem and by the construction of β_n, g_n, we can show that

$$\begin{aligned} \beta_n(u_n) &\to \beta(u) \quad \text{in } L^2(\partial\Omega), \\ g_n(u_n) &\to g(u) \quad \text{in } L^2(\partial\Omega). \end{aligned}$$

Thus this u gives a solution of original problem (1). □

3.2 Proof of Theorem 2.2

For this case, we use simpler approximations for β, g than previous ones. We set $\beta(u) = |u|^{q-1}u$ and

$$\beta_n(u) = \begin{cases} \beta(n) & u > n, \\ \beta(u) & |u| \le n, \\ \beta(-n) & u < -n, \end{cases} \qquad g_n(u) = \begin{cases} g(n) & u > n, \\ g(u) & |u| \le n, \\ g(-n) & u < -n. \end{cases}$$

step1: Approximation problem

We again rely on the variational approach. The associated functional is defined by

$$I_n(u) = \int_\Omega \frac{1}{2}(|\nabla u|^2 + bu^2)dx + \int_{\partial\Omega}(j_n(u) - G_n(u))dS - \int_\Omega F(u)dx.$$

We can easily see the existence of critical point u_n of I_n by the following mountain pass lemma.

Lemma 3.3 ([4]). *Let E be a real Banach space and $I \in C^1(E;\mathbb{R})$ satisfying* (PS)*-condition. Suppose $I(0) = 0$ and*

(I_1) *there are constant $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \ge \alpha$,*
(I_2) *there is $e \in E \setminus B_\rho$ such that $I(e) \le 0$,*

where $B_\rho = \{z \in E; \|z\|_E < \rho\}$. Then I possesses a critical value $c \ge \alpha$. Moreover c can be characterized as

$$c = \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} I(u),$$

where $\Gamma = \{\gamma \in C([0,1];E); \gamma(0) = 0, \gamma(1) = e\}$.

In fact by (g2)' and (g3), there exists $R > 0$ independent of n such that

$$\begin{aligned} j_n(u) - G_n(u) &\ge -\epsilon u^2 - C_\epsilon |u|^{q^*} && \text{for } |u| \le R, \\ j_n(u) - G_n(u) &\ge 0 && \text{for } |u| \ge R, \end{aligned} \tag{10}$$

where $q^* = \frac{2(N-1)}{N-2}$. Thus by using the fact $H^1(\Omega) \subset L^{q^*}(\partial\Omega)$, we have

$$\begin{aligned} \int_{\partial\Omega}(j_n(u) - G_n(u))dS &\ge \int_{\partial\Omega\cap\{|u(x)|\le R\}}(j_n(u) - G_n(u))dS, \\ &\ge -\int_{\partial\Omega\cap\{|u(x)|\le R\}}(\epsilon u^2 + C_\epsilon u^{q^*})dS, \\ &\ge -\epsilon\|u\|^2_{H^1(\Omega)} - C_\epsilon\|u\|^{q^*}_{H^1(\Omega)}. \end{aligned}$$

Thus there exist $\rho, \alpha > 0$, which are independent of n, such that

$$I_n|_{\partial B_\rho} \ge \alpha. \tag{11}$$

Let ϕ_1 be a first eigenfunction of $-\Delta\phi = \lambda\phi$, $\phi|_{\partial\Omega} = 0$. If we take $\|\phi_1\|_{H^1(\Omega)}$ large enough, $I_n(\phi_1) \leq 0$. Hence (I_1) and (I_2) are verified.

step2: H^1-estimates

Since u_n is a critical point of I_n, we have

$$\begin{aligned} I_n(u_n) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega (|\nabla u_n|^2 + bu_n^2)\,dx \\ &\quad + \int_{\partial\Omega}\left(j_n(u) - \frac{1}{p+1}\beta_n(u)u - \left(G_n(u) - \frac{1}{p+1}g_n(u)\right)\right)dS. \end{aligned}$$

By the construction of β_n, g_n and (g2)',

$$\begin{aligned} &j_n(u) - \tfrac{1}{p+1}\beta_n(u)u - \left(G_n(u) - \tfrac{1}{p+1}g_n(u)\right) \geq 0 && \text{for } |u| > n, \\ &j_n(u) - \tfrac{1}{p+1}\beta_n(u)u - \left(G_n(u) - \tfrac{1}{p+1}g_n(u)\right) \geq (C-\epsilon)|u|^{q+1} - C_\epsilon && \text{for } |u| \leq n. \end{aligned}$$

Thus we get

$$I_n(u_n) \geq \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega (|\nabla u_n|^2 + bu_n^2)dx - C. \tag{12}$$

To obtain H^1-estimate, we need the boundedness of $I_n(u_n)$. But we note that $e = \phi_1$ can be taken independent of n in Lemma 3.3. Hence since $t\phi_1 \in \Gamma$ for all n, we get

$$I_n(u_n) = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_n(\gamma(t)) \leq \max_{t\in[0,1]} I_n(t\phi_1) = \max_{t\in[0,1]} I(t\phi_1). \tag{13}$$

Combing (12) and (13) we obtain the following H^1-estimate.

$$\|u_n\|_{H^1(\Omega)} \leq C, \tag{14}$$

where C is independent of n.

step3: L^∞-estimates

Here we consider the following linear equation.

$$\begin{cases} -\Delta V + bV = f(x) & \text{in } \Omega, \\ -\frac{\partial V}{\partial n} = \beta_n(V) - g_n(V) & \text{on } \partial\Omega, \end{cases} \tag{15}$$

where $b \geq 0$.

Lemma 3.4. *Assume* (g1)*and* (g2) *and* $r > N$. *Let* $V \in H^1(\Omega)$ *be a weak solution of* (15), *then*

$$\|V\|_{L^\infty(\Omega)} \leq C\left(1 + \|V\|_{L^2(\Omega)} + \|f\|_{L^{\frac{r}{2}}(\Omega)}\right), \tag{16}$$

where C is independent of n.

Proof. Let V be a weak solution of (15) and set

$$w = (V - R)^+ + k,$$

where R is large enough constant such that $\beta(u) - g(u) \geq 0$ for $u \geq R$, and $k \geq 0$, $\gamma \geq 1$ are chosen later.
We put

$$\xi(t) = \begin{cases} 0 & |t| < k, \\ |t|^\gamma - k^\gamma & k \leq |t|, \end{cases}$$

and we use the test function $\phi = \xi \circ w$. Since $\phi = 0$ if $V \leq R$ and $(\beta_n(V) - g_n(V))\phi \geq 0$ if $V > R$,

$$\int_\Omega (\nabla V \cdot \nabla \phi + bV\phi)dx \leq \int_\Omega f\phi \, dx.$$

Hence, since $V \leq w$ if $k \geq R$, we have

$$\gamma \int_\Omega w^{\gamma-1}|\nabla w|^2 dx \leq \int_\Omega |f| w^\gamma dx, \tag{17}$$

for $k \geq R$.
We choose $k = \|f\|_{L^{\frac{r}{2}}(\Omega)} + R$. Let $z = w^{\frac{\gamma+1}{2}}$, then Hölder' s inequality gives

$$\int_\Omega |f| w^\gamma dx \leq \int_\Omega \frac{|f|}{k} w^{\gamma+1} dx = \int_\Omega \frac{|f|}{k} z^2 dx \leq \|z\|^2_{L^{\frac{2r}{r-2}}(\Omega)}. \tag{18}$$

Since $2 < \frac{2r}{r-2} < 2^* = \frac{2N}{N-2}$, we can use the interpolation inequality,

$$\|z\|_{L^{\frac{2r}{r-2}}(\Omega)} \leq \epsilon \|z\|_{L^{2^*}(\Omega)} + \epsilon^{-\sigma}\|z\|_{L^2(\Omega)},$$

where $\sigma = \frac{N}{r-N}$. Thus (18) is rewritten as

$$\begin{aligned} \int_\Omega |f| w^\gamma dx &\leq \epsilon^2 \left(\int_\Omega \left(w^{\frac{\gamma+1}{2}} \right)^{2^*} dx \right)^{\frac{2}{2^*}} + C\epsilon^{-2\sigma} \int_\Omega \left(w^{\frac{\gamma+1}{2}} \right)^2 dx, \\ &\leq \epsilon^2 \int_\Omega \left| \nabla \left(w^{\frac{\gamma+1}{2}} \right) \right|^2 dx + C\epsilon^{-2\sigma} \int_\Omega \left(w^{\frac{\gamma+1}{2}} \right)^2 dx, \end{aligned}$$

where we used Sobolev's inequality. Choosing $\epsilon^2 = \frac{1}{\gamma+1}$, we plug this formula into (17) to get

$$\int_\Omega w^{\gamma-1}|\nabla w|^2 dx \leq C\gamma^{\sigma-1} \int_\Omega w^{\gamma+1} dx, \tag{19}$$

which implies

$$\|w\|_{L^{\frac{2^*}{2}(\gamma+1)}(\Omega)} \leq C^{\frac{1}{\gamma+1}} \|w\|_{L^{\gamma+1}(\Omega)}.$$

We set $\gamma_0 = 1$, $\gamma_{i+1} + 1 = \frac{2^*}{2}(\gamma_i + 1)$,

$$\|w\|_{L^{\gamma_{i+1}+1}(\Omega)} \le C^{\frac{1}{\gamma_i+1}} \|w\|_{L^{\gamma_i+1}(\Omega)} \le C^{\frac{1}{\gamma_0+1}\sum_{j=0}^{\infty}(\frac{2}{2^*})^j} \|w\|_{L^{\gamma_0+1}(\Omega)}.$$

Thus we get

$$\|V^+\|_{L^\infty(\Omega)} \le C\left(1 + \|V\|_{L^2(\Omega)} + \|f\|_{L^{\frac{N}{2}}(\Omega)}\right).$$

By the quite same argument, we can obtain the estimate for V^-. □

Next we give an estimate for nonlinear problem (4).

Lemma 3.5. *Assume* (g1) *and* (g2). *For any $\gamma \ge 1$ there exist $C > 0$ and $\gamma^* \ge 1$ which are independent of n such that any weak solution u_n of* (4) *satisfies*

$$\|u_n\|_{L^\gamma(\Omega)} \le C\left(\|u_n\|^{\gamma^*}_{L^{2^*}(\Omega)} + 1\right). \tag{20}$$

Proof. We repeat almost the same procedure as above.
We set $w_n = (u_n - R)^+ + R$, and take $\phi_n = \xi \circ w_n$ as a test function where R, ξ are given in previous lemma. By the same reasoning as before, we get

$$\int_\Omega w_n^{\gamma-1}|\nabla w_n|^2 \le \int_\Omega |u_n|^p w_n^\gamma.$$

We note that $|u_n| \le w_n$ and $w_n \ge 1$ by the definition of w_n, thus

$$\int_\Omega w_n^{\gamma-1}|\nabla w_n|^2 \le \int_\Omega w_n^{\gamma+p}.$$

By Sobolev's inequality, we get

$$\|w_n\|_{L^{\frac{2^*}{2}(\gamma+1)}(\Omega)} \le C^{\frac{1}{\gamma+1}} \|w_n\|^{\frac{\gamma+p}{\gamma+1}}_{L^{\gamma+p}(\Omega)}.$$

We set $\gamma_1 + p = 2^*$, $\gamma_{i+1} + p = \frac{2^*}{2}(\gamma_i + 1)$,

$$\|w_n\|_{L^{\gamma_{i+1}+p}(\Omega)} \le C^{\frac{1}{\gamma_i+1}} \|w_n\|^{\frac{\gamma_i+p}{\gamma_i+1}}_{L^{\gamma_i+p}(\Omega)}.$$

Hence for any $\delta \ge 2^*$, there exist $C, \gamma^* > 0$ which are independent of n such that

$$\|w_n\|_{L^\delta(\Omega)} \le C\|w_n\|^{\gamma^*}_{L^{2^*}(\Omega)}.$$

By the same way as in the previous lemma, we obtain (20). □

step4: H^2-estimates

We apply (8) by regarding the nonlinear term $|u|^{p-1}u$ as the given external term f. By Lemma 3.5,

$$\begin{aligned} \|u_n\|^2_{H^2(\Omega)} &\leq C\left(\|u_n\|^2_{H^1(\Omega)} + \|f(u_n)\|^2_{L^2(\Omega)}\right) = C\left(\|u_n\|^2_{H^1(\Omega)} + \|u_n\|^{2p}_{L^{2p}(\Omega)}\right), \\ &\leq C\left(\|u_n\|^2_{H^1(\Omega)} + \|u_n\|^{\gamma^*}_{L^{2^*}(\Omega)} + 1\right), \end{aligned} \tag{21}$$

where C, γ^* are independent of n.

step5: Convergence to the original problem

By Lemma 3.4, (14) and (21), $\{u_n\}_{n\in\mathbf{N}}$ is bounded in $H^2(\Omega) \cap L^\infty(\Omega)$. By the quite same argument as before, there exist $u \in H^2(\Omega) \cap L^\infty(\Omega) \cap D(j)$ and a subsequence $\{u_n\}_{n\in\mathbf{N}}$ such that $u_n \rightharpoonup u$ weakly in $H^2(\Omega)$. Thus u turns out to be our desired solution of (4).

3.3 Proof of Theorem 2.3

In this proof, we use the following symmetric mountain pass lemma.

Lemma 3.6 (symmetric mountain pass lemma). *Let E be a real Banach space and $E_m = \text{span}\{e_1, e_2, \cdots, e_m\} \subset E$ where $\{e_i\}_{i=1}$ are any linearly independent vectors in E. We assume*

(1) *$I \in C^1(E; \mathbb{R})$ is even and satisfies* (PS)*-condition,*
(2) *there exists $\alpha, \rho > 0$ such that $I|_{\partial B_\rho} \geq \alpha$,*
(3) *there exists $R_m > 0$ such that $I \leq 0$ on $E_m \setminus B_{R_m}$.*

Then there exist infinitely many critical points $\{u_j\}_{j=1}$ of I satisfying,

$$\lim_{j\to\infty} I(u_j) = \infty.$$

Moreover the critical value is characterized as

$$I(u_j) = \inf_{h\in\Gamma} \max_{u\in E_j} I_n(h(u)),$$

where $E_j = span\{e_1, e_2, \cdots, e_j\}$ and $\Gamma = \{h \in C(E; E); h$ is odd, $h(u) = u\ \forall u \in E_m \backslash B_{R_m}\}$.

In our case, we can take ϕ_i as e_i independent of n in Lemma 3.6, where ϕ_i is the i-th eigenfunction of $-\Delta\phi = \lambda\phi$, $\phi|_{\partial\Omega} = 0$. By Lemma 3.6 for any $n \in \mathbb{N}$ there exist infinitely many critical points $\{u_n^j\}_{j\in\mathbf{N}}$ of I_n satisfying

$$\lim_{j\to\infty} I_n(u_n^j) = \infty. \tag{22}$$

Here we set $c_n^j = I_n(u_n^j)$, then this sequences $\{c_n^j\}$ are expressed as

$$\begin{array}{llllllllll} n=1 & c_1^1 & \le & c_1^2 & \le & c_1^3 & \le \cdots & \le & c_1^j & \to \infty, \\ n=2 & c_2^1 & \le & c_2^2 & \le & c_2^3 & \le \cdots & \le & c_2^j & \to \infty, \\ & \cdots & \le & \cdots & \le & \cdots & \le \cdots & \le & \cdots & \to \infty. \end{array}$$

First we show that $\lim_{n\to\infty} c_n^j$ exists for any $j \in \mathbb{N}$.
By assumption (g2)' we find that $I_n(u) \le I_{n+1}(u)$ for all $u \in H^1(\Omega)$ for large $n \in \mathbb{N}$. Thus without loss of generality, we can assume

$$c_n^j \le c_{n+1}^j. \tag{23}$$

Moreover since we note that e_i can be chosen independent of n in Lemma 3.6,

$$c_n^j = \inf_{h\in\Gamma} \max_{u\in E_j} I_n(h(u)) \le \max_{u\in E_j} I(u) = C^j.$$

Thus $c_*^j = \lim_{n\to\infty} c_n^j$ exists for all $j \in \mathbb{N}$.
By the same argument as before, there exist solutions $\{u_*^j\}_{j=1}$ of (4) satisfying $u_*^j \in H^2(\Omega) \cap L^\infty(\Omega) \cap D(j)$ and $I(u_*^j) = c_*^j$. If $\lim_{j\to\infty} c_*^j = \infty$, the proof is finished.
In fact, for c_*^1 there exists $c_1^{l_1}$ such that $c_*^1 < c_1^{l_1}$ by (22). Similarly we can find $c_1^{l_2}$ satisfying $c_*^{l_1} < c_1^{l_2}$. Repeating this procedure, we get sequences $\{c_*^{l_j}\}_{j=1}$ satisfying

$$c_*^1 < c_1^{l_1} \le c_*^{l_1} < \cdots \le c_*^{l_j} < c_1^{l_{j+1}} \le c_*^{l_{j+1}} \le \cdots.$$

This implies $\lim_{j\to\infty} c_*^j = \infty$. □

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