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# Swaption pricing: A Binomial Approach

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## 1 Introduction

The Libor market model and Swap market model are inconsistent with each other in that they can not be simultaneously described by log-normal processes.

The market quotes the at-the-money caps in term of their Black implied volatilities. From these, one can infer caplet volatilities. Caplet implied volatilities give information about the distribution of forward Libor. The market seems to assume that it is log-normal with volatility. At-the-money European swaptions are also quoted in term of their Black implied volatilities which give information about distribution of swap rate. The Black model pricing is assumed that forward swap rate follows log-normal distribution.

The purpose of this paper is to build an arbitrage-free lattice model for swaption, which is consistent with Libor market model and which provides an implementation method for the theoretical closed-form formula which is difficult to get numerical solution. We furthermore compare the approximations of swaption pricing between swap market model and binomial lattice in theoretical and numerical aspect.

There are several papers on solving inconsistency of two market models. We can see the swap volatility approximation by Libor volatility in Rebonate[5] or Brigo[1]. These approaches are mainly to adjust the swap volatility by using Libor volatility. Recently, however, Davis and Mataix-Pastor[2] have shown the possibility of negative forward Libor rate from coexistence of Libor market model and Swap market model. This negative forward Libor could give us arbitrage opportunity. Our approximation by lattice would make it possible to get arbitrage opportunity.

The rest of this paper is organized as follows. In Section 2 we provide notation and introduce Libor market model which is based on HJM model. In Section 3, we derive European payer swaption price formula for Gaussian volatility. The formula is a weighted average of discount bonds with Gaussian distribution weight. However, it is not easy task to find numerical solution of function which satisfy positivity of swaption. In section 4, we propose the numerical method to get a solution of this function by binomial lattice, which uses the change of measure technique based on Jamishidan [3]. In Section 5 is devoted to numerical example of flat term structure of Libor and volatility. After providing Swap market model with European payer swaption formula, we compare numerical values of coefficient for discount bonds in the portfolio of bonds replicating the swap. Finally, we discuss the replication strategy for arbitrage and closing remarks.

## 2 Libor model

We assume HJM-model for discounted bond prices of maturity  $T_i$ ,  $\{B_i(t)\}_{0 \leq t \leq T_i}$  under risk neutral measure  $Q$ ,

$$dB_i(t)/B_i(t) = r(t)dt + \sigma^i(t)dW(t)$$

where the time span is  $\delta = t_{i+1} - t_i$ , for  $i = 0, \dots, N - 1$ . Let  $r(t)$  be spot rate and  $\sigma^i(t)$  be the volatility of discount bond. The bond price of maturity of  $T_i$  at time  $T_m$  is, for  $t \leq T_m \leq T_i$

$$B_i(T_m) = B_i(t) \exp \left( \int_t^{T_m} r(s) ds + \int_t^{T_m} \sigma^i(s) dW(s) - \frac{1}{2} \int_t^{T_m} |\sigma^i(s)|^2 ds \right), \quad (2.1)$$

and for the bond price of maturity  $T_{i+1}$  is

$$B_{i+1}(T_m) = B_{i+1}(t) \exp \left( \int_t^{T_m} r(s) ds + \int_t^{T_m} \sigma^{i+1}(s) dW(s) - \frac{1}{2} \int_t^{T_m} |\sigma^{i+1}(s)|^2 ds \right). \quad (2.2)$$

Let  $L_i(t)$  be a forward Libor from  $T_i$  to  $T_{i+1}$ , then the Libor process is defined as

$$L_i(t) = \delta^{-1} \left( \frac{B_i(t)}{B_{i+1}(t)} - 1 \right).$$

Dividing (2.1) by (2.2) and from the definition of Libor we get

$$\frac{1 + \delta L_i(T_m)}{1 + \delta L_i(t)} = \exp \left( \int_t^{T_m} [\sigma^i(s) - \sigma^{i+1}(s)] dW(s) - \frac{1}{2} \int_t^{T_m} [|\sigma^i(s)|^2 - |\sigma^{i+1}(s)|^2] ds \right) \quad (2.3)$$

In HJM-model the forward process of settlement time  $T$  is modeled as

$$df_T(t) = \mu_T(t) dt + \sigma_T(t) dW(t)$$

where  $\sigma_T$  is the volatility of forward process  $\{f_T(t)\}$ . For the settlement time  $T_i$  we write the volatility  $\sigma_i(t)$  instead of  $\sigma_{T_i}(t)$ . The bond price at  $t$  of maturity  $T$  in (2.1) divided by (2.2) and let  $B_t(t) = 1$ , then

$$B_T(t) = \frac{B_T(0)}{B_t(0)} \exp \left( \int_0^t (\sigma^T(s) - \sigma^t(s)) dW(s) - \frac{1}{2} \int_0^t (|\sigma^T(s)|^2 - |\sigma^t(s)|^2) ds \right)$$

Forward rate is defined as  $f_T(t) = -\frac{\partial}{\partial T} \log B_T(t)$  and then

$$df_T(t) = \sigma^T(t) \frac{\partial}{\partial T} \sigma^T(t) dt - \frac{\partial}{\partial T} \sigma^T(t) dW(t)$$

By Itô's division rule

$$\frac{d(B_i(t)/B_{i+1}(t))}{B_i(t)/B_{i+1}(t)} = (\sigma^i(t) - \sigma^{i+1}(t))(dW(t) - \sigma^{i+1}(t)dt)$$

Under the risk adjusted measure  $Q^{i+1}$

$$\frac{dL_i(t)\delta}{1 + \delta L_i(t)} = (\sigma^i(t) - \sigma^{i+1}(t)) dW^{i+1}(t),$$

when we define the risk adjusted Measure  $Q^{i+1}$  by  $dQ^{i+1}/dQ = \mathcal{E}(\int_0^{T_{i+1}} \sigma^{i+1}(t) dW(t))$ , then  $W^{i+1}(t) = W(t) - \int_0^t \sigma^{i+1}(s) ds$  is Brownian motion under  $Q^{i+1}$  where  $\mathcal{E}(\cdot)$  is stochastic exponential.

Therefore Libor  $L_i(t)$  is  $Q^{i+1}$ -martingale as,

$$E_t^{i+1}[L_i(T_m)] = L_i(t).$$

The Bond Volatility  $\sigma^T(t) = -\int_t^T \sigma_s(t)ds$  and let  $v_i(t)$  be the volatility of Libor  $L_i(t)$  ;

$$v_i(t) = \sigma^i(t) - \sigma^{i+1}(t) = -\int_t^{T_i} \sigma_u(t)du + \int_t^{T_{i+1}} \sigma_u(t)du = \int_{T_i}^{T_{i+1}} \sigma_u(t)du$$

In section 4 of binomial lattice model we assume  $v_i$  is constant for  $(T_i, T_{i+1})$  and in numerical experiment section 5 assume a constant  $v = v_i, \forall i$ . Libor process is expressed under  $Q^{i+1}$  from (2.3) as follows,

$$L_i(T_m) = \delta^{-1}(1 + \delta L_i(t)) \exp\left\{\int_t^{T_m} v_i(s)dW^{i+1}(s) - \frac{1}{2}\int_t^{T_m} |v^i(s)|^2 ds\right\} - 1$$

### 3 Swaption price of Gaussian volatility

The payer swaption is the option with strike swap rate  $k$  and the maturity  $T_n$ , where the underlying swap contract starts from  $T_n$  to  $T_N$  and payment period  $\delta = T_i - T_{i-1}, i = n+1, \dots, N$ . The payment at the maturity is

$$A(T_n) = \max(B_n(T_n) - B_N(T_n) - k\delta \sum_{i=n+1}^N B_i(T_n), 0)$$

where,  $B_i(T_j)$  denotes the price at  $T_j$  of bond of maturity time  $T_i$ .

The bond price of maturity  $T_n$  is 1 at time  $T_n$  then  $A(T_n) = \max(1 - V(T_n), 0)$  is a put option on bonds portfolio, where

$$V(T_n) = B_N(T_n) + k\delta \sum_{i=n+1}^N B_i(T_n)$$

Under risk neutral measure  $Q$ , the price of swaption at time 0 is

$$S(0) = E^Q[\exp\{-\int_0^{T_n} r(s)ds\}A(T_n)]$$

**Theorem 1** The swaption price of Gaussian volatility HJM model is given as follows,

$$S(0) = B_n(0)N(d_n) - B_N(0)N(d_N) - k\delta \sum_{i=n+1}^N B_i(0)N(d_i) \quad (3.1)$$

where  $d_i = d_n - \int_0^{T_n} (\sigma^i(s) - \sigma^n(s))ds, i = n+1, \dots, N$  and  $d_n$  is the solution of equation;

$$\begin{aligned} f(x) &= \frac{B_N(0)}{B_n(0)} \exp\{v(0, T_n, T_N)\sqrt{T_n}x - \frac{1}{2}v(0, T_n, T_N)^2T_n \\ &+ k\delta \sum_{i=n+1}^N \frac{B_i(0)}{B_n(0)} \exp\{v(0, T_n, T_i)\sqrt{T_n}x - \frac{1}{2}v(0, T_n, T_i)^2T_n\} - 1 = 0 \end{aligned} \quad (3.2)$$

where let the variance process  $v(t, T_n, T_i)^2 = \frac{1}{T_n-t} \int_t^{T_n} |\sigma^i(t) - \sigma^n(t)|^2 dt$ .

*Proof.* Taking  $B_n(t)$  as the numeraire for the payoff at time  $T_n$ ;

$$\begin{aligned} \frac{V(T_n) - 1}{B_n(T_n)} &= \frac{B_N(0)}{B_n(0)} \exp\left\{\int_0^{T_n} (\sigma^N(t) - \sigma^n(t))dW^n(t) - \frac{1}{2} \int_0^{T_n} |\sigma^N(t) - \sigma^n(t)|^2 dt\right\} \\ &\quad + k\delta \sum_{i=n+1}^N \frac{B_i(0)}{B_n(0)} \exp\left\{\int_0^{T_n} (\sigma^i(t) - \sigma^n(t))dW^n(t) - \frac{1}{2} \int_0^{T_n} |\sigma^i(t) - \sigma^n(t)|^2 dt\right\} - 1 \end{aligned}$$

Let  $U_n$  be a standard normal distributed variate, i.e.  $U_n \sim N(0, 1)$  and define the function;

$$\begin{aligned} f(U_n) &= \frac{B_N(0)}{B_n(0)} \exp\{v(0, T_n, T_n)\sqrt{T_n}U_n - \frac{1}{2}v(0, T_n, T_n)^2T_n\} \\ &\quad + k\delta \sum_{i=n+1}^N \frac{B_i(0)}{B_n(0)} \exp\{v(0, T_n, T_i)\sqrt{T_n}U_n - \frac{1}{2}v(0, T_n, T_i)^2T_n\} - 1 \end{aligned}$$

where the normal variate  $\int_0^{T_n} (\sigma^i(t) - \sigma^n(t))dW^n(t) \sim N(0, v(0, T_i, T_n)^2T_n)$ .

The swaption price under risk neutral becomes as follows, with using the change of numeraire technique as  $dQ^i/dQ = B_i(T_n)/B_i(0) \exp\{-\int_0^{T_n} r(s)ds\}$ ,  $i = n, \dots, N$ ;

$$\begin{aligned} S(0) &= E^Q[\exp\{-\int_0^{T_n} r(s)ds\} \max(1 - V(T_n), 0)] \\ &= E^Q[\exp\{-\int_0^{T_n} r(s)ds\} (1 - V(T_n)) \mathbf{1}_{\{1 \geq V(T_n)\}}] \\ &= E^Q[\exp\{-\int_0^{T_n} r(s)ds\} (B_n(T_n) - B_N(T_n) - k\delta \sum_{i=1}^n B_i(T_n)) \mathbf{1}_{\{1 \geq V(T_n)\}}] \\ &= B_n(0)Q^n(V(T_n) \leq 1) - B_N(0)Q^N(V(T_n) \leq 1) - k\delta \sum_{i=n+1}^N B_i(0)Q^i(V(T_n) \leq 1) \end{aligned}$$

To compute  $Q^i(V(T_n) \leq 1)$  we use the function  $f(x)$ ,

$$Q^n(V(T_n) \leq 1) = Q^n(f(U_n) \leq f(d_n))$$

Since  $f(d_n) = 0$  and  $f(\cdot)$  is a monotone increasing function and  $U_n$  is a standard normal variate,

$$Q^n(V(T_n) \leq 1) = N(d_n).$$

On the other hand,  $Q^i(V(T_n) \leq 1) = Q^i(f(U_n) \leq f(d_n))$ ,

$$\begin{aligned} \frac{dQ^i}{dQ^n} \Big|_{\mathcal{F}_t} &= \frac{B_i(t)}{B_i(0)} \exp\{-\int_0^t r(s)ds\} / \left( \frac{B_n(t)}{B_n(0)} \exp\{-\int_0^t r(s)ds\} \right) \\ &= \frac{B_i(t) B_n(0)}{B_n(t) B_i(0)} \\ &= \exp\left\{\int_0^t (\sigma^i(s) - \sigma^n(s))dW^n(s) - \frac{1}{2} \int_0^t |\sigma^i(s) - \sigma^n(s)|^2 ds\right\} \end{aligned}$$

By Girsanov theorem,  $W^i(t) = W^n(t) - \int_0^t (\sigma^i(s) - \sigma^n(s))ds$  is Brownian motion under  $Q^i$ .

$$\begin{aligned} Q^i(V(T_n) \leq 1) &= Q^i(f(U_n - \int_0^{T_n} (\sigma^i(s) - \sigma^n(s))ds) \leq f(d_n - \int_0^{T_n} (\sigma^i(s) - \sigma^n(s))ds)) \\ &= N(d_n - \int_0^{T_n} (\sigma^i(s) - \sigma^n(s))ds) = N(d_i) \end{aligned}$$

□

## 4 Lattice model

The above described one factor swaption model has difficulty to find the solution of equation (3.2) but we can easily get the numerical solution by Binomial approximation of Libor model. First see the main theorem of Libor model.

**Theorem 2** The following equations are satisfied in transition probability in Libor binomial model between  $Q^i$  and  $Q^{i+1}$  which are respectively martingale measures for  $L_i(t)$  and  $L_{i+1}(t)$ , where  $q_i$  is upward transistional probability in binomial tree in the measure  $Q^i$ , and  $q_{i+1}$  is that in  $Q^{i+1}$ .

$$q_i = q_{i+1} \frac{1 + \delta L_i^u(t)}{1 + \delta L_i(t)} \quad (4.1)$$

$$1 - q_i = (1 - q_{i+1}) \frac{1 + \delta L_i^d(t)}{1 + \delta L_i(t)}, \quad (4.2)$$

where the binomial states are  $L_i^u(t)$  and  $L_i^d(t)$ .

*Proof.* From Jamshidian's theorem\*

$$E_t^i[L_i(t + \Delta t)] = E_t^{i+1}[L_i(t + \Delta t) \frac{1 + \delta L^i(t + \Delta t)}{1 + \delta L^i(t)}]$$

Since  $L_i(t)$  is  $Q^{i+1}$ -martingale,

$$E_t^i[L_i(t + \Delta t)] = \frac{L_i(t) + \delta E^{i+1}[L_i^2(t + \Delta t)]}{1 + \delta L_i(t)}$$

By Binomial modeling assumption, the Libor moves in one step for measures  $Q^{i+1}$  and  $Q^i$ ;

$$L_i(t + \Delta t) = \begin{cases} L^u & Q_t^{i+1}(\omega_u) = q_{i+1} & Q_t^i(\omega_u) = q_i \\ L^d & Q_t^{i+1}(\omega_d) = 1 - q_{i+1} & Q_t^i(\omega_d) = 1 - q_i \end{cases}$$

To simplify notations, we use  $L^i$  instead of  $L_i(t)$ .

$$\begin{aligned} q^i L^u + (1 - q^i) L^d &= \frac{L_i}{1 + \delta L_i} + \frac{\delta}{1 + \delta L_i} \left( (L^u)^2 q_{i+1} + (L^d)^2 (1 - q_{i+1}) \right) \\ q_i (L^u - L^d) &= \frac{L_i - L^d (1 + \delta L_i) + \delta (L^d)^2}{1 + \delta L_i} + \delta q_{i+1} \frac{(L^u)^2 - (L^d)^2}{1 + \delta L_i} \end{aligned}$$

Using the martingale measure  $q_{i+1} = (L_i - L^d)/(L^u - L^d)$ ,

$$q_i = q_{i+1} \left( \frac{1 - \delta L^d}{1 + \delta L_i} + \frac{\delta L^u + \delta L^d}{1 + \delta L_i} \right).$$

Then we get (4.1). The equation (4.2) is also obtained by using the martingale measure,

$$1 - q_i = 1 - q_{i+1} \frac{1 + \delta L^u}{1 + \delta L_i} = (1 - q_{i+1}) \frac{1 + \delta L^d}{1 + \delta L_i}$$

□

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\*see Jamshidian[3]pp.25-26

The forward bond price from  $T_n$  to  $T_N$  at  $t$  is

$$B(t; T_n, T_N) = \frac{B_N(t)}{B_n(t)} = \frac{1}{\prod_{i=n}^{N-1} (1 + \delta L_i(t))}, \quad t \leq T_n$$

In the binomial lattice, the forward bond price at  $t + \Delta t$  has two states,

$$B_N^u = \frac{1}{\prod_{i=n}^{N-1} (1 + \delta L_i^u)}$$

$$B_N^d = \frac{1}{\prod_{i=n}^{N-1} (1 + \delta L_i^d)}$$

$Q^N$  is called the terminal measure and the transition probability  $Q^n$  of Libor  $L_n$  is changed to  $Q^N$ ,

$$q_n/q_N = \prod_{i=n}^{N-1} \frac{1 + \delta L_i^u}{1 + \delta L_i}$$

for upward state and

$$(1 - q_n)/(1 - q_N) = \prod_{i=n}^{N-1} \frac{1 + \delta L_i^d}{1 + \delta L_i}$$

for the downward state. Swaption payoff at  $T_n$  is  $\max(1 - V(T_n), 0)$  and the price at time 0 is

$$S(0)/B_n(0) = E^n \left[ \frac{(1 - V(T_n))^+}{B_n(T_n)} \right]$$

$$= E^n [\mathbf{1}_{\{1 \geq V(T_n)\}}] - E^n \left[ \frac{B_N(T_n)}{B_n(T_n)} \mathbf{1}_{\{1 \geq V(T_n)\}} \right] - k\delta \sum_{i=n+1}^N E^n \left[ \frac{B_i(T_n)}{B_n(T_n)} \mathbf{1}_{\{1 \geq V(T_n)\}} \right] \quad (4.3)$$

Using change of measure as (4.1),

$$E^n [B(t + \Delta t; T_n, T_N) \mathbf{1}_{\{1 \geq V(T_n)\}} | \mathcal{F}_t] = (q_n B_N^u + (1 - q_n) B_N^d) \mathbf{1}_{\{1 \geq V(T_n)\}}$$

$$= \frac{q_N \mathbf{1}_{\{1 \geq V(T_n)\}}^u + (1 - q_N) \mathbf{1}_{\{1 \geq V(T_n)\}}^d}{\prod_{i=n}^{N-1} (1 + \delta L_i(t))}$$

$$= B(t; T_n, T_N) Q^N (\mathbf{1}_{\{1 \geq V(T_n)\}})$$

Then the unconditional expectation becomes

$$E^n \left[ \frac{B_N(T_n)}{B_n(T_n)} \mathbf{1}_{\{1 \geq V(T_n)\}} \right] = B(0, T_n, T_N) Q^N (\{1 \geq V(T_n)\})$$

In general, by change of measure to  $Q^i$  from  $Q^n$ ,

$$E^n \left[ \frac{B_i(T_n)}{B_n(T_n)} \mathbf{1}_{\{1 \geq V(T_n)\}} \right] = B(0, T_n, T_i) Q^i (\{1 \geq V(T_n)\})$$

Therefore (4.3) becomes

$$S(0)/B_n(0) = Q^n (\{1 \geq V(T_n)\}) - B(0, T_n, T_N) Q^N (\{1 \geq V(T_n)\}) - k\delta \sum_{i=n+1}^N B(0, T_n, T_i) Q^i (\{1 \geq V(T_n)\})$$

Then we get swaption pricing formula like (3.1),

$$S(0) = B_n(0) Q^n (\{1 \geq V(T_n)\}) - B_N(0) Q^N (\{1 \geq V(T_n)\}) - k\delta \sum_{i=n+1}^N B_i(0) Q^i (\{1 \geq V(T_n)\}) \quad (4.4)$$

**Theorem 3** The payer swaption price is in binomial model as follows,

$$S(0) = B_n(0)F_n(l^*) - B_N(0)F_N(l^*) - k\delta \sum_{i=n}^N B_i(0)F_i(l^*) \quad (4.5)$$

where  $l^*$  is the smallest integer which satisfies

$$1 - \frac{1}{\prod_{i=n}^{N-1} (1 + \delta L_i(T_n))} - k\delta \sum_{i=n}^N \frac{1}{\prod_{i=n}^{j-1} (1 + \delta L_i(T_n))} \geq 0 \quad (4.6)$$

$$1 + \delta L_i(T_n) = 1 + \delta L_i(0)u_i^{l^*} d_i^{n-l^*}$$

where  $L_i^u(T_{k+1}) = L_i(T_k)u_i$  and  $L_i^d(T_{k+1}) = L_i(T_k)d_i$  are for  $k \leq i$ . The binomial distribution function is defined as

$$F_i(l) = 1 - \sum_{j=0}^l \binom{n}{j} q_i^j (1 - q_i)^{n-i}.$$

*Proof.* For binomial lattice the probability in (4.4) is binomial distribution  $F_i(l)$ . The positive payoff condition  $1 \geq V(T_n)$  satisfies

$$1 - B(0, T_n, T_N) - k\delta \sum_{i=n+1}^N B(0, T_n, T_i) \geq 0$$

and it is (4.6). □

## 5 Example of flat term structure and volatility

The simplest case is of flat term structure and flat volatility structure so as  $L_i(t) = L(t)$  and  $u_i = u$ ,  $d_i = d$ . The bond price at time 0 is for the maturity  $T_i$  due to flat term structure,

$$B_i(0) = \frac{1}{(1 + \delta L(0))^i}.$$

The Libor is at  $T_n$  is

$$L_i(T_n) = L(0)u^l d^{n-l}, \quad l = 0, \dots, n$$

Because of assumption of flat volatility structure, the forward bond price at  $T_n$  is

$$B(T_n, T_n, T_j) = \frac{1}{\prod_{i=n}^j (1 + \delta L_i(T_n))} = \frac{1}{(1 + \delta L(0)u^l d^{n-l})^{j-n}}, \quad j = n+1, \dots, N$$

The minimal integer to satisfy (4.6) is

$$1 - \frac{1}{(1 + \delta L(0)u^l d^{n-l})^{N-n}} - k\delta \sum_{j=n+1}^N \frac{1}{(1 + \delta L(0)u^l d^{n-l})^{j-n}}$$

$$= (1 + \delta L(0)u^l d^{n-l})^{N-n} - k\delta \sum_{j=n+1}^N (1 + \delta L(0)u^l d^{n-l})^{N-j} - 1 \geq 0$$

Let  $a_0 = -(1 + k\delta)$ ,  $a_{N-n} = 1$ ,  $a_i = -k\delta$ , and  $x = (1 + \delta L(0)u^l d^{n-l})$ , then

$$\sum_{i=0}^{N-n} a_i x^i = 0$$



There exist a positive solution  $x^*$  because only  $a_{N-n} > 0$  and others  $a_i < 0$ , by Decartes' rule of signs. The number of upward moves becomes

$$l^* = \min\{l \geq \log(x^* - 1)/\delta L(0) - n \log d / (\log u - \log d)\}$$

From (4.5) for flat term and volatility structure, the positive payment condition  $l^*$  is same for all binominal distributions. Thus

$$S(0) = B_n(0)F_n(l^*) - B_N(0)F_N(l^*) - k\delta \sum_{i=n}^N B_i(0)F_i(l^*) \quad (5.1)$$

where  $F_i(l^*) = 1 - \sum_{j=0}^{l^*} \binom{n}{j} q_i^j (1-q_i)^{n-j}$  and  $q_n = (1-d)/(u-d)$ ,  $q_i = q_{i+1}(1+\delta Lu)/(1+\delta L)$

### 5.1 Swap market model

Swap market model is utilized for calibration of implied volatility term structure. Let  $B_{nN}(t)$  the portfolio value of discount bonds whose maturities are from  $T_{n+1}$  to  $T_N$ .

$$B_{nN}(t) = \sum_{i=n+1}^N B_i(t)$$

There exists the martingale measure  $Q^{nN}$  whose numeraire is this portfolio. For any attainable portfolio process  $\{C(t)\}$

$$E^{nN} \left[ \frac{C(T)}{B_{nN}(T)} \mid \mathcal{F}_t \right] = \frac{C(t)}{B_{nN}(t)}$$

Payer Swaption payoff of swap rate  $k$  at Maturity  $T_n$  is  $\max(B_n(T_n) - B_N(T_n) - k\delta B_{nN}(T_n), 0)$ , by taking the portfolio  $B_{nN}(t)$  as numeraire, the swaption premium at 0 is

$$\begin{aligned} \frac{C(0)}{B_{nN}(0)} &= E^{nN} \left[ \frac{\max(B_n(T_n) - B_N(T_n) - k\delta B_{nN}(T_n), 0)}{B_{nN}(T_n)} \right] \\ &= \delta E^{nN} [\max(S_{nN}(T_n) - k, 0)] \end{aligned}$$

where  $S_{nN}(t) = \frac{B_n(t) - B_N(t)}{\delta B_{nN}(t)}$  is swap rate at  $t$  ( $0 \leq t \leq T_n$ ). The swap rate is also  $Q^{nN}$ -martingale and in the swap market model the swap rate is assumed to be the log normal process;

$$dS_{nN}(t) = \theta(t)S_{nN}(t)dW_{nN}(t)$$

where  $W_{nN}(t)$  is Brownian process under  $Q^{nN}$ . The swap rate at  $T_n$  is

$$S_{nN}(T_n) = S_{nN}(0) \exp \left\{ -\frac{1}{2} \int_0^{T_n} \theta^2(s) ds + \int_0^{T_n} \theta(s) dW_{nN}(s) \right\}.$$

From this simplified assumption the swaption price is given by Black formula,

$$C(0) = \delta B_{nN}(0) (S_{nN}(0) N(d_1) - k N(d_2))$$

where  $d_1 = \log(S_{nN}(0)/k)/v_{nN}(T_n) + v_{nN}(T_n)/2$ ,  $d_2 = d_1 - v_{nN}(T_n)$ . The volatility is  $v_{nN}(T_n) = \int_0^{T_n} \theta^2(s) ds$ . We compare the swaption premium (3.1)

$$\begin{aligned} C(0) &= \frac{\delta B_{nN}(0)}{\delta B_{nN}(0)} (B_n(0) - B_N(0)) N(d_1) - k \delta B_{nN}(0) N(d_2) \\ &= B_n(0) N(d_1) - B_N(0) N(d_1) - \delta k \sum_{i=n+1}^N B_i(0) N(d_2) \end{aligned} \quad (5.2)$$

The difference is coefficients of bond prices  $B_i(0)$ .

$$\begin{aligned} d_1 &= (\log(B_n(0) - B_N(0)) - \log(k\delta B_{nN})) / v_{nN}(T_n) + \frac{1}{2}v_{nN}(T_n) \\ d_2 &= d_1 - v_{nN}(T_n) \end{aligned}$$

The payer swaption price could take the general equation form;

$$C(0) = B_n(0)c_n - B_N c_N - k\delta \sum_{i=n+1}^N B_i c_i \quad (5.3)$$

We juxtapose coefficients of discount bonds in equations of (3.1),(4.5) and(5.2) in Table 1.

Bond maturity	$T_n$	$T_N$	$T_i$	
coefficient	$c_n$	$c_N$	$c_i$	equation
Gaussian closed-form	$N(d_n)$	$N(d_N)$	$N(d_i)$	(3.1)
Binomial approx.	$F_n(l^*)$	$F_N(l^*)$	$F_i(l^*)$	(4.5)
Swap market model	$N(d_1)$	$N(d_1)$	$N(d_2)$	(5.2)

Table 1: Payer swaption coefficients

## 5.2 The hedging strategy and numerical example

We calculate the payer swaption of  $3 \times 7$  and  $5 \times 5$  cases of flat Libor and volatilities structure, where Libor are (i) 2% (ii) 5% and the volatilities are (a)0.4 (b)0.2. These maturities are  $3 \times 7$  swaption for strike swap-rates for case of (i) are 1%, 2% and 3%. For the case of (ii) strike swap-rates are 4%, 5% and 6%. We compare the binomial lattice, Monte Carlo method and Black formula which assumption is Swap market model.

Table 2: Payer Swaption prices

Vol=40%	3 × 7 Swaption		Strike rate
Initial Libor=2%	0.01	0.02	0.03
Lattice	672.23	335.85	173.34
M.C.	660.20	331.33	173.03
Black	665.76	335.63	176.31
Vol=20%	3 × 7 Swaption		Strike rate
Initial Libor=5%	0.04	0.05	0.06
Lattice	647.42	360.99	188.38
M.C.	625.22	346.17	179.56
Black	648.30	363.39	191.06
Vol=40%	5 × 5 Swaption		Strike rate
Initial Libor=2%	0.01	0.02	0.03
Lattice	491.18	289.99	182.16
M.C.	495.04	297.95	191.95
Black	496.43	298.72	192.08
Vol=20%	5 × 5 Swaption		Strike rate
Initial Libor=5%	0.04	0.05	0.06
Lattice	485.65	308.81	191.81
M.C.	475.94	302.70	188.79
Black	489.86	313.75	196.87

all swaption prices are Basis point(1/100%) unit and M.C.are Monte Carlo Method which are provided by Dr. Yasuoka, Mizuho Information & Reseach Insitute, (100,000 runs)

Lattice method prices are 1000 node for a year and total steps are  $1000 \times x$  for swaption maturity  $x$  years.

From inconsistency of Libor market model and Swap market model, we could have arbitrage opportunity if we had constructed a hedging strategy. Davis [2] has shown the existence of negative libor rate in the case of coexistense of Libor and swap market models.

For the swaption if we take Gaussian model, the hedging strategy is as follows,

$$dC(t) = N(d_n)dB_n(t) - N(d_N)dB_N(t) - k\delta \sum_{i=n+1}^N N(d_i)dB_i(t)$$

which is easily shown. Delta hedging is change of the portfolio which is  $N(d_i)$  unit of bond of maturity  $T_i$ .

The hedging strategy of swap market model is obtained from (5.2)

$$dC(t) = N(d_1)dB_n(t) - N(d_1)dB_N(t) - k\delta \sum_{i=n+1}^N N(d_2)dB_i(t)$$

Provided the longer term interest is changed, the delta hedging of swap market model is not sensitive due to the same delta  $N(d_2)$  for all  $dB_i(t)$ . The price differences are caused by coefficients  $c_i$  as Table 1. We calculate coefficient for 3x7 swaption (volatility =40%, interest=2%, strike=2%).

Bond maturity	$T_3$	$T_{10}$	$T_i(i = 3.5, \dots, 10)$
coefficient	$c_n$	$c_N$	$c_i$
Binomial approx.	0.57170	0.53369	0.56919, ..., 0.53643
Swap market model	0.63819	0.63819	0.36723

Table 3: Payer swaption coefficients

In this data case, we can see in Table 3 the  $B_n(t)$  trading amount is excessive and other maturity bonds trading is insufficient in swap market model. For this case we could take arbitrage opportunity if change of longer term interest shift upward and we trade swaption and the hedging strategy of bonds.

## References

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