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A cohomology group of a $\mathbb{Z}_2$-orbifold model of the symplectic fermionic vertex operator superalgebra [1]

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1 Introduction

In this report we calculate a cohomological group of a model of an irrational C_2-cofinite simple vertex operator algebra. The cohomological group is considered by Miyamoto in a study on the category of modules for C_2-cofinite vertex operator algebras, and this result is just a calculation of a concrete example. In my talk, I introduced a homology of a certain functor. But the functor we considered is left exact, and hence the homology should be considered as a cohomology[2]. In this report we consider the cohomological group of the simple vertex operator algebra SF^+ which is one of examples of irrational C_2-cofinite vertex operator algebra.

2 Preliminaries

We do not state the definition of vertex operator algebras and its modules. For them, please refer to the literatures [LL], [MN] and [FHL]. Let $(V, Y(\cdot, x), \mathbf{1}, \omega)$ be a simple vertex operator algebra over $\mathbb{C}$, and $(M, Y(\cdot, x))$ a weak V-module. We write $Y(a, x) = \sum_{n\in\mathbb{Z}} a_{(n)}x^{-n-1}$ for $a \in V$ following [MN], where $a_{(n)} \in \operatorname{End} M$. We also write L_n for the n-th mode $\omega_{(n)}$ of the *Virasoro vector* ω. The *vacuum vector* $\mathbf{1}$ satisfies that for any $a \in V$ and $i \in \mathbb{Z}_{\geq 0}$, $a_{(i)}\mathbf{1} = 0$ and $a_{(-1)}\mathbf{1} = a$.

A *vacuum-like vector* $u \in M$ is a vector $u \in M$ satisfying $a_{(i)}u = 0$ for any $a \in V$ and $i \in \mathbb{Z}_{\geq 0}$. We set $\mathrm{Vac}(M)$ to be the set of all vacuum-like vectors in M. It is known that

$$\mathrm{Vac}(M) = \operatorname{Ker} L_{-1} = \{u \in M \mid L_{-1}u = 0\}.$$

Actually, $L_{-1} = \omega_{(0)}$ shows that $\mathrm{Vac}(M) \supset \operatorname{Ker} L_{-1}$. On the other hand if $u \in \operatorname{Ker} L_{-1}$, then $\binom{-i-1}{k}a_{(i)}u = \frac{1}{k!}L_{-1}^{k}a_{(i+k)}u$. Since $a_{(j)}u = 0$ for sufficiently

[1] The original title is "A homology group of a $\mathbb{Z}_2$-orbifold model of the symplectic fermionic vertex operator superalgebra.

[2] After my talk, Professors Matsuo and Arakawa gave me this advice. I apologize that I made audience confused a lot according to my knowledgeless.

large positive integer j and $\binom{-i-1}{k} \neq 0$ for any $i, k \in \mathbb{Z}_{\geq 0}$, we see that $a_{(i)}u = 0$ and that $u \in \mathrm{Vac}\,(M)$.

We note that $\mathrm{Vac}\,(M)$ is included in the L_0-eigenspace M_0 of weight 0 because $L_0 = \omega_{(1)}$. Thus if L_0 does not have any eigenvector in M, then $\mathrm{Vac}\,(M) = 0$.

Proposition 2.1. ([Li]) *Let $u \in \mathrm{Vac}\,(M)$, and suppose that $u \neq 0$. Then the V-submodule $\langle u \rangle$ of M generated from u is isomorphic to V. A linear map $V \to \langle u \rangle$ defined by $a \mapsto a_{(-1)}u$ is a V-module isomorphism.*

Proof. Let $f : V \to \langle u \rangle$ be a linear map given by $f(a) = a_{(-1)}u$ It is known that $\langle u \rangle$ is spanned by vectors of the form $a_{(m)}u$ with $a \in V$ and $m \in \mathbb{Z}$. Since $u \in \mathrm{Vac}\,(M)$, we see that $\langle u \rangle$ is in fact spanned by $a_{(-m)}u$ with $a \in V$ and $m \in \mathbb{Z}_{>0}$. Thus f is surjective. We also see that $\langle u \rangle = \{a_{(-1)}u \mid a \in V\}$ because $(m-1)!a_{(-m)}u = (L_{-1}^{m-1}a)_{(-1)}u$ for $m \in \mathbb{Z}_{>0}$.

Now we see that

$$\begin{aligned} f(a_{(n)}b) = (a_{(n)}b)_{(-1)}u &= \sum_{i=0}^{\infty} \binom{n}{i} (-1)^i (a_{(n-i)}b_{(-1+i)}u - (-1)^n b_{(n-1-i)}a_{(i)}u) \\ &= a_{(n)}b_{(-1)}u \\ &= a_{(n)}f(b) \end{aligned}$$

for $a, b \in V$ and $n \in \mathbb{Z}$. Therefore, f is a V-module homomorphism. Finally $\ker f$ is a proper ideal of V and hence $\ker f = 0$ because V is simple. Thus f is a V-module isomorphism. □

3 A cohomological group associated to V

Suppose that the adjoint module V has an injective resolution;

$$0 \to V \to X^0 \xrightarrow{f_0} \cdots \to X^n \xrightarrow{f_n} X^{n+1} \xrightarrow{f_{n+1}} \cdots \quad (exact).$$

Then we have a cochain complex

$$0 \to \mathrm{Vac}\,(X^0) \xrightarrow{r_0} \mathrm{Vac}\,(X^1) \xrightarrow{r_1} \cdots \to \mathrm{Vac}\,(X^n) \xrightarrow{r_n} \mathrm{Vac}\,(X^{n+1}) \xrightarrow{r_{n+1}} \cdots,$$

where $r_n = f_n|_{P^n}$. We denote the corresponding cohomological group by $H(V) = \bigoplus_{n=0}^{\infty} H^n(V)$;

$$H^n(V) = \ker r_n / \mathrm{Im}\, r_{n-1}$$

for $n \in \mathbb{Z}_{\geq 0}$, where $r_{-1} = 0$. The cohomological group is independent of the choice of injective resolutions.

A vertex operator algebra V is called *C_2-cofinite* if the subspace $C_2(V)$ spanned by vectors of the form $a_{(-2)}b$ with $a, b \in V$ has finite codimension in V. If V is C_2-cofinite then we can show that any finitely generated weak V-module has projective cover. Therefore, the contragredient module V' has a projective resolution. In particular, V has an injective resolution.

4 The vertex operator algebra SF^+

Let $\mathfrak{h}$ be a finite dimensional vector space of dimension $2d$ with a nondegenerate skew-symmetric bilinear form $\langle \cdot , \rangle$. Then the vector space $\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K$ has a Lie super-algebra structure as follows; the even part is $\mathbb{C}K$ and the odd part is $\mathfrak{h} \otimes \mathbb{C}[t^{\pm 1}]$, and the super-commutation relations are

$$\{\psi \otimes t^m, \phi \otimes t^n\} = m\langle \psi, \phi \rangle \delta_{m,-n} K, \quad [K, \widehat{\mathfrak{h}}] = 0$$

for $\phi, \psi \in \mathfrak{h}$ and $m, n \in \mathbb{Z}$.

Now we consider the super-algebra $\mathcal{A} := U(\widehat{\mathfrak{h}})/\langle K - 1\rangle$, where $U(\widehat{\mathfrak{h}})$ is the universal enveloping algebra of $\widehat{\mathfrak{h}}$ and $\langle K - 1\rangle$ is the two-sided ideal of $U(\widehat{\mathfrak{h}})$ generated by $K - 1$. Let $I_{\geq 0}$ be the left ideal of $\mathcal{A}$ generated by $\psi \otimes t^n$ for all $\psi \in \mathfrak{h}$ and $n \in \mathbb{Z}_{\geq 0}$. We then have a left $\mathcal{A}$-module $\mathcal{A}/I_{\geq 0}$ and denote it by SF.[3] It is clear that SF is isomorphic to the exterior algebra $\Lambda(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}])$ as vector spaces. We write $\psi_{(n)}$ for the left multiplication on SF by $\psi \otimes t^n$ for $\psi \in \mathfrak{h}$ and $n \in \mathbb{Z}$. Let $\mathbf{1}$ be the image of the unit of $\mathcal{A}$ in SF. Then SF is spanned by vectors of the form

$$\psi^1_{(-n_1)}\psi^2_{(-n_2)} \cdots \psi^r_{(-n_r)}\mathbf{1}, \quad (\psi^i \in \mathfrak{h},\ n_i \in \mathbb{Z}_{>0}).$$

We define the vertex operator map $Y(\cdot , z) : SF \to (\operatorname{End} SF)[[z, z^{-1}]]$ by

$$\begin{aligned}
&Y(\mathbf{1}, z) = \mathrm{id}_T,\\
&Y(\psi_{(-1)}\mathbf{1}, z) = \sum_{n \in \mathbb{Z}} \psi_{(n)} z^{-n-1},\\
&Y(\psi^1_{(-n_1)}\psi^2_{(-n_2)} \cdots \psi^r_{(-n_r)}\mathbf{1}, z)\\
&\quad = {}^{\circ}_{\circ}\, \partial^{(n_1-1)} Y(\psi^1_{(-1)}\mathbf{1}, z) \cdots \partial^{(n_r-1)} Y(\psi^r_{(-1)}\mathbf{1}, z) \,{}^{\circ}_{\circ},
\end{aligned}$$

[3]The notation SF comes from "Symplectic Fermion".

for $\psi, \psi^i \in \mathfrak{h}$, $n, n_i \in \mathbb{Z}_{>0}$, where $\partial^{(k)} := \frac{1}{k!}\frac{d^k}{dz^k}$ for $k \in \mathbb{Z}_{\geq 0}$.

Let $\{e^i, f^i\}_{i=1,\cdots,d}$ be a basis of $\mathfrak{h}$ satisfying

$$\langle e^i, e^j\rangle = \langle f^i, f^j\rangle = 0 \quad \text{and } \langle e^i, f^j\rangle = -\delta_{i,j}$$

for $1 \leq i, j \leq d$. Then the Virasoro element ω is given by

$$\omega = \sum_{i=1}^{d} e^i_{(-1)} f^i_{(-1)}\mathbf{1}.$$

Finally we have a vertex operator superalgebra $(SF, Y(\,\cdot\,, z), \mathbf{1}, \omega)$ of central charge $-2d$.

The vertex operator superalgebra SF has canonically an automorphism θ defined by

$$\theta(\psi^1_{(-n_1)}\psi^2_{(-n_2)}\cdots\psi^r_{(-n_r)}\mathbf{1}) = (-1)^r\psi^1_{(-n_1)}\psi^2_{(-n_2)}\cdots\psi^r_{(-n_r)}\mathbf{1}$$

for any $\psi_i \in \mathfrak{h}, n_i \in \mathbb{Z}_{>0}$. The fixed point set SF^+ of SF for θ is the even part of the vertex operator superalgebra SF and the -1-eigenspace SF^- is the odd one. The even part SF^+ becomes a simple vertex operator algebra of central charge $-2d$, and SF^- is an irreducible SF^+-module.

5 Projective and injective resolutions of SF^+

It is known that SF^+ has four irreducible modules (see [A]). These are given by $SF^\pm$ and irreducible components of the unique irreducible θ-twisted SF-module. The lowest weights of SF^+ and SF^- are 0 and 1 respectively. Those of other two irreducible SF^+-modules are $-\frac{d}{8}$ and $\frac{4-d}{8}$.

The two irreducible modules given as submodules of the irreducible θ-twisted SF-module are projective and injective. This fact is not so easy but can be shown by using the structure of Zhu's algebra of SF^+ studied in [A]. On the other hand, $SF^\pm$ are not projective nor injective. Their projective covers can be constructed as follows.

First we consider the SF-module $\widehat{SF} = \mathcal{A}/I_{>0}$, where $I_{>0}$ is a left ideal of $\mathcal{A}$ generated by $\psi \otimes t^n$ with $\psi \in \mathfrak{h}$ and $n \in \mathbb{Z}_{>0}$. We see that $\widehat{SF}$ is generated from the vector $\widehat{\mathbf{1}} = 1 + I_{>0}$ and that $\widehat{SF} \cong \Lambda(\mathfrak{h} \otimes \mathbb{C}[t^{-1}])$ as vector spaces. We define the action of θ on $\widehat{T}$ by

$$\theta(\psi^1_{(-n_1)}\psi^2_{(-n_2)}\cdots\psi^r_{(-n_r)}\widehat{\mathbf{1}}) = (-1)^r\psi^1_{(-n_1)}\psi^2_{(-n_2)}\cdots\psi^r_{(-n_r)}\widehat{\mathbf{1}}$$

for any $\psi_i \in \mathfrak{h}, n_i \in \mathbb{Z}_{\geq 0}$. We denote by $\widehat{SF}^{\pm}$ by the ± 1-eigenspace for θ. We note that they are SF^{+}-modules and $(\widehat{SF}^{\pm})' \cong \widehat{SF}^{\pm}$ respectively. We use the following conjecture.

Conjecture. The SF-modules $\widehat{SF}^{\pm}$ are projective and injective.

Assuming this conjecture is true, we can find that $\widehat{SF}^{\pm}$ are projective covers of the SF^{+}-module $SF^{\pm}$ respectively as follows. By construction, we have an SF-module epimorphism $\phi_0 : \widehat{SF} \to SF$ defined by

$$\phi_0(\psi^1_{(-n_1)}\psi^2_{(-n_2)}\cdots\psi^r_{(-n_r)}\widehat{\mathbf{1}}) = \psi^1_{(-n_1)}\psi^2_{(-n_2)}\cdots\psi^r_{(-n_r)}\mathbf{1}$$

for $\psi_i \in \mathfrak{h}, n_i \in \mathbb{Z}_{\geq 0}$. By definition ϕ_0 gives epimorphisms $\widehat{SF}^{\pm} \to SF^{\pm}$ respectively. We set $W_0 = \ker\phi_0$. Then W_0 is an SF-submodule of $\widehat{SF}$ generated from $e^i_{(0)}\widehat{\mathbf{1}}$ and $f^i_{(0)}\widehat{\mathbf{1}}$ for $1 \leq i \leq d$. We also see that $W_0 = (W_0 \cap \widehat{SF}^{+}) \oplus (W_0 \cap \widehat{SF}^{-})$ and the submodules $W_0 \cap \widehat{SF}^{\pm}$ are indecomposable. Hence $\widehat{SF}^{\pm}$ are projective covers of $SF^{\pm}$ respectively.

We now state that SF has the following projective resolution.

Theorem 5.1. *The SF^{+}-module SF has a projective resolution*

$$\cdots \to P^{n+1} \to P^{n} \to \cdots \to P^{0} \to SF \to 0,$$

with $P^n = \widehat{SF}^{\oplus h(n+1)}$; the direct sum of $h(n)$-copies of $\widehat{SF}$.

The number $h(n)$ is given as follows: Let

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & \binom{2d}{0} \\ -1 & 0 & \cdots & 0 & \binom{2d}{1} \\ 0 & -1 & \cdots & 0 & \binom{2d}{2} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \binom{2d}{2d-1} \end{pmatrix}$$

be a $2d \times 2d$-matrix, and set

$$v^{(n)} = A^{n-1}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Then $h(n)$ is the $2d$-th component of $v^{(n)}$. Hence

$$h(1) = 1, \quad h(2) = 2d, \quad h(3) = d(2d+1), \quad \cdots .$$

In the case $d = 1$, we have $d(n) = n$.

Since $\widehat{SF}'$, the contragredient module to $\widehat{SF}$, is isomorphic to $\widehat{SF}$, by this theorem, we have an injective resolution

$$0 \to SF \to P^0 \to P^0 \to \cdots \to P^n \to \cdots .$$

By studying the structure of $\widehat{SF}$ in detail, we get

Theorem 5.2. *The irreducible SF^+-modules $SF^\pm$ have injective resolutions*

$$0 \to SF^\pm \to P^{0,\pm} \to \cdots \to P^{n,\pm} \to P^{n+1,\pm} \to \cdots$$

respectively, where

$$P^{n,\pm} = \begin{cases} (\widehat{SF}^{\pm})^{\oplus h(n+1)} & \text{if } n \text{ is even,} \\ (\widehat{SF}^{\mp})^{\oplus h(n+1)} & \text{if } n \text{ is odd.} \end{cases}$$

6 Cohomological group $H^\bullet(SF^+)$

By Theorem 5.2, we get the cochain complex

$$0 \to \mathrm{Vac}\,(P^{0,+}) \xrightarrow{r_0} \mathrm{Vac}\,(P^{1,+}) \xrightarrow{r_1} \cdots$$
$$\cdots \to \mathrm{Vac}\,(P^{n,+}) \xrightarrow{r_n} \mathrm{Vac}\,(P^{n+1,+}) \xrightarrow{r_{n+1}} \cdots .$$

We note that $\mathrm{Vac}\,(\widehat{SF}^{+}) = \mathbb{C}e_0^1 \cdots e_{(0)}^d f_{(0)}^1 \cdots f_{(0)}^d \mathbf{1}$ and $\mathrm{Vac}\,(\widehat{SF}^{-}) = 0$. Hence

$$\mathrm{Vac}\,(P^{n,+}) \cong \begin{cases} \mathbb{C}^{h(n+1)} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

for $n \geq 1$. We can observe that

$$\mathrm{Im}\, r_n = 0 \quad \text{for } n \in \mathbb{Z}_{\geq 0},$$
$$\ker r_n = \begin{cases} \mathrm{Vac}\,(P^{n,+}) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Therefore, we have

Theorem 6.1.

$$\begin{aligned} H^i(SF^+) &\cong \mathbb{C}^{h(i+1)} && \textit{if } i \textit{ is even,} \\ H^i(SF^+) &= 0 && \textit{if } i \textit{ is odd.} \end{aligned}$$

Remark 6.2. We can also define $H^i(SF^-)$. Then we have $H^i(SF^-) \cong 0$ if i is even and $H^i(SF^-) \cong \mathbb{C}^{h(i+1)}$ if i is odd.

7 A projective resolution in the case $d = 1$

We explain the projective resolution of SF in the case $d = 1$. For simplicity, we set $e = e^1$ and $f = f^1$. In this case, the submodule $\ker \phi_0 = W_0$ is generated by $e_{(0)}\hat{\mathbf{1}}$ and $f_{(0)}\hat{\mathbf{1}}$, and the submodule generated from $e_{(0)}f_{(0)}\hat{\mathbf{1}}$ is isomorphic to SF because $e_{(0)}f_{(0)}\hat{\mathbf{1}}$ is a vacuum-like vector. Therefore, we have the following sequence of submodules;

$$0 \subset SF \subset W_0 \subset \widehat{SF}.$$

One sees that $\widehat{SF}/W_0 \cong SF$ and $W_0/SF \cong SF \oplus SF$.

Now we consider the SF-module epimorphism $\phi_1 : \widehat{SF} \oplus \widehat{SF} \to W_0$ defined by

$$\phi_1(u\hat{\mathbf{1}}, v\hat{\mathbf{1}}) = ue_{(0)}\hat{\mathbf{1}} + vf_{(0)}\hat{\mathbf{1}},$$

where $u, v \in \Lambda(\mathfrak{h} \otimes \mathbb{C}[t^{-1}])$. Then we see that the kernel of ϕ_1, denoted by W_1, is the SF-submodule of $\widehat{SF}^{\oplus 2}$ generated by the vectors $(e_{(0)}\hat{\mathbf{1}}, 0)$, $(f_{(0)}\hat{\mathbf{1}}, e_{(0)}\hat{\mathbf{1}})$ and $(0, f_{(0)}\hat{\mathbf{1}})$.

If we draw an extension of X by Y as

$$\begin{array}{c} X \\ \downarrow \\ Y \end{array},$$

then we have the following pictures;

$$\widehat{SF} = \begin{array}{ccccc} & & SF & & \\ & \swarrow & & \searrow & \\ SF & & & & SF \\ & \searrow & & \swarrow & \\ & & SF & & \end{array}$$

and

$$W_0 = \begin{array}{ccccc} & & SF & & \\ & \swarrow & & \searrow & \\ SF & & & & SF \end{array}.$$

We also see that

$$\widehat{SF}^{\oplus 2} = \begin{array}{ccccc} & & SF & & \\ & \swarrow & & \searrow & \\ SF & & & & SF \\ & \searrow & & \swarrow & \\ & & SF & & \end{array} \quad \begin{array}{ccccc} & & SF & & \\ & \swarrow & & \searrow & \\ SF & & & & SF, \\ & \searrow & & \swarrow & \\ & & SF & & \end{array}$$

and

$$W_1 = \begin{array}{ccccccccc} & & SF & & & & SF & & \\ & \swarrow & & \searrow & & \swarrow & & \searrow & \\ SF & & & & SF & & & & SF \end{array}.$$

By the same way, for $n \in \mathbb{Z}_{>0}$, we consider a SF-module homomorphism $\phi_{n-1} : \widehat{SF}^{\oplus n} \to \widehat{SF}^{\oplus(n-1)}$ defined by

$$\begin{aligned} &\phi_{n-1}(u^1\widehat{\mathbf{1}}, \ldots, u^n\widehat{\mathbf{1}}) \\ &= (u^1 e_{(0)}\widehat{\mathbf{1}} + u^2 f_{(0)}\widehat{\mathbf{1}}, u^2 e_{(0)}\widehat{\mathbf{1}} + u^3 f_{(0)}\widehat{\mathbf{1}}, \ldots, u^{n-1} e_{(0)}\widehat{\mathbf{1}} + u^n f_{(0)}\widehat{\mathbf{1}}) \end{aligned}$$

with $u^1, \ldots, u^n \in \Lambda(\mathfrak{h} \otimes \mathbb{C}[t^{-1}])$. Then we can show that

$$\operatorname{Im} \phi_n = \ker \phi_{n-1}$$

for $n \in \mathbb{Z}_{>0}$ and we have the exact sequence

$$\cdots \xrightarrow{\phi_{n+1}} \widehat{SF}^{\oplus(n+1)} \xrightarrow{\phi_n} \widehat{SF}^{\oplus n} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} \widehat{SF} \xrightarrow{\phi_0} SF \to 0.$$

We recall the action of θ on $\widehat{SF}$. We extend the action of θ to that on $\widehat{SF}^{\oplus n}$ with diagonal action. Then it is easy to see that $\theta \circ \phi_n \circ \theta = -\phi_n$ for any $n \in \mathbb{Z}_{\geq 0}$. Therefore, the projective resolution above gives rise to two projective resolutions

$$\begin{aligned} \cdots \xrightarrow{\phi_{n+1}} (\widehat{SF}^{\varepsilon^{\pm}_{n+1}})^{\oplus(n+1)} \xrightarrow{\phi_n} (\widehat{SF}^{\varepsilon^{\pm}_{n}})^{\oplus n} \xrightarrow{\phi_{n-1}} \cdots \\ \cdots \to (\widehat{SF}^{\mp})^{\oplus 2} \xrightarrow{\phi_1} \widehat{SF}^{\pm} \xrightarrow{\phi_0} SF^{\pm} \to 0, \end{aligned}$$

where $\varepsilon_n^\pm$ is defined by

$$\varepsilon_n^\pm = \begin{cases} \mp & \text{if } n \text{ is even} \\ \pm & \text{if } n \text{ is odd.} \end{cases}$$

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