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Arithmetical rank of squarefree monomial ideals

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Introduction

This report complements our paper [4]. Throughout this report, k is an infinite field and R is a polynomial ring over k. We denote variables of R by x_i and y_i $(i = 1, 2, \ldots)$. Let I denote a squarefree monomial ideal (i.e., the ideal generated by monomials in which the exponent of each variable is at most 1). For example, $I = (x_1x_2, x_2x_3x_4, x_1x_4x_5)$ is a squarefree monomial ideal.

The *arithmetical rank* of I, denoted by $\operatorname{ara} I$, is defined by

$$\operatorname{ara} I = \min\left\{r \;:\; \text{there exists } a_1, \ldots, a_r \in I \text{ such that } \sqrt{(a_1, \ldots, a_r)} = \sqrt{I}\right\}.$$

That is, $\operatorname{ara} I$ is the minimal number of elements in I which generate I up to radical. The arithmetical rank has the following geometric interpretation. Assume k is an algebraic closed field, and put $R = k[x_1, \ldots, x_n]$. Then the algebraic variety associated to I is defined by

$$V(I) = \{(z_1, \ldots, z_n) \in k^n \;:\; f(z_1, \ldots, z_n) = 0 \text{ for all } f \in I\}.$$

As $V(\sqrt{I}) = V(I)$, if $r = \operatorname{ara} I$ and $\sqrt{(a_1, \ldots, a_r)} = \sqrt{I}$, then

$$V(I) = V((a_1, \ldots, a_r)) = V((a_1)) \cap \cdots \cap V((a_r)). \tag{0.1}$$

So, $V(I)$ can be written as an intersection of just r hypersurfaces set-theoretically. Moreover, (0.1) shows an importance to know explicitly r elements generate I up to radical.

In general, it is difficult to determine the arithmetical rank. If we find r elements which generate I up to radical, then such r gives an upper bound for $\operatorname{ara} I$, and in particular, $\mu(I)$, the minimal number of generators of I, is a trivial upper bound. On the other hand, the following fact is known (see Lyubeznik [5]).

Fact 0.1. *If I is a squarefree monomial ideal, then*

$$\operatorname{pd}_R R/I \leq \operatorname{ara} I, \tag{0.2}$$

where $\operatorname{pd}_R R/I$ *is the projective dimension of* R/I.

The projective dimension is easy to compute. So, the importance of this inequality is to give a lower bound for the arithmetical rank. Here, we consider the following problem.

Problem 0.2. *Does* $\operatorname{ara} I = \operatorname{pd}_R R/I$ *hold?*

If $\mu(I) - \operatorname{height} I = 0$, then the problem is trivially true. Moreover, it is also known that $\operatorname{ara} I = \operatorname{pd}_R R/I$ holds in the case $\mu(I) - \operatorname{height} I = 1, 2$; see [4].
But, in general, there is a counter-example for this problem.

Example 0.3 ([11]). Let I be the Stanley–Reisner ideal of Reisner's triangulation of $\mathbb{P}^2(\mathbb{R})$ (see Figure 1). That is, I is the squarefree monomial ideal in

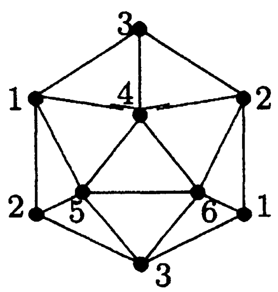

FIGURE 1. Reisner's triangulation of $\mathbb{P}^2(\mathbb{R})$

$k[x_1, \ldots, x_6]$ generated by following 10 monomials:

$$x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6.$$

Then, $\mu(I) = 10$ and $\operatorname{height} I = 3$, so, the difference is rather big. If the character of k is not 2, then R/I is Cohen–Macaulay and $\operatorname{pd}_R R/I = 3$. But Z. Yan [11] showed $\operatorname{ara} I = 4$ using the étale cohomology. Therefore $\operatorname{pd}_R R/I < \operatorname{ara} I$.

Now let us explain the organization of this report. In Section 1, we recall the notion of the Alexander duality, and explain the following inequality:

$$\operatorname{indeg} I \leq \operatorname{reg} I \leq \operatorname{arithdeg} I.$$

In Section 2, we prove the main theorem of this report, which asserts that Problem 0.2 is true in the case $\operatorname{arithdeg} I = \operatorname{reg} I$ by giving $\operatorname{ara} I$ generators (up to radical). See also [4, Theorem 4.1]. In Section 3, we construct another $\operatorname{ara} I$ generators in special cases, which is different from ones constructed in Section 2. These generators do not contain no redundant elements in some sense. Finally, as an appendix, we consider the analytic spread in the case $\operatorname{arithdeg} I = \operatorname{indeg} I$. Note that contents in Section 3 and Appendix A are not included in [4].

1. Alexander duality

In this section, we recall the notion of the Alexander duality and introduce an inequality corresponding to (0.2).

Set $R = k[x_1, \dots, x_n]$ and $[n] = \{1, \dots, n\}$. Let $\Delta \subset 2^{[n]}$ be a *simplicial complex* with the vertex set $[n]$, that is, (a) $\{i\} \in \Delta$ for all $i \in [n]$; (b) $F \in \Delta$, $G \subset F$ implies $G \in \Delta$. The *Alexander dual complex* of Δ, denoted by Δ^*, is defined by

$$\Delta^* = \{F \subset [n] \ : \ [n] \setminus F \notin \Delta\},$$

and the *Stanley–Reisner ideal* $I_\Delta \subset R$ associated to Δ is defined by

$$I_\Delta = (x_{i_1} \cdots x_{i_r} \ : \ 1 \le i_1 < \cdots < i_r \le n \text{ such that } \{i_1, \dots, i_r\} \notin \Delta).$$

It is clear that I_Δ is a squarefree monomial ideal. Conversely, for any squarefree monomial ideal $I \subset R$, there exists the unique simplicial complex Δ on $[n]$ such that $I = I_\Delta$ when $\operatorname{indeg} I \ge 2$. Then if $\operatorname{height} I \ge 2$, we can define $I^* = I_{\Delta^*}$, the *Alexander dual ideal* of I. It is known that $I^{**} = I$.

We shall see the correspondence between Alexander dual ideals and original ones. If I admits the prime decomposition

$$I = \bigcap_{\ell=1}^{q} \left(x_{t_{\ell 1}}, x_{t_{\ell 2}}, \dots, x_{t_{\ell j_\ell}}\right),$$

then $I^* = (m_1, \dots, m_q)$, where $m_\ell = \prod_{i=1}^{j_\ell} x_{t_{\ell i}}$. It is easy to see that $\mu(I^*) = \sharp \operatorname{Ass}_R R/I$ and $\operatorname{height} I = \operatorname{indeg} I^*$, where $\operatorname{Ass}_R R/I$ is the set of the *associated prime ideal* of R/I and $\operatorname{indeg} I$, the *initial degree* of I, is the minimal degree of minimal generators of I. Since I is a squarefree monomial ideal, the *arithmetic degree* of I, denoted by $\operatorname{arithdeg} I$, is equal to $\sharp \operatorname{Ass}_R R/I$.

Example 1.1. Consider

$$I = (x_1, x_2) \cap (x_2, x_3, x_4) \cap (x_1, x_4, x_5) = (x_1x_2, x_1x_3, x_1x_4, x_2x_4, x_2x_5),$$

then

$$I^* = (x_1x_2, x_2x_3x_4, x_1x_4x_5) = (x_1, x_2) \cap (x_1, x_3) \cap (x_1, x_4) \cap (x_2, x_4) \cap (x_2, x_5).$$

So, $\mu(I^*) = \operatorname{arithdeg} I = 3$ and $\operatorname{height} I = \operatorname{indeg} I^* = 2$.

We now recall the following inequalities:

$$\operatorname{height} I \le \operatorname{pd}_R R/I \le \mu(I). \tag{1.1}$$

Then the notion which corresponds to the projective dimension is the *regularity* $\operatorname{reg} I$ of I:

$$\operatorname{reg} I = \max\left\{j - i \ : \ \left(\operatorname{Tor}_i^R(I, k)\right)_j \ne 0\right\}.$$

Theorem 1.2 (N. Terai [9, Corollary 0.3]). *Let I be a squarefree monomial ideal with* $\operatorname{height} I \ge 2$. *Then we have*

$$\operatorname{reg} I^* = \operatorname{pd}_R R/I.$$

From (1.1), we obtain the following corollary.

Corollary 1.3 (Hoa–Trung [3, Theorem 1.1], Frühbis-Krüger–Terai [2, Theorem 3.8]). *Let I be a squarefree monomial ideal. Then we have*

$$\operatorname{indeg} I \leq \operatorname{reg} I \leq \operatorname{arithdeg} I. \tag{1.2}$$

2. Main theorem

We consider Problem 0.2 in the case $\operatorname{arithdeg} I = \operatorname{reg} I$.

Theorem 2.1 ([4, Theorem 4.1]). *Let I be a squarefree monomial ideal with* $\operatorname{arithdeg} I = \operatorname{reg} I$. *Then we have*

$$\operatorname{ara} I = \operatorname{pd}_R R/I.$$

Remark 2.2. This theorem has been already proved by Terai [10, Theorem 3.3], but our proof gives $\operatorname{ara} I$ generators of I up to radical.

Remark 2.3. The case $\operatorname{arithdeg} I = \operatorname{indeg} I$, which is contained in this case because of (1.2), is solved by Schenzel–Vogel [7] and Schmitt–Vogel [8]. In this case, $\operatorname{ara} I$ generators have been already known; see Section 3.

From now on, we prove Theorem 2.1. We use the following lemma.

Lemma 2.4 (Hoa–Trung [3, Theorem 2.6]). *Let I be as in Theorem 2.1. Then I can be rewritten in the following form by changing the notation of variables in R:*

$$I = (y_1, x_{t_{11}}, \dots, x_{t_{1j_1}}) \cap (y_2, x_{t_{21}}, \dots, x_{t_{2j_2}}) \cap \cdots \cap (y_q, x_{t_{q1}}, \dots, x_{t_{qj_q}}),$$

where y_ℓ and x_t are variables of R, and y_ℓ is different from other $y_{\ell'}$ and x_t.

From this lemma, we also have

$$\operatorname{pd}_R R/I = \sharp \left\{ x_{t_{11}}, \dots, x_{t_{1j_1}}, x_{t_{21}}, \dots, x_{t_{2j_2}}, \dots, x_{t_{q1}}, \dots, x_{t_{qj_q}} \right\} + 1.$$

Now to prove the theorem, it is enough to find $\operatorname{pd}_R R/I$ generators.

Proof of Theorem 2.1. By Lemma 2.4, we can write

$$I = Q_1 \cap \cdots \cap Q_q, \quad Q_\ell = (y_\ell, x_{t_{\ell 1}}, \dots, x_{t_{\ell j_\ell}}).$$

We denote the number of variables x_t appearing in I by s, that is,

$$s = \sharp \left\{ x_{t_{11}}, \dots, x_{t_{1j_1}}, \dots, x_{t_{q1}}, \dots, x_{t_{qj_q}} \right\}.$$

Then $\operatorname{pd}_R R/I = s + 1$. Set

$$P_{s-\ell} = \left\{ x_{i_1} \cdots x_{i_\ell} \prod_j y_j \ : \ 1 \leq i_1 < \cdots < i_\ell \leq s \right\}, \quad 0 \leq \ell \leq s,$$

where j runs through $x_{i_1} \cdots x_{i_\ell} \notin Q_j$, and set

$$g_\ell = \sum_{a \in P_\ell} a, \qquad P = \bigcup_{\ell=0}^{s} P_\ell.$$

Then Schmitt–Vogel lemma (Lemma 2.5) means $\sqrt{(g_0, g_1, \dots, g_s)} = \sqrt{(P)}$. Since P generates I, we have $\sqrt{(g_0, g_1, \dots, g_s)} = \sqrt{I}$. Therefore $\operatorname{ara} I \le \operatorname{pd}_R R/I$. This complete the proof. □

Lemma 2.5 (Schmitt–Vogel [8, Lemma, pp.249]). *Let R be a ring and P is a finite subset of R. Suppose subsets $P_0, P_1, \dots, P_s$ of P satisfy the following conditions:*

(SV-1) $P = \bigcup_{\ell=0}^{s} P_\ell$;
(SV-2) $\sharp P_0 = 1$;
(SV-3) *For all ℓ $(0 < \ell \le s)$ and for all $a, a'' \in P_\ell$, $a \neq a''$, there exist ℓ' $(0 \le \ell' < \ell)$ and $a' \in P_{\ell'}$ such that $a \cdot a'' \in (a')$.*

Then setting $g_\ell = \sum_{a \in P_\ell} a^{e(a)}$ $(\ell = 0, 1, \dots, s)$, where $e(a)$ is an arbitrary element in $\mathbb{Z}_{>0}$, we have

$$\sqrt{(g_0, g_1, \dots, g_s)} = \sqrt{(P)}.$$

Example 2.6. Consider $I = (y_1, x_1, x_2) \cap (y_2, x_1, x_3) \cap (y_3, x_3)$. Then $\operatorname{pd}_R R/I = \sharp\{x_1, x_2, x_3\} + 1 = 4$. In this case,

$$\begin{aligned} P_0 &= \{x_1x_2x_3\}, \\ P_1 &= \{x_1x_2y_3, x_1x_3, x_2x_3\}, \\ P_2 &= \{x_1y_3, x_2y_2y_3, x_3y_1\}, \\ P_3 &= \{y_1y_2y_3\}. \end{aligned}$$

Let check conditions of Schmitt–Vogel lemma. From our setting, (SV-1) and (SV-2) are clear. We shall see (SV-3). For example, we take $x_1x_2y_3, x_1x_3 \in P_1$, then their product is

$$x_1x_2y_3 \cdot x_1x_3 = x_1^2x_2x_3y_3 \in (x_1x_2x_3), \quad \text{and } x_1x_2x_3 \in P_0.$$

Take $x_1y_3, x_2y_2y_3 \in P_2$, then their product is

$$x_1y_3 \cdot x_2y_2y_3 = x_1x_2y_2y_3^2 \in (x_1x_2y_3), \quad \text{and } x_1x_2y_3 \in P_1.$$

Thus, the product of 2 elements $a, a'' \in P_\ell$ increase the variety of variables x_i, and if the element $a' \in P_{\ell'}$ divisible by y_j, then each elements $a, a'' \in P_\ell$ also divisible by the same variable y_j.

Moreover, if we set

$$\begin{aligned} e(x_1x_2x_3) &= e(y_1y_2y_3) = 1, \\ e(x_1x_2y_3) &= e(x_2y_2y_3) = 2, \\ e(x_1x_3) &= e(x_2x_3) = e(x_1y_3) = e(x_3y_1) = 3, \end{aligned}$$

then we have homogeneous generators.

3. Irredundant generators in the case $\operatorname{arithdeg} I = \operatorname{reg} I = \operatorname{indeg} I + 1$

In the previous section we constructed $\operatorname{ara} I$ generators in the case $\operatorname{arithdeg} I = \operatorname{reg} I$. However, we needed many "redundant" elements in I there in some sense; see Example 3.3. In this section, we will give another generators which consists of irredundant elements of I in the case $\operatorname{arithdeg} I = \operatorname{reg} I = \operatorname{indeg} I + 1$.

Before stating our result, we now consider the case $\operatorname{arithdeg} I = \operatorname{indeg} I$. Notice that this condition implies that $\operatorname{arithdeg} I = \operatorname{reg} I$. Thus our method in the previous section (see the proof of Theorem 2.1) gives at least one $\operatorname{ara} I$ generators of I (up to radical). On the other hand, if $\operatorname{arithdeg} I = \operatorname{indeg} I$, then it is known that I can be written by the following form:

$$I = (x_{11}, \ldots, x_{1j_1}) \cap (x_{21}, \ldots, x_{2j_2}) \cap \cdots \cap (x_{q1}, \ldots, x_{qj_q}),$$

where $x_{11}, \ldots, x_{1j_1}, x_{21}, \ldots, x_{2j_2}, \ldots, x_{q1}, \ldots, x_{qj_q}$ are distinct variables of R. Schenzel–Vogel [7] and Schmitt–Vogel [8] showed that any squarefree monomial ideal I with $\operatorname{arithdeg} I = \operatorname{indeg} I$ satisfies $\operatorname{ara} I = \operatorname{pd}_R R/I$ using this fact. Indeed, Schenzel–Vogel [7, Lemma 2] showed that such an ideal I satisfies $\operatorname{pd}_R R/I = s + 1$, where $s = \sum_{i=1}^{q} j_i - q$, and Schmitt–Vogel [8, Proposition, pp.248] showed that if we set

$$P_\ell = \left\{x_{1\ell_1} x_{2\ell_2} \cdots x_{q\ell_q} \;:\; \ell_1 + \ell_2 + \cdots + \ell_q = q + \ell\right\}, \quad g_\ell = \sum_{a \in P_\ell} a,$$

for $\ell = 0, 1, \ldots, s$, then $\sqrt{(g_0, g_1, \ldots, g_s)} = \sqrt{I}$.

Each polynomial appearing in the $\operatorname{ara} I$ generators given by Schmitt–Vogel is described as a sum of several elements in the minimal set of monomial generators of I. Therefore they consist of irredundant terms in some sense.

We now consider the case $\operatorname{arithdeg} I = \operatorname{reg} I = \operatorname{indeg} I + 1$. In this case, we can classify the following two cases; see [4, Lemma 5.2].

$$\begin{aligned} \textit{case } 1\colon\quad I_1 \;=\;& (x_{11}, x_{12}, \ldots, x_{1j_1}) \cap (x_{21}, x_{22}, \ldots, x_{2j_2}) \cap \cdots \cap (x_{q1}, x_{q2}, \ldots, x_{qj_q}) \\ & \cap (x_{q+11}, x_{q+12}, \ldots, x_{q+1j_{q+1}}, \\ & \quad x_{11}, x_{12}, \ldots, x_{1i_1}, x_{21}, x_{22}, \ldots, x_{2i_2}, \ldots, x_{p1}, x_{p2}, \ldots, x_{pi_p}), \end{aligned}$$

where $1 \le p \le q$, $1 \le i_\ell < j_\ell$ $(\ell = 1, 2, \ldots, p)$, $j_{p+1}, \ldots, j_q, j_{q+1} \ge 1$.

$$\begin{aligned} \textit{case } 2\colon\quad I_2 \;=\;& (x_{11}, x_{12}, \ldots, x_{1j_1}, y_1, y_2, \ldots, y_p) \\ & \cap (x_{21}, x_{22}, \ldots, x_{2j_2}, y_1, y_2, \ldots, y_p) \\ & \cap (x_{31}, x_{32}, \ldots, x_{3j_3}) \cap \cdots \cap (x_{q1}, x_{q2}, \ldots, x_{qj_q}) \\ & \cap (x_{q+11}, x_{q+12}, \ldots, x_{q+1j_{q+1}}, x_{11}, x_{12}, \ldots, x_{1i_1}, x_{21}, x_{22}, \ldots, x_{2i_2}), \end{aligned}$$

where $q \ge 2$, $p \ge 1$, $1 \le i_\ell < j_\ell$ $(\ell = 1, 2)$, $j_3, \ldots, j_q, j_{q+1} \ge 1$.

Set

$$s_1 = \sum_{i=1}^{q+1} j_i - (q + 1), \quad s_2 = \sum_{i=1}^{q+1} j_i + p - (q + 1),$$

then $\operatorname{pd}_R R/I_i = s_i + 1$ for $i = 1, 2$.

Proposition 3.1. *Consider the ideal I_1. For $\ell = 0, 1, \dots, s_1$, we set*

$$P_\ell = \left\{ x_{1\ell_1} x_{2\ell_2} \cdots x_{q\ell_q} \ : \ \begin{array}{l} \ell_1 + \cdots + \ell_q = \ell + q \\ \ell_t \le i_t \text{ for some } t = 1, 2, \dots, p \end{array} \right\}$$
$$\bigcup \left\{ x_{1\ell_1} x_{2\ell_2} \cdots x_{q\ell_q} x_{q+1 l_{q+1}} \ : \ \begin{array}{l} \ell_1 + \cdots + \ell_q + \ell_{q+1} = \ell + q + 1 \\ i_t < \ell_t \le j_t \text{ for all } t = 1, 2, \dots, p \end{array} \right\},$$

and $g_\ell = \sum_{a \in P_\ell} a$. Then we have

$$\sqrt{(g_0, g_1, \dots, g_{s_1})} = \sqrt{I_1}.$$

Proof. It is clear that $P = \bigcup_{\ell=0}^{s_1} P_\ell$ generates I. Hence it is enough to check conditions of Schmitt–Vogel lemma. (SV-1) is nothing. Since $P_0 = \{x_{11} x_{21} \cdots x_{q1}\}$, (SV-2) is clear. For (SV-3), we set the former set of P_ℓ as $P_\ell^{(1)}$ and the latter one as $P_\ell^{(2)}$. For any $\ell > 0$, we take $a, a'' \in P_\ell$ $(a \neq a'')$. If both a and a'' lie in $P_\ell^{(1)}$, then we can write

$$a = x_{1\ell_1} x_{2\ell_2} \cdots x_{q\ell_q}, \quad a'' = x_{1\ell_1''} x_{2\ell_2''} \cdots x_{q\ell_q''}.$$

Since $\ell_1 + \ell_2 + \cdots + \ell_q = \ell_1'' + \ell_2'' + \cdots + \ell_q''$, and $(\ell_1, \ell_2, \dots, \ell_q) \neq (\ell_1'', \ell_2'', \dots, \ell_q'')$, there exists $u \in \{1, 2, \dots, q\}$ such that $\ell_u > \ell_u''$. Then $a' = x_{1\ell_1} \cdots x_{u\ell_u''} \cdots x_{q\ell_q}$ satisfies the condition. The case both a and a'' lie in $P_\ell^{(2)}$ can be checked similarly. If $a \in P_\ell^{(2)}$ and $a'' \in P_\ell^{(1)}$, then we can write

$$a = x_{1\ell_1} x_{2\ell_2} \cdots x_{q\ell_q} x_{q+1\ell_{q+1}}, \quad a'' = x_{1\ell_1''} x_{2\ell_2''} \cdots x_{q\ell_q''},$$

where $\ell_1 + \ell_2 + \cdots + \ell_q + \ell_{q+1} = \ell + q + 1$, $\ell_1'' + \ell_2'' + \cdots + \ell_q'' = \ell + q$, and there exists $t \in \{1, 2, \dots, p\}$ such that $\ell_t'' \le i_t$. Then $\ell_t'' < \ell_t$ and therefore

$$\ell_1 + \cdots + \ell_t'' + \cdots + \ell_q < \ell + q + 1 - \ell_{q+1} \le \ell + q.$$

So, $a' = x_{1\ell_1} \cdots x_{t\ell_t''} \cdots x_{q\ell_q}$ satisfies the condition. □

Proposition 3.2. *Consider the ideal I_2. For $i = 1, 2, \dots, p$, we set $y_i = x_{1j_1+i} = x_{2j_2+i}$. For $\ell = 0, 1, \dots, s_2$, we set*

$$P_\ell = \left\{ x_{1\ell_1} x_{2\ell_2} \cdots x_{q\ell_q} \ : \ \begin{array}{l} \ell_1 + \cdots + \ell_q = \ell + q \\ \ell_1 \le i_1 \text{ or } \ell_2 \le i_2 \end{array} \right\}$$
$$\bigcup \left\{ x_{1\ell_1} x_{2\ell_2} \cdots x_{q\ell_q} x_{q+1 l_{q+1}} \ : \ \begin{array}{l} \ell_1 + \cdots + \ell_q + \ell_{q+1} = \ell + q + 1 \\ i_t < \ell_t \le j_t \text{ for all } t = 1, 2 \end{array} \right\}$$
$$\bigcup \left\{ y_i x_{3\ell_3} \cdots x_{q\ell_q} x_{q+1 l_{q+1}} \ : \ \begin{array}{l} j_1 + j_2 + i + \ell_3 + \cdots + \ell_q + \ell_{q+1} = \ell + q + 1 \\ 1 \le i \le p \end{array} \right\},$$

and $g_\ell = \sum_{a \in P_\ell} a$. Then we have

$$\sqrt{(g_0, g_1, \dots, g_{s_2})} = \sqrt{I_2}.$$

Since the proof of this proposition is similar to Proposition 3.1, we omit here.

Example 3.3. Let us compare the ara$I$ generators in previous section and these proposition.

Consider

$$I = (x_1, x_2, x_3) \cap (x_4, x_5, x_6) \cap (x_7, x_1, x_2, x_4).$$

To use the method of previous section, we can set $y_1 = x_3$, $y_2 = x_6$, $y_3 = x_7$. Then other variables are x_1, x_2, x_4, and x_5. Thus $\mathrm{pd}_R\, R/I = 4 + 1 = 5$. $g_0, g_1, \ldots, g_4$ are followings:

$$\begin{cases} g_0 = x_1x_2x_4x_5, \\ g_1 = x_1x_2x_4 + x_1x_2x_5 + x_1x_4x_5 + x_2x_4x_5, \\ g_2 = x_1x_2 \cdot x_6 + x_1x_4 + x_1x_5 + x_2x_4 + x_2x_5 + x_4x_5 \cdot x_3, \\ g_3 = x_1 \cdot x_6 + x_2 \cdot x_6 + x_6 \cdot x_3 + x_5 \cdot x_3x_7, \\ g_4 = x_3x_6x_7. \end{cases}$$

There are 16 elements of I in the summand of $g_0, g_1, \ldots, g_4$.

While Proposition 3.1 shows

$$\begin{cases} g_0 = x_1x_4, \\ g_1 = x_1x_5 + x_2x_4, \\ g_2 = x_2x_5 + x_1x_6 + x_3x_4, \\ g_3 = x_3x_5x_7 + x_2x_6, \\ g_4 = x_3x_6x_7. \end{cases}$$

So, there are only 9 elements of I in the summand of $g_0, g_1, \ldots, g_4$. These are minimal generators of I.

We consider another example corresponding to Proposition 3.2. Set

$$I = (x_1, x_2, x_3, x_4) \cap (x_5, x_6, x_4) \cap (x_7, x_1, x_5).$$

For the method of previous section, we can set $y_1 = x_3$, $y_2 = x_6$, $y_3 = x_7$, then other variables are x_1, x_2, x_4, and x_5. So $\mathrm{pd}_R\, R/I = 5$.

$$\begin{cases} g_0 = x_1x_2x_4x_5, \\ g_1 = x_1x_2x_4 + x_1x_2x_5 + x_1x_4x_5 + x_2x_4x_5, \\ g_2 = x_1x_2 \cdot x_6 + x_1x_4 + x_1x_5 + x_2x_4 \cdot x_7 + x_2x_5 + x_4x_5, \\ g_3 = x_1 \cdot x_6 + x_2 \cdot x_6x_7 + x_4 \cdot x_7 + x_5 \cdot x_3, \\ g_4 = x_3x_6x_7. \end{cases}$$

There are 16 elements of I in the summand of $g_0, g_1, \ldots, g_4$.

While Proposition 3.2 shows

$$\begin{cases} g_0 = x_1x_5, \\ g_1 = x_1x_6 + x_2x_5, \\ g_2 = x_1x_4 + x_3x_5 + x_2x_6x_7, \\ g_3 = x_4x_5 + x_3x_6x_7, \\ g_4 = x_4x_7. \end{cases}$$

There are only 9 elements of I in the summand of $g_0, g_1, \ldots, g_4$, and these are minimal generators of I.

Appendix A. Analytic spread

In this section, we state the result that we get after the talk. We have considered the inequality

$$\operatorname{height} I \leq \operatorname{pd}_R R/I \leq \operatorname{ara} I \leq \mu(I).$$

But there is an invariant which lies between $\operatorname{ara} I$ and $\mu(I)$, and that is the analytic spread $l(I)$ of I.

Definition A.1. Let I be a homogeneous ideal of R. Let $R[It]$ be a Rees ring of I, that is, a subring of a polynomial ring $R[t]$. Then $\ell(I) = \dim R[It]/\mathfrak{m}R[It]$ is called the *analytic spread* of I.

An ideal J is called a *reduction* of I if $J \subset I$ and $I^{n+1} = JI^n$ for some $n \geq 1$. Moreover, J is called a *minimal reduction* of I if J is a reduction of I, and J itself does not have any proper reductions. It is known that the cardinality of the minimal set of generators of minimal reductions of I is constant, and this number is equal to the analytic spread of I.

As stated in the beginning of this section, the following inequality is known:

$$\operatorname{height} I \leq \operatorname{pd}_R R/I \leq \operatorname{ara} I \leq l(I) \leq \mu(I). \tag{A.1}$$

We prove $l(I) = \operatorname{pd}_R R/I$ for the squarefree monomial ideal I with $\operatorname{arithdeg} I = \operatorname{indeg} I$.

Theorem A.2. *Let I be a squarefree monomial ideal with* $\operatorname{arithdeg} I = \operatorname{indeg} I$. *Then we have*

$$l(I) = \operatorname{pd}_R R/I.$$

We prove this theorem by showing that $\operatorname{ara} I$ generators as in previous section generate minimal reduction of I. This result is stronger than $\operatorname{ara} I = \operatorname{pd}_R R/I$.

References

[1] M. Barile, *On the number of equations defining certain varieties*, manuscripta math. **91** (1996), 483–494.

[2] A. Frühbis-Krüger and N. Terai, *Bounds for the regularity of monomial ideals*, Mathematiche (Catania) **53** (1998), 83–97.

[3] L. T. Hoa and N. V. Trung, *On the Castelnuovo–Mumford regularity and the arithmetic degree of monomial ideals*, Math. Z. **229** (1998), 519–537.

[4] K. Kimura, N. Terai and K. Yoshida, *Arithmetical rank of squarefree monomial ideals of small arithmetic degree*, preprint.

[5] G. Lyubeznik, *On the local cohomology modules $H^i_{\mathfrak{a}}(R)$ for ideals $\mathfrak{a}$ generated by monomials in an R-sequence*, in Complete Intersections, Acireale, 1983 (S. Greco and R. Strano eds., Lecture Notes in Mathematics No. 1092, Springer-Verlag, 1984 pp. 214–220.

[6] P. Schenzel, *Applications of Koszul homology to numbers of generators and syzygies*, J. Pure Appl. Algebra **114** (1997), 287–303.

[7] P. Schenzel and W. Vogel, *On set-theoretic intersections*, J. Algebra **48** (1977), 401–408.

[8] T. Schmitt and W. Vogel, *Note on set-theoretic intersections of subvarieties of projective space*, Math. Ann. **245** (1979), 247–253.

[9] N. Terai, *Stanley–Reisner rings of Alexander dual complexes*, In: Proceedings of the 19th symposium on Commutative Algebra in Japan, **19** (1997), 53–66.

[10] N. Terai, *Arithmetical rank of squarefree monomial ideals of small arithmetic degree*, In: Proceedings of the 26th symposium on Commutative Algebra in Japan, **26** (2004), 11–18.

[11] Z. Yan, *An étale analog of the Goresky-Macpherson formula for subspace arrangements*, J. Pure Appl. Algebra **146** (2000), 305–318.