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Dynamics of an SIS epidemic model with general incidence rate and treatment*

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Abstract: In this paper, an SIS epidemic model with treatment is proposed. The incidence rate of the model, which can include bilinear incidence rate and standard incidence rate, is a general nonlinear incidence rate. We give some conditions for the existence of multiple endemic equilibria and backward bifurcations. From the model, we can understand the effect of the capacity for treatment.

Key Words: SIS; Treatment; Backward bifurcation; Global stability

1 Introduction

Recently, the phenomenon of backward bifurcation has played an important role in disease control. In such a scenario, the classical requirement of the reproduction number being less than 1 becomes only a necessary, but not sufficient condition for disease elimination ([4], [7]). Thus it is important to identify backward bifurcation to obtain conditions for disease control.

For disease control, the treatment is an important factor. In this paper, we consider the following SIS model

$$\begin{cases} \frac{dS}{dt} = A - dS - \lambda(S+I)^{\alpha-1}SI + rI + h(I), \\ \frac{dI}{dt} = \lambda(S+I)^{\alpha-1}SI - (d+r+e)I - h(I) \ (0 \le \alpha \le 1) \end{cases} \tag{1.1}$$

with the same treatment function as [7]:

$$\begin{cases} h(I) = kI, & 0 \le I \le I_0, \\ h(I) = m, & I > I_0, \end{cases} \tag{1.2}$$

where $m = kI_0$, A is the recruitment rate of the population, d is the natural death rate of the population, r is the natural recovery rate of the infective individuals and e is the additional death rate causing by the disease. Treatment are frequently done for some infections, such as the group of those responsible for the common cold, which do not confer any long lasting immunity. Hence we use an SIS model here. Since mass action law may be not suitable for human diseases, we use a general incidence force $\lambda N^{\alpha}SI/N$, where

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N is the total population number and α is a constant between 0 and 1. Data for five human diseases in communities with population sizes from 1,000 to 400,000 ([1, p. 157], [2, p. 306]) imply that α is between 0.03 and 0.07. Obviously, this general incidence characterize the continuous transitions from the bilinear incidence to the standard incidence and can simulate behavior changes of populations from random mobility in a fixed area to the mobility with a fixed population density [5].

The organization of this paper is as follows. In the next two Sections, we analyze the model of the cases $e = 0$ and $e > 0$, respectively. And in Section 4, we end the paper with a brief discussion on mathematical results and epidemiological implications.

2 Analysis of the model when $e = 0$

Since some diseases, such as common cold, catarrh, measle and water pox, etc., are not lethal or whose death rate can be ignored, we first consider the case where $e = 0$,i.e.,

$$\begin{cases} \frac{dS}{dt} = A - dS - \lambda(S+I)^{\alpha-1}SI + rI + h(I), \\ \frac{dI}{dt} = \lambda(S+I)^{\alpha-1}SI - (d+r)I - h(I). \ (0 \le \alpha \le 1) \end{cases} \tag{2.1}$$

There always exits the disease free equilibrium, $E_0 = (A/d, 0)$. The total population size, $N = S + I$, satisfies that $dN/dt = dS/dt + dI/dt = A - dN$. Hence $\lim_{t\to\infty} N(t) = A/d$. This implies that the ω-limit set of all positive solutions of Eq. (2.1) lies on the set $\Omega = \{(S, I) \in R_+^2 \mid S + I = A/d\}$. Substituting $S = A/d - I$ into the second equation of system (2.1), we obtain the following one-dimension system,

$$\frac{dI}{dt} = -\lambda(\frac{A}{d})^{\alpha-1}I^2 + (\lambda(\frac{A}{d})^{\alpha} - d - r)I - h(I) \triangleq f(I), \tag{2.2}$$

where $f(I)$ is continuous. Let $R_0 = \lambda(A/d)^{\alpha}/(d + r + k)$, When $0 < I \le I_0$, $f(I) = 0$ admits a unique solution $I^* = \frac{A}{d}(1 - \frac{1}{R_0})$ where $1 < R_0 \le \frac{1}{1-\frac{dI_0}{A}}$. When $I > I_0$ and $R_0 \ge 1 + \frac{2\sqrt{\lambda m(\frac{A}{d})^{\alpha-1}} - k}{d+r+k} \triangleq P_0$, $f(I) = 0$ has two positive solutions I_1 and I_2 where

$$I_1 = \frac{\lambda(\frac{A}{d})^{\alpha} - d - r - \sqrt{\Delta}}{2\lambda(\frac{A}{d})^{\alpha-1}}, I_2 = \frac{\lambda(\frac{A}{d})^{\alpha} - d - r + \sqrt{\Delta}}{2\lambda(\frac{A}{d})^{\alpha-1}}.$$

Set $S_i = A/d - I_i$ and $E_i = (S_i, I_i)$ for $i = 1, 2$. It is easy to check that $I_i < A/d$ and E_i is an endemic equilibrium of Eq. (2.1) if $I_i > I_0, i = 1, 2$. Further, $I_1 > I_0$ holds if and only if $R_0 > 1 + \frac{2\lambda I_0(\frac{A}{d})^{\alpha-1} - k}{d+r+k} \triangleq P_1$, and $R_0 < 1 + \frac{\lambda I_0(\frac{A}{d})^{\alpha-1}}{d+r+k} \triangleq P_2$. Thus, $I_1 \le I_0$ if $R_0 \le P_1$ or $R_0 \ge P_2$.

By similar arguments as above, we can get that $I_2 > I_0$ if $R_0 > P_1$ or $P_2 < R_0 \le P_1$ and further $I_2 \le I_0$ if $R_0 \le \min\{P_1, P_2\}$. So we have:

Theorem 2.1. *E_1, E_2 do not exist if $R_0 < P_0$, and if $R_0 \ge P_0$, we have:*

(i) *If $\lambda I_0 < k(\frac{A}{d})^{1-\alpha}$, then both E_1 and E_2 exist when $P_1 < R_0 < P_2$.*

(ii) *If $\lambda I_0 < k(\frac{A}{d})^{1-\alpha}$, then E_1 does not exist but E_2 exists when $R_0 \ge P_2$.*

(iii) *If $\lambda I_0 \ge k(\frac{A}{d})^{1-\alpha}$, then E_1 does not exist. Further, E_2 exists when $R_0 > P_2$, and E_2 does not exist when $R_0 \le P_2$.*

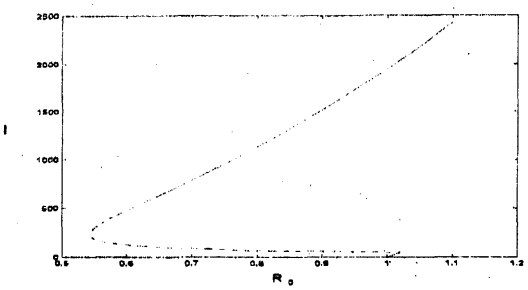

Figure 1: I^*, I_1 and I_2 versus R_0 when $d = 0.1, \lambda = 0.02, \alpha = 0.5, r = 0.1, k = 0.8, I_0 = 50$, it shows a backward bifurcation with endemic equilibria when $R_0 < 1$.

Equilibria	Global asymptotical stability conditions	
	$P_1 < P_2$	$P_1 \geq P_2$
$E_0(A/d, 0)$	$R_0 \leq 1$ and further $R_0 < P_0$ or $R_0 \leq P_1$	$R_0 \leq 1$
$E^*(S^*, I^*)$	$1 < R_0 < P_0$ and $R_0 \leq P_2$	$1 < R_0 \leq P_2$
$E_1(S_1, I_1)$	Unstable	
$E_2(S_2, I_2)$	$R_0 > P_2$	

Table 1: The global dynamics of of system (2.1)

As a simple consequence of Theorem 2.1 (i) , Eq.(1.1) has a backward bifurcation with endemic equilibria when $R_0 < 1$ if $P_1 < P_2$ and $P_0 < 1$ (See Fig.1). i.e., the disease does not die out when $R_0 < 1$. This is a very important conclusion for disease control. From this we know that there still are much work to do for eradicating the disease besides driving R_0 below 1. Further, the increasing of P_0 can eliminate the backward bifurcation, and the increasing of I_0 can lead to that of P_0. So we can draw a conclusion from Fig.1 that an insufficient capacity for treatment is a source of the backward bifurcation.

By analyzing the Jacobian matrices of right hand side of Eq. (2.1) at the equilibria, we can obtain:

Theorem 2.2. *E_0 is asymptotically stable if $R_0 < 1$ and is unstable if $R_0 > 1$, E^* is asymptotically stable if $1 < R_0 \leq P_2$. E_1 is a saddle and E_2 is asymptotically stable whenever they exist.*

From the theory of limit system [3, 6] and Theorem 2.1, we can easily obtain the global dynamics of system (2.1)which can be summarized as Table 1.

From Table 1 we can see that, the smaller R_0 is, the bigger the opportunity of E_0 being global stable is, and the more possibly the disease is to die out. The valid things we can do to control the disease are to diminish R_0 and enlarge the parameter P_0 or P_1, that is to say, in order to eradicate the disease or control it to a lower level, we should improve our ability of cure, and enlarge the capacity of treatment for patients.

3 Analysis of the model when $e > 0$

In this section, we study model (1.1) with $e > 0$ when it is lethal for some kind of diseases, such as cholera, malaria, and cancer, etc.

The disease free equilibrium $E_0 = (A/d, 0)$ still exists. According to $dN/dt = A - dN - eI$, system (1.1) is equivalent to

$$\begin{cases} \frac{dN}{dt} = A - dN - eI, \\ \frac{dI}{dt} = \lambda N^{\alpha-1}(N-I)I - (d+r+e)I - h(I). \ (0 \le \alpha \le 1) \end{cases} \tag{3.1}$$

When $0 < I \le I_0$, the endemic equilibrium satisfies

$$\begin{cases} A - dN - eI = 0, \\ \lambda N^{\alpha-1}(N-I)I - (d+r+e)I - kI = 0. \end{cases} \tag{3.2}$$

Note that $I = \frac{A-dN}{e}$, so $I > 0$ if and only if $0 < N < A/d$. Substituting $I = \frac{A-dN}{e}$ into the second equation of Eq. (3.2), we get

$$(d+r+e+k)N^{1-\alpha} - \lambda(1+\frac{d}{e})N + \lambda\frac{A}{e} = 0. \tag{3.3}$$

This equation can not be solved explicitly. Hence we discuss the existence of its roots.

Let $R_{0e} = \lambda(A/d)^\alpha/(d+r+k+e)$. Suppose that $E^* = (N^*, I^*)$ is an endemic equilibrium of (3.1), we get $I^* = N^*(1 - \frac{(A/d)^\alpha (N^*)^{-\alpha}}{R_{0e}})$ from the second equation of Eq. (3.2). Because $0 < I^* \le I_0$ and $0 < N^* < A/d$, we have

$$1 < (A/d)^\alpha (N^*)^{-\alpha} < R_{0e} \le (A/d)^\alpha (N^*)^{-\alpha} \cdot \frac{1}{1 - \frac{I_0}{N^*}}.$$

Set $g(N) = (d+r+e+k)N^{1-\alpha} - \lambda(1+\frac{d}{e})N + \lambda\frac{A}{e}$. Obviously, $g(0) = \lambda\frac{A}{e} > 0$, and $g(A/d) = \frac{\lambda A}{d} \cdot (\frac{1}{R_{0e}} - 1) < 0$, so $g(0+) > 0$, $g(\frac{A}{d}-) < 0$. Thus, there is at lest a root on $(0, A/d)$ which makes $g(N) = 0$.

Further, let $g'(N) = 0$, we get a unique positive solution $\bar{N} = (\frac{(d+r+e+k)(1-\alpha)}{\lambda(1+\frac{d}{e})})^{\frac{1}{\alpha}}$. At the same time, $g''(\bar{N}) < 0$, and $g(\bar{N}) > 0$. Therefore, Eq. (3.3) has only one root N^* on $(0, A/d)$ and $\bar{N} < A/d$. So when $0 < I \le I_0$, system (3.1) has only one endemic equilibrium $E^* = (N^*, I^*)$ where $I^* = \frac{A-dN^*}{e}$.

When $I > I_0$, the endemic equilibria satisfiy

$$\begin{cases} A - dN - eI = 0, \\ \lambda N^{\alpha-1}(N-I)I - (d+r+e)I - m = 0. \end{cases} \tag{3.4}$$

Substitute $I = \frac{A-dN}{e}$ into the second equation, we get

$$N^2 - BN^{2-\alpha} - CN + DN^{1-\alpha} + E = 0, \tag{3.5}$$

where $B = \frac{d+r+e}{\lambda(1+\frac{d}{e})}, C = \frac{A(e+2d)}{d(e+d)}, D = \frac{(d+r+e)A+em}{d\lambda(1+\frac{d}{e})}, E = \frac{A^2}{d(e+d)}$. And Eq.(3.5) is equivalent to

$$N^2 - CN + E = (BN - D)N^{1-\alpha}. \tag{3.6}$$

If

$$(\frac{D}{B})^2 - C(\frac{D}{B}) + E = 0, \tag{3.7}$$

then $N = \frac{D}{B}$ is a root of (3.6). But note that $\frac{D}{B} = \frac{(d+r+e)A+em}{d(d+r+e)} > \frac{A}{d}$, so we discuss the other root of (3.6) on $(0, A/d)$. When (3.7) holds, we have $N^2 - CN + E = (N - \frac{D}{B})(N - v)$, where $v = \frac{BE}{D} = C - \frac{D}{B}$. So (3.6) can be changed into $N - \frac{BE}{D} = BN^{1-\alpha}$. Let $p(N) = N - \frac{BE}{D} - BN^{1-\alpha}$, then $p'(N) = 0$ has a unique solution $N_1 = (\frac{(d+r+e)(1-\alpha)}{\lambda(1+\frac{d}{e})})^{\frac{1}{\alpha}}$. At the same time, $p''(N_1) > 0$, $p(0) < 0$, and $N_1 = (\frac{(d+r+e)(1-\alpha)}{\lambda(1+\frac{d}{e})})^{\frac{1}{\alpha}} < \bar{N} < \frac{A}{d}$,. So equation $p(N) = 0$ has a unique root on $(0, A/d)$ if and only if $p(\frac{A}{d}) = \frac{A}{d} - \frac{(d+r+e)A^2}{(e+d)[(d+r+e)A+em]} - (\frac{A}{d})^{1-\alpha} > 0$.

That is to say, the endemic equilibria still exist under some conditions. For example, fix $A = 100, d = 0.1, \lambda = 0.02, \alpha = 0.5, r = 0.1, k = 0.8, e = 0.01, I_0 = 50$. Thus, $R_{0e} = 0.626 < 1$. (3.1) have two endemic equilibria, one is stable, the other is a saddle.

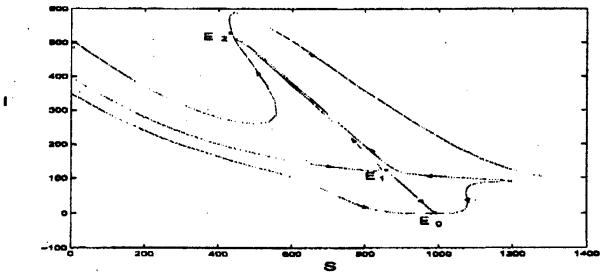

Figure 2: One region of disease persistence and one region of disease extinction when $A = 100, d = 0.1, \lambda = 0.02, \alpha = 0.5, r = 0.1, k = 0.8, e = 0.01, I_0 = 50$.

From this example, we can see in Fig.2 that the disease does not die out even if $R_{0e} < 1$, that is to say, system (3.1) still has the backward bifurcation with endemic equilibria when $R_{0e} < 1$ in this case.

By calculating the Jacobian matrices of Eq. (3.1) at the equilibria, we have:

Theorem 3.1. *The disease free equilibrium E_0 is asymptotically stable if $R_{0e} < 1$ and unstable if $R_{0e} > 1$.*

Theorem 3.2. *If $1 < (A/d)^{\alpha}(N^*)^{-\alpha} < R_{0e} \leq (A/d)^{\alpha}(N^*)^{-\alpha} \cdot \frac{1}{1-\frac{I_0}{N^*}}$, the endemic equilibrium E^* is asymptotically stable*

Proof. Note that $\lambda(N^*)^{\alpha-1}(N^* - I^*) = d + r + e + k$, we can obtain that the Jacobian matrix of Eq. (3.1) at $E_0 = (A/d, 0)$ is

$$J = \begin{pmatrix} -d & -e \\ (\alpha-1)(N^*)^{-1}I^*(d+r+e+k) + \lambda(N^*)^{\alpha-1}I^* & -\lambda(N^*)^{\alpha-1}I^* \end{pmatrix}.$$

Obviously, $tr(J(E^*)) < 0$, and $R_{0e} > (A/d)^{\alpha}(N^*)^{-\alpha}$ is equivalent to $\frac{\lambda(N^*)^{\alpha}}{d+r+e+k} > 1$, so $\frac{\lambda(N^*)^{\alpha}}{d+r+e+k} > 1 - \alpha$, certainly $det(J(E^*)) > 0$, and E^* is asymptotically stable. □

Lastly, we discuss the conditions under which we can exclude the limit cycle. Take a Dulac function $D = 1/I$ in R^2_+, then system (1.1) have $T = \frac{\partial(Df_1)}{\partial S} + \frac{\partial(Df_2)}{\partial I} = -\frac{d}{I} - \lambda(S + I)^{\alpha-1} < 0$ if $0 < I < I_0$. If $I > I_0$, it is easy to see that $T < 0$ if $d > k$. Hence, by Wang [7, Lemma 3.2], system (1.1) does not have a limit cycle when $d > k$.

4 Discussion

In this paper, we have studied an SIS epidemic model, the incidence of which is a general incidence including bilinear and standard incidence. When $e = 0$, We have shown that backward bifurcation occurs because of the insufficient capacity for treatment. So in order to eradicate the disease, it is not enough to drive the basic reproduction number below 1, we should improve our medical technology and invest more medicines, beds for the patients to enlarge the capacity of treatment. When $e > 0$, the backward bifurcation still exists under some conditions, but the equilibria cannot be solved explicitly. So it is difficult for us to establish threshold for the control of the disease. How to find out the critical parameter values at the turning point, how is the global dynamics of the system, and how to eliminate the backward bifurcation? We shall do these in our future work.

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