Title	Arabic Versions and Reediting Apollonius' Conics(Study of the History of Mathematics)
Author(s)	RASHED, Roshdi
Citation	数理解析研究所講究録 (2007), 1546: 128-139
Issue Date	2007-04
URL	http://hdl.handle.net/2433/80776
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
第 1546 巻 2007 年 128-139

Arabic Versions and Reediting Apollonius' *Conics*

Roshdi RASHED

I. The *Conics* is the principal work written by Apollonius, and one of the summits of Greek geometry. To ignore the *Conics* is to prohibit oneself from understanding anything about the development of mathematical research, particularly in Arabic, from the 9th century onwards. This was the century in which Apollonius' work began to be read, commented, and developed, as is attested such names as the Banū Mūsā, Thābit ibn Qurra, Ibrāhīm ibn Sinān, al-Qūhī, and Ibn al-Haytham, among many others. This interest became noticeable once again in the 17th century, in the works of Mydorge, Descartes, Fermat, Roberval, Desargues, Pascal, and Barrow, to cite only these names. It is as if every time research in classical mathematics is reborn, scholars have returned to the *Conics* of Apollonius, as they did to the works of Archimedes.

It is therefore easy to understand the importance of the history of this text, and the role it played, in the history of mathematical research. In this presentation, I would like to examine what the Arabic manuscript tradition has contributed to the history of Apollonius' *Conics*.

In the prologue to the *Conics*, Apollonius recalls that he had composed his work in eight books. He also recalls that some of these books went through more than one version, and that the first authorized version of the first three books was sent to his friend Eudemus. After the death of Eudemus, he sent the remaining books, beginning with the fourth, to a certain Attalus.

The fate of the *Conics* after Apollonius is in a way similar to that of the *Arithmetics* of Diophantes. Very early, and probably before Pappus, the eighth book was already lost. Of the seven remaining books, Eutocius, in the sixth century, knew only four, of which he provided an edition. For the first three books, this edition was carried out primarily from the version sent to Eudemus. Fortunately, a Greek manuscript containing the seven books was translated into Arabic in the 9th century.[1]

Eutocius' version of the first four books was first edited by Commandino at Bologna in 1566. The astronomer E. Halley repeated the task in 1710; and finally I. L. Heiberg, with the competence for which he is known, provided the first truly critical edition in 1891. This is the edition on which eminent historians

---

[1] Apollonius' *Conics* were translated into Arabic not only once, but twice. For the history of the translation, see my introduction to the new critical edition and translation of Apollonius, *Œuvres mathématiques* (grecques, arabes), in collaboration with M. Decorps and M. Fiederspiel (vol. I), to be published in 2007.

have worked. The Arabic translation of the totality of the seven books has not experienced so fortunate a fate. The first four books of this translation have not attracted the attention of editors or of historians. Only the last three translated books (5, 6, 7), lost in Greek, have, since the 17th century, been the object of their concern.

To tackle the Latin translation of the last three books on the basis of the Arabic, while neglecting the first four books of the Arabic translation, as E. Halley had done; or to consider it possible to provide a critical edition of these three books directly, without concerning oneself with the first four, as Ludwig Nix thought he could do in 1889, and more recently G. Toomer, are not insignificant acts. They all reveal the same prejudice, agreed upon by editors and historians of mathematics past and present, which has become a common opinion. Its constituent elements are the following:

1. Eutocius' edition gives us Apollonius' very text of the first four books of the *Conics*;

2. Eutocius' edition of the first four books, as it has come down to us in a manuscript from the 12th century – the Vat. graec. 206 – is all of one piece, and relatively homogeneous with the exception of a few linguistic considerations, as if Eutocius had had available a basic text consisting in these four books, which he used as the foundation for his editorial work.

3. The fourth book of Euctocius' edition therefore has the same status as the three preceding ones.

4. The Arabic translation of the first four books – which has never been examined – is that of this same edition by Eutocius.

In the course of work carried out over more than a decade with a view to providing a critical edition of the *Conics*, we have been able to show several results, the most important of which are:

1. The common opinion we have just recalled does not stand up to scrutiny. Differences that are rather remarkable and often irreducible are observed between Eutocius' edition of the first four books and the Arabic translation.

2. The Greek manuscript tradition from which the Arabic translation was carried out is different from that known to Eutocius. In all probability, we have to do with an authorized version sent by Apollonius to Attalus. Apollonius seems to have proceeded to some improvements on the version of the first three books sent to Eutocius.

3. There is a division in Eutocius' edition, which has not yet been noticed, between the third and fourth book.

These results, together with many others, not only renew our knowledge of the history of the text of the *Conics*, but also shed light on the text itself.

To illustrate this situation in the time that has been allotted to me, I have chosen the case of book four.

II. The fourth book of the *Conics* represents the culmination of the research carried out at Alexandria in the entourage of Conon. Here, Apollonius studies the maximum number of points common to two conics. This is how he himself presents the matter:

> In the fourth book, we have shown in how many ways the sections of cones meet one another and meet the circumference of the circle, the arc of the circle, and many other things as well. None of our predecessors has shown at how many points the section of a cone and the arc of a circle meet[2].

Eutocius' edition of this important book raises many questions, centered on several anomalies that have not been noticed until now. However, before we briefly examine some of these anomalies, let us first recall that this fourth book, like the following ones, are, as we have said, addressed to Attalus, in an authorized version, after the death of Eudemus, to whom the first three had been addressed. However, whereas the first three books, in whole or in part, were the object of more than one edition, I do not know of any evidence that suggests the existence of any edition of the last four other than the one sent to Attalus. One would therefore expect a perfect correspondence between the edition of Eutocius and the Arabic translation. But there is no such correspondence. Let us proceed to a few comparisons.

1. In Eutocius' edition, the book consists of 57 propositions, whereas in the Arabic translation there are only 53.

2. Some propositions of Eutocius' edition are missing from the Arabic translation: 4.7, 4.21, 4.23. An attentive examination of these propositions shows that they are all defective. Proposition 4.7 depends on a hypothesis – that the secant is parallel to the asymptote – that is supposed to have been given in the preceding proposition, 4.6. But this hypothesis does not exist. The redaction of proposition 4.21 is that of a summary, and the text is quite obviously uncertain.

3. There are two propositions in the Arabic translation that do not figure in Eutocius' edition – 4.2 and 4.34, according to the numbering of the translation. Proposition 4.2 deals with the parabola. This is not merely a case of the omission of a proposition, but of a difference in presentation. Proposition 4.34 of the Arabic translation, also missing from Eutocius' edition, nevertheless

---

[2] See the prologue of Book one in *Apollonius Pergaeus quae graece exstant*, edidit et latine interpretatus est I. L. Heiberg, 2 vol., Stuttgart, Teubner, 1974, I, p. 4, 17-22.

offers some affinities with the latter's proposition 4.37. However, Eutocius' proposition 4.37 is written in an imprecise way: it lacks a condition on the concavity of hyperbolas, and the asymptotes do not play a role, as they should. Everything indicates that the two propositions of the Arabic translation that are missing from Eutocius' edition disappeared at one moment or another of the history of the texts utilized by Euctocius. However – and this is the most important point – this loss affects contexts where the Greek text reproduced by Eutocius would have been manipulated to such an extent that it cannot, in its present state, be attributed to Apollonius.

4. The order of propositions differs, sometimes to a surprising extent. For instance:

Eutocius	17	19	22	18	20	24
Arabic	15	16	17	18	19	20

It is easy to verify that it is the order of the Arabic translation that is logical.

5. The figures and their letters differ in a certain number of propositions.

6. More seriously, there are different demonstrations. It can happen that the demonstrations are simply missing from Eutocius' editions (for instance, in propositions 2, 3, 10, 11, 19...), but never from the Arabic translation. Yet such lacunae do not correspond at all to Apollonius' demonstrative norms.

7. Even more serious is the fact that some demonstrations in Eutocius' edition are false. Proposition 4.43 is a significant illustration of this. The demonstration of this proposition takes place by a reduction to the absurd, and contains an error in reasoning already noted by Commandino. It seems hardly likely that such an elementary error could be due to Apollonius.

In the Arabic translation, by contrast, the demonstration of this proposition is direct and perfectly correct. Moreover, in this translation we find a demonstration of a variant transmitted by Eutocius, not in his edition of the *Conics*, but in his *Commentary on the Conics*, which was never translated into Arabic. As Eutocius writes, the variant in question runs as follows:

> When a hyperbola intersects opposite sections respectively at two points, while having its convexity in an opposite direction to each of them, its opposite section will not meet any of these opposite sections.

Let $\mathscr{H}$ be a hyperbola, $\mathscr{H}_1$ and $\mathscr{H}_2$ two opposite section, $\mathscr{H}$ intersects $\mathscr{H}_1$ in two points A and B, and $\mathscr{H}$ intersects $\mathscr{H}_2$ in two points Δ and E; the hyperbola $\mathscr{H}'$, opposite to $\mathscr{H}$, will intersect neither $\mathscr{H}_1$ nor $\mathscr{H}_2$.

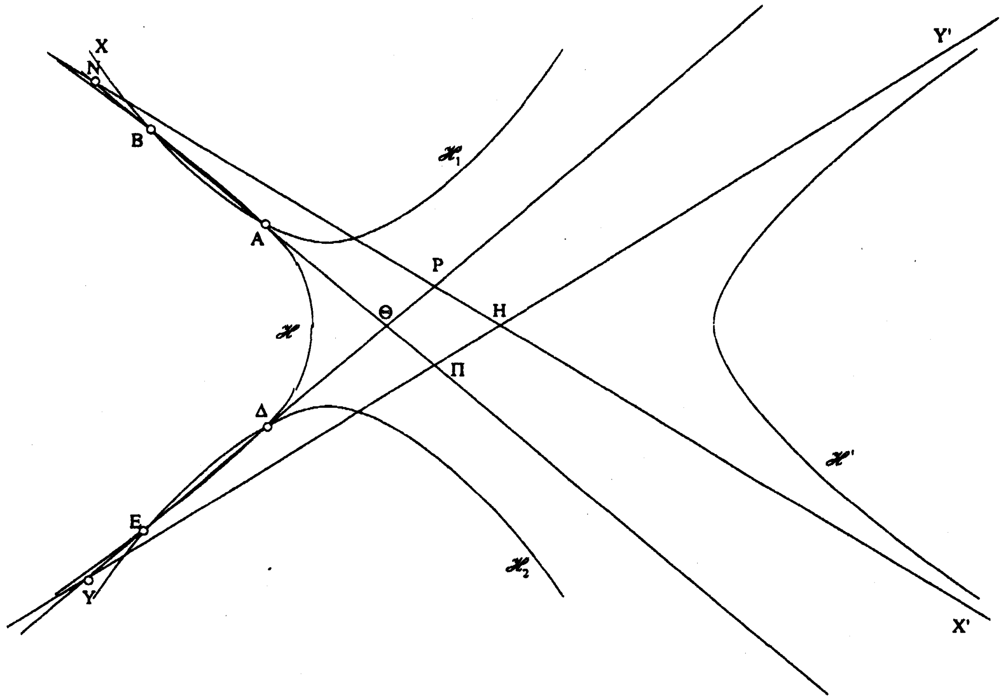
This proposition is number 39 in the Arabic text while it is number 43 in the Greek one. In the Arabic version its proof is deduced from the proposition 37 of the same text (i.e. 41 of the Greek text) by means of the proposition 33 of the second book of *Conics*. In this last one, Apollonius proves that the straight line AB intersects the asymptotes of $\mathcal{H}$, it intersects HX in point N and HY in Π; it will be then in the angles HXY, HXY', YHX' but will not be in the angle $X'HY'$. Consequently, it will not intersect $\mathcal{H}'$.

But according to the proposition 4.37 of the Arabic text, if a hyperbola intersects one of the two opposite sections ($\mathcal{H}_1$, $\mathcal{H}_2$); let it be $\mathcal{H}_1$, in two points A and B, and if the convexity of $\mathcal{H}$ and that of $\mathcal{H}_1$ are different, then the hyperbola $\mathcal{H}'$ opposite to $\mathcal{H}$ will not intersect $\mathcal{H}_2$.

Indeed, according to 2.33 the straight line AB which intersects $\mathcal{H}$ does not intersect $\mathcal{H}'$. In the same way, because the straight line AB intersects $\mathcal{H}_1$, it does not intersect $\mathcal{H}_2$. The sections $\mathcal{H}'$ and $\mathcal{H}_2$ are on both sides of AB.

We can come back now to the proposition 4.39. The straight line AB is between $\mathcal{H}'$ and $\mathcal{H}_2$. In the same way the straight line ΔE is between $\mathcal{H}'$ and $\mathcal{H}_1$, therefore $\mathcal{H}'$ intersects neither $\mathcal{H}_1$ nor $\mathcal{H}_2$.

The alternative demonstration in Eutocius' *Commentary* is quite close to the demonstration of the proposition 4.37 of the Arabic version, i.e. of the proposition 41 of the Greek text.



Thus we have: in the Arabic translation, a correct demonstration, deduced according to the logical order from a preceding proposition of the same group, that is, 4.41; in Eutocius' edition, a false demonstration, testifying to the obvious corruption of the text; and finally, a variant preserved by Eutocius, in his *Commentary*, which is close precisely to proposition 4.41, and in the same style. All these are reasons for concluding to the existence of two different textual traditions of this same book 4.

8. It sometimes happens that the statements of certain propositions as they appear in Eutocius' edition are imprecise and faulty if they are taken literally. Let us consider the first proposition, among many other examples:

> When a point is taken external the section of a cone or of the circumference of a circle; if from this point two straight lines meet the section, one tangent, the other intersecting at two points [...][3].

This statement is false, unless it is specified that the figure in question can only be a parabola, an ellipse, or the circumference of a circle, and not a "section of a cone" in general; and unless the condition is specified that this point must be at the angle formed by the asymptotes. One of the consequences, and not the least important, of this lack of precision is the marked difference between the beginning of book IV in Eutocius' edition and its beginning in the Arabic tradition. In the Arabic translation, the statement is perfectly correct. We find another example of this type in proposition 4.9.

9. Some propositions are identical in the Arabic translation and in Eutocius' edition:

Arabic manuscript:	41	42	43	44	45	46	47	48	50	52
Edition of Eutocius :	45	46	48	49	50	51	52	53	54	56

10. There is another group of propositions that are almost identical in Eutocius' edition and in the Arabic translation, or whose differences are insignificant, for instance:

Arabic manuscript:	28	29	30
Edition of Eutocius:	33	34	35

11. There is a group of about twenty propositions that differ only slightly, whereas the remaining ones are notably different in the two versions.

12. It often happens that an abridged version is found in Eutocius' edition: this is observed in propositions 4, 5, 8, and 20, among many others. This is contrary to the redaction of Apollonius, not only in the other books of the *Conics*, but also even within this fourth book for the group of identical propositions (45 to 54 of Eutocius' edition) and the group of propositions that are almost identical (33 to 35 of Eutocius' edition).

This comparison between the Arabic translation and Eutocius' edition, of which we have given only a few elements, allows the following facts to be established:

---

[3] Ed. Heiberg, II, p. 4, 22-25.

1. Whereas the Arabic translation is unitary, homogeneous and integrally written in the geometrical style of Apollonius, things are otherwise for several parts of Eutocius' edition (except for the group 45-56, in particular), where the contours of an abridged and less rigorous redaction become visible.

2. Eutocius' edition is split by a veritable division, which occurs at the level of propositions 44-45. The last group corresponds perfectly to the propositions of the Arabic translation. Although it may happen that a group from Eutocius' edition corresponds to a group from the Arabic translation before proposition 44, this is nevertheless not the rule. This division is completely absent from the Arabic translation.

These facts are easily verifiable: would it not be arbitrary, if not absurd, to make Apollonius responsible for this division and rather lax redaction? Why would he have begin to write differently, and less well, to change notation, to abridge his version, and to commit errors, all of this in a redaction he intended to be authorized? It would be just as unfair to make Eutocius bear all the responsibility. Why would he, after the first three books, have changed the style of his presentation and the quality of his edition? It is more reasonable to seek the reason for this elsewhere; but such research, in the current state of our knowledge, can result only in conjectural conclusions, not all of which have the same value of likelihood. Yet let us begin by noting a few troubling facts.

1. In his *Collection*, Pappus wrote lemmas for all the books of the *Conics*, except for the fourth. Was this a deliberate decision, an oversight, or was he not familiar with this book?

2. Eutocius himself, in his *Commentary on the Conics*, devotes only the most meager share to the fourth book. It suffices to count the number of pages devoted to it in Heiberg's edition of this *Commentary*: three and one-half pages, whereas the first three books occupy sixty, twelve, and twenty pages respectively.

3. In these three and a half pages, Eutocius gives two variants on a proposition from the first group, prior to the line of division: proposition 24; and two variants at the location of this line and slightly after 43 and 51. The two variants of 24 correspond to 20 in the Arabic translation. The first one deals with a conclusion from the latter, and the second is the same demonstration of the proposition in the Arabic translation. The variants of 43 and 51 are merely alternative demonstrations.

These anomalies, among many others, lead us to add a third fact to the two we have established: as it presents itself in Eutocius' edition, the fourth book is not all of one piece. It is a gravely contaminated text, a composite made up of two sources. The first text, before the line of division, has been substantially reworked, while the second, after this line, is a redaction by Apollonius, with slight vicissitudes due to copying and edition.

All these are simply well established facts. The rest can only be conjectural. It is likely that a commentator (before Pappus?) tried to abridge the first part of the text, while copying one or another proposition here and there. Yet this is scarcely important. Everything indicates that it took place after the third book, and not, as the common doctrine teaches, after the fourth book. Eutocius intended to reconstitute an edition of the elements of the *Conics*, and therefore of the first four books. He certainly had available the first three books in the redaction addressed to Eudemus. For book 4, he had recourse to a copy – composed by himself or by one of his predecessors - of a part that derived from a different tradition, on the basis of an edition that was more or less abridged, and a text of Apollonius (the last part). What is certain, in contrast, is that on the occasion of his redaction of the fourth book, Eutocius did not have available the totality of the redaction addressed by Apollonius to Attalus, but his version was seriously contaminated.

If, therefore, Eutocius had only the first three books of Apollonius' own redaction addressed to Eudemus, his lack of familiarity with books 5, 6 and 7 would be even more understandable. This is what we shall now discuss.

Why did Eutocius stop with the fourth book, whereas he knew perfectly well that according to Apollonius himself, it was the last three that were the most novel, not to mention the eighth book, which had been lost since Antiquity? No doubt, as we have mentioned, he wanted to edit the books devoted to the "elements of the theory of conics". Yet this does not seem to be the only reason. We have seen the knowledge he had of the fourth book, and we conjectured the existence of a break after the third book. There is no doubt that Eutocius knew of the existence of these last four books (5 to 8), at the very least through Apollonius' own introduction. He implies as much when he writes to Anthemius of Tralles:

> If you want me to set forth the following ones (five, six, seven and eight) along the same lines, I will do so, with the help of God[4].

If we can believe Eutocius, then, we would be dealing with a mere deliberate choice, that of a "professor" anxious to limit himself to the elements

[4] Ed. Heiberg, II, p. 356, 1-5.

of conics. However, whatever may be the reasons for such a choice, the question remains: what knowledge did Eutocius have of the last books? Was he really familiar, as he suggests, with the eighth book, which is believed to have been long lost? Frankly, this may be doubted.

One thing is odd: Eutocius' only precise reference to one of these books is found in a text known from his *Commentary* on Archimedes' *Equilibrium of plane figures*. Here he writes: "In book 6 of the *Conics*, Apollonius has defined as similar segments [...]"[5]; the definition follows. It has been claimed that Eutocius was citing Apollonius directly. Let us recall this passage, and compare it to the corresponding text from Apollonius in the translation by Thābit ibn Qurra:

Eutocius, *Commentary* on prop. 2.3 of Archimedes' *Equilibrium of plane figures* (ed. Mugler, p. 178, 8-14)	Book 6 (Arabic translation)
Τὰ ὅμοια τμήματα τῶν τοῦ κώνου τομῶν 'Απολλώνιος ὡρίσατο ἐν τῷ ἕκτῳ βιβλίῳ τῶν Κωνικῶν, ἐν οἷς ἀχθεισῶν ἐν ἑκάστῳ παραλλήλων τῇ βάσει ἴσων τὸ πλῆθος αἱ παράλληλοι καὶ αἱ βάσεις πρὸς τὰς ἀποτεμνομένας ἀπὸ τῶν διαμέτρων πρὸς ταῖς κορυφαῖς ἐν τοῖς αὐτοῖς λόγοις εἰσὶ καὶ αἱ ἀποτεμνόμεναι πρὸς τὰς ἀποτεμνομένας.	والقطع التي يقال إنها متشابهة هي التي تحيط قواعدها مع أقطارها بزوايا متساوية؛ وقد تخرج في كل واحد منها خطوط موازية لقاعدته* متساوية العدد ، وتكون نسبها ونسبة القاعدة إلى ما تفصل من القطر، مما يلي رأس القطع في كل واحد من القطع، نسبًا متساوية؛ وكذلك نسبة ما ينفصل من أقطار بعضها إلى ما ينفصل من أقطار الأخر. *لقاعدته: لقاعدتها.
[Apollonius defined similar segments of conic segments in book six of the *Conics*] if parallels equal in number to the base are extended in each one, the parallels and the bases have the same ratio to the <segments> that have been cut out from the diameters toward the vertices, as the ratio of the <segments> cut out to the <segments> cut out; and he showed that all parabolas are similar.	Sections called similar are those whose bases surround equal angles with their diameters; and it is possible to extend in each of them straight lines parallel to its base in an equal number, so that their ratios, as well as the ratio of the base to that which they separate from the diameter, on the side of the vertex, are equal ratios, and the same holds true for the ratios of what is separated from the diameter of one to what is separated from the diameter of the other.

The two texts clearly differ, particularly by the lack of the condition concerning the equality of the angles, which could not have escaped Eutocius himself if he had studied the sixth book carefully. No doubt he had a global and cursory knowledge of some of the definitions and results contained in this book, but that, it seems, is all. Thus, he recalls that all parabolas are similar among themselves (proposition 6.11 of the *Conics*); and that if perpendiculars are

[5] Archimède, *Commentaires d'Eutocius et Fragments*, texte établi et traduit par Charles Mugler, Paris, Les Belles Lettres, 1972, t. IV, p. 178, 8-10.

dropped to the axis of a parabola or a hyperbola, the segments cut out on either side of the axis by any two perpendiculars of the same section will not be similar to the first (proposition 6.19 of the *Conics*). Yet this knowledge remains implicit: Eutocius does not cite Apollonius. Nothing, however, denotes the slightest familiarity on part with its mathematical contents – a good indication that the second part of Apollonius' work, infinitely more difficult than the first, was already extinct, or at least on the way to extinction, in the Greek domain.

This impression is only confirmed with the fifth and sixth books, for there is nothing to suggest that Eutocius really worked on them. He is allusive at best in his *Commentary* on the *Conics*. Thus, when commenting on Apollonius' preface to the *Conics*, he writes that the fifth book "includes the study of minima and maxima"[6]; and he gives the example of the circle according to Euclid, which is far from reflecting the richness and novelty of Apollonius' fifth book, and in particular the study it contains of *extrema* as well as normals to conic sections, their existence and their number. On the other hand, Eutocius affirms that "it is the same research (as that of Euclid) that Apollonius carries out in book 5 on the sections of the cone" [7], which verges on the ridiculous. As far as the other books are concerned, he devotes to them, as a grand total, the following phrase:

> Finally, the goal of books 6, 7 and 8 is clearly set forth by Apollonius himself[8].

On the basis of such comments, vague and allusive, we cannot even be sure that Eutocius knew the last three books of the *Conics* – 5, 6, and 7 – at first hand. And it can be said that, even if he had them before him in some form or another – which we doubt – Eutocius had not studied these books, as is suggested by his allusion to the fifth book. Eutocius' knowledge of the various books of the *Conics* thus appears to be variable and of heterogeneous origin, and the text of the fourth book is to a large degree contaminated. As far as the other books are concerned, there is nothing to confirm that he really knew them.

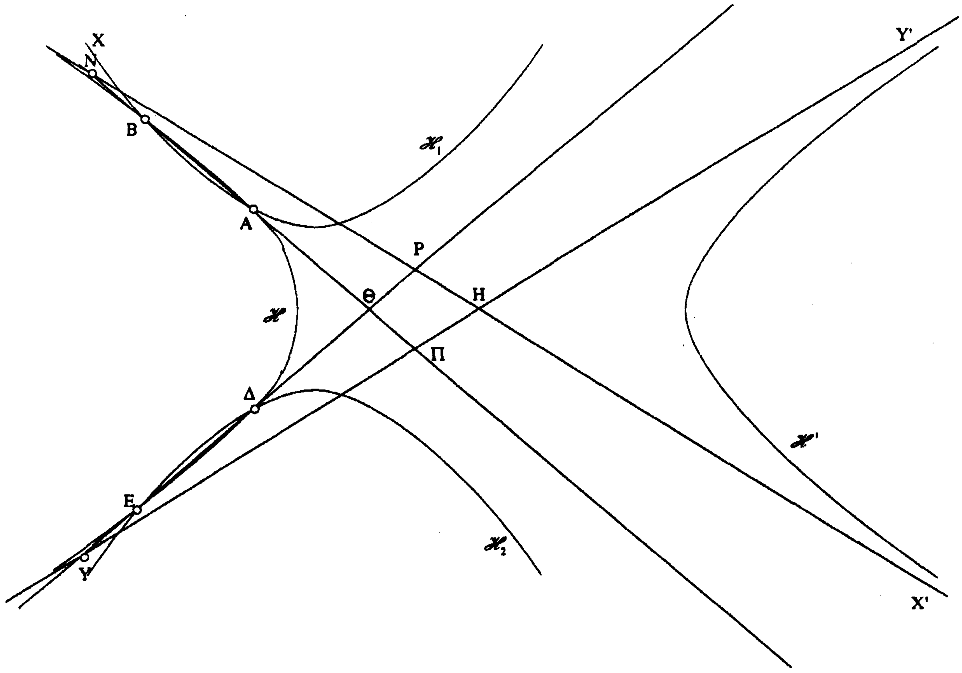
We have thus shown that the widespread opinion with regard to the textual history of the *Conics* is incorrect, and that one cannot rely on Eutocius' edition alone to establish the text of the fourth book. As far as the first three books are concerned, which we have not dealt with here, they are not without raising problems either. For this reason, in order to give a rigorous edition of all seven books of the *Conics*, we have taken up all the textual traditions, Greek as well as Arabic.

---

[6] Ed. Heiberg, II, p. 186, 12-13.

[7] Ed. Heiberg, II, p. 186, 13-14. A careful reading of the fifth book of *Conics* shows that Apollonius proves a necessary and sufficient condition for an *extrema* to be a normal. See propositions 5.27 to 5.33. The propositions 5.35 and 5.36 are devoted to the study of the angle between the normal and the axis.

[8] Ed. Heiberg, II, p. 186, 19-21.


X
N
B
$\mathscr{H}_1$
Y'
A
P
Θ
H
$\mathscr{H}$
Π
Δ
$\mathscr{H}'$
E
Y
$\mathscr{H}_2$
X'

والقطع التي يقال إنها متشابهة هي التي تحيط قواعدها مع أقطارها بزوايا متساوية؛ وقد تخرج في كل واحد منها خطوط موازية لقاعدته* متساوية العدد، وتكون نسبها ونسبة القاعدة إلى ما تفصل من القطر، مما يلي رأس القطع في كل واحد من القطع، نسبًا متساوية؛ وكذلك نسبة ما ينفصل من أقطار بعضها إلى ما ينفصل من أقطار الأخر.

* لقاعدته: لقاعدتها.