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Traveling wave solutions of the Allen-Cahn equations*

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1 Introduction

The Allen-Cahn equation

$$u_t = \Delta u - f(u), \qquad \mathbf{x} \in \mathbb{R}^N,\ t > 0 \tag{1.1}$$

is one of the most simple and popular parabolic nonlinear equations, because this equation is often appeared in the several fields (see [2]). This is also called the Nagumo equation for the nerve axon. The typical example of the nonlinear term f is

$$f(u) = (u+1)(u-1)(u-a).$$

We are interested in solutions having interfaces that travel upwards in the vertical z direction with a constant speed c. For simplicity, we introduce $(x, z) = (x_1, \cdots, x_n, z)$ for the spatial coordinates with dimension $N = n + 1 \geqslant 2$. Thus we rewrite the Allen-Cahn equation for $u = u(x, z, t)$ as

$$u_t = u_{zz} + \Delta' u - f(u), \qquad x \in \mathbb{R}^n,\ z \in \mathbb{R},\ t > 0. \tag{1.2}$$

Hereafter we use $\Delta' = \sum_{i=1}^n \partial_{x_i}^2$. If a solution is of the form $u(x, z, t) = U(x, z - ct)$, then (c, U) is called a *traveling wave solution* with its profile U and the speed c. The traveling wave solution (c, U) satisfies

$$\begin{cases} cU_z + U_{zz} + \Delta' U = f(U) & \forall x \in \mathbb{R}^n, z \in \mathbb{R}, \\ \lim\limits_{z \to \pm\infty} U(x, z) = \pm 1 & \forall x \in \mathbb{R}^n. \end{cases} \tag{1.3}$$

A function $W(y)$ is called cylindrically symmetric if $W(x, z) = \tilde{W}(|x|, z)$ for some $\tilde{W}$. For simplicity, we abuse the notation $W(x, z) = W(|x|, z)$. A function $W(y)$ is radially symmetric if $W(y) = \tilde{W}(|y|)$ for some $\tilde{W}$. For radially symmetric functions of $y = (x, z)$, $\partial_{zz} + \Delta' = \frac{n}{\rho}\partial_\rho + \partial_{\rho\rho}$. We shall look for cylindrically symmetric traveling wave solutions.

Set

$$F(u) = \int_{-1}^{u} f(s)ds.$$

*This is based on the joint works with Chen, Guo, Hamel, Roquejoffre [15] and Taniguchi [37, 38]

When $F(1) \neq F(-1)$, the existence of a traveling wave with asymptotic planar interface was proved by Fife [22] in dimension $n+1 = 2$ (see also [34]). Solutions having asymptotic conical level sets with any positive aperture angle were constructed by Ninomiya and Taniguchi [39, 40] in dimension $N = n+1 = 2$ and by Hamel, Monneau, and Roquejoffre [31] in any dimension $N = n+1 \geq 2$, where the nonlinearities f is assumed to have exactly three zeros ± 1, a ($|a| < 1$). See also the works of Bonnet and Hamel [8] and Hamel, Monneau, and Roquejoffre [30] for the "ignition temperature" type of the combustion problem (i.e., $f = 0$ in $[-1, \theta]$ and $f > 0$ in $(\theta, 1)$ for some $\theta \in (-1,1)$) in dimension $N = n+1 = 2$, and Hamel and Nadirashvili [33] for the mono-stable case (i.e., $f > 0$ in $(-1,1)$) and for solutions with general level sets in any dimension $n+1 \geq 2$. Other related works can be found in [28, 29, 34, 37, 38].

We consider the case $F(1) = 0$ called *balanced bistable*; more precisely,

(A) $\quad f = F' \in C^2(\mathbb{R}),\ F(\pm 1) = 0 < F(s)\ \forall s \neq \pm 1,\ F''(\pm 1) > 0.$

The one-dimensional stationary wave Φ is the unique solution to

$$\Phi'' = f(\Phi) \quad \text{on } \mathbb{R}, \qquad \Phi(\pm\infty) = \pm 1, \quad \Phi(0) = \alpha \tag{1.4}$$

where α is a constant specified later, see (2.1). Actually

$$[\Phi'^2 - 2F(\Phi)]' = 0,$$

then we have

$$\Phi' = \sqrt{2F(\Phi)}, \qquad \int_{\alpha}^{\Phi(\xi)} \frac{ds}{\sqrt{2F(s)}} = \xi \quad \forall \xi \in \mathbb{R}.$$

Theorem 1.1 ([15]) *Assume* **(A)**. *For any $c > 0$, (1.3) admits a cylindrically symmetric solution U with the* **monotonicity property***:*

$$U_z > 0 \quad on \quad \mathbb{R}^{n+1} \quad and \quad U_r < 0 \quad on \quad (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}. \tag{1.5}$$

One of the motivation of our study of (1.3) is the De Giorgi conjecture [18] which asserts that

> *when $c = 0$ and $f(U) = U^3 - U$, all z-monotonic solutions of (1.3) are planar*

at least in dimension $N = n+1 \leq 9$. Here planar means that there exist a unit vector $\mathbf{n} \in \mathbb{R}^{n+1}$ and a function $\Psi : \mathbb{R} \to [-1,1]$ such that $U(x,z) = \Psi(\mathbf{n} \cdot (x,z))$ for all (x,z); in this conjecture, the radial symmetry in x is not assumed. This conjecture was proven recently by Savin [42] (see also [1, 3, 5, 26]). More general nonlinearities of type **(A)** can also be considered in the spirit of [26, 42].

In view of the De Giorgi conjecture, a natural extension is to ask *whether planar solutions are the only solutions to the corresponding parabolic equation*

$$u_t = u_{zz} + \Delta' u + u - u^3, \qquad (x,z) \in \mathbb{R}^n \times \mathbb{R},\ t \in \mathbb{R} \tag{1.6}$$

subject to the monotonicity conditions

$$(1.7) \quad \lim_{z\to\pm\infty} u(x,z,t) = \pm 1, \qquad u_z(x,z,t) > 0 \qquad \forall (x,z,t) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}.$$

In the literature, a solution to a parabolic equation that is defined for all $t \in \mathbb{R}$ is called an *entire solution.* Since traveling waves are special entire solutions, Theorem 1.1 clearly provides an example, when $N = n+1 \geqslant 2$, of an entire solution that satisfies the monotonicity conditions (1.7) and that is not planar. Thus, for the elliptic equation (1.3) with $c \neq 0$ or for the parabolic equation (1.6) additional conditions are needed for an entire monotone solution to be planar. See also Lemma 2.3.

The monotonicity property (1.5) and the boundary values of U imply that the interface can be represented as a graph $z = H(|x|)$ or $|x| = R(z)$ where R is the inverse of H. We can describe the asymptotic shape of the interface as follows.

Theorem 1.2 *Assume* **(A)**. *Let* (c, U) *be as in Theorem 1.1 and* Γ *be the* 0*-level set of* U.

(i) *If* $n > 1$, Γ *is asymptotically a paraboloid, i.e.*

$$\lim_{z\to\infty,\, U(x,z)=0} \frac{|x|^2}{2z} = \frac{n-1}{c}.$$

(ii) *If* $n = 1$, Γ *is asymptotically a hyperbolic cosine curve, i.e., for some* $A = A(f) > 0$,

$$\lim_{z\to\infty,\, U(x,z)=0} \frac{\cosh(2\mu x)}{\mu z} = \frac{A}{c}, \qquad \mu := \sqrt{f'(1)}.$$

2 Outline of proof

The condition **(A)** is assumed hereafter. It implies the existence of constants $\alpha \in (0,1)$ and $\hat{\alpha} \in (-1,0)$ satisfying

$$(2.1) \quad f' = F'' > 0 \text{ on } [-1,\hat{\alpha}] \cup [\alpha, 1], \qquad F(\alpha) = F(\hat{\alpha}) < F(s)\ \forall s \in (\hat{\alpha}, \alpha).$$

In the sequel, α and $\hat{\alpha}$ are thus fixed. Also fixed is the wave speed $c > 0$. Note that all wells (roots to $f(\cdot) = 0$) other than ± 1 lie either in $(\hat{\alpha}, \alpha)$ or in $(-\infty, -1) \cup (1, \infty)$ where the latter is not our concern at all. The depth (the value of F) of any well in $(-1,1)$ is higher than $F(\alpha) > 0 = F(\pm 1)$.

For definiteness, we use notation

$$x \in \mathbb{R}^n, \quad z \in \mathbb{R}, \quad y = (x,z) \in \mathbb{R}^{n+1}, \quad r = |x|, \quad \rho = |y| = \sqrt{z^2 + |x|^2}.$$

2.1 Existence of the traveling wave solutions

Set

$$f_\varepsilon(u) := f(u) + \varepsilon\sqrt{2F(u)}, \quad F_\varepsilon(u) := \int_{-1}^{u} f_\varepsilon(s)ds.$$

For any $\varepsilon > 0$, f^ε is unbalanced; in particular $F_\varepsilon(s) > 0$ for all $s \in (-1, 1]$, which attains its deepest well only at $u = -1$. It is easy to verify that for any $\varepsilon > 0$, Φ is also the profile of a one dimensional traveling wave of speed ε to

$$\varepsilon\Phi' + \Phi'' = f_\varepsilon(\Phi) \quad \text{on } \mathbb{R}. \tag{2.2}$$

Furthermore, one assumes that $\varepsilon > 0$ is small enough so that $f'_\varepsilon(\pm 1) > 0$ and the profile of Φ is then a unique solution to (2.2) up to shift such that $\Phi(\pm\infty) = \pm 1$. (cf. [4]).

Hence, according to [39] when $n+1 = 2$, and [31] when $n+1 \geq 3$, for any given speed $c > 0$, there exists a cylindrically symmetric traveling wave $U^\varepsilon = U^\varepsilon(x, z)$ satisfying

$$\begin{cases} cU_z^\varepsilon + U_{zz}^\varepsilon + \Delta' U^\varepsilon = f_\varepsilon(U^\varepsilon) & \text{on } \mathbb{R}^{n+1}, \\ U^\varepsilon(0,0) = \alpha, \quad U^\varepsilon(\cdot, \pm\infty) \equiv \pm 1, \quad U_z^\varepsilon > 0 \geqslant U_r^\varepsilon & \text{on } \mathbb{R}^{n+1}, \end{cases} \tag{2.3}$$

where $r = |x|$. Since $|U^\varepsilon| \leqslant 1$, by a standard elliptic estimate [27], $\{U^\varepsilon\}_{0<\varepsilon\ll 1}$ is a bounded family in $C^3(\mathbb{R}^{n+1})$. Thus it is a compact family in $C^2_{\text{loc}}(\mathbb{R}^{n+1})$. Along a sequence $\varepsilon \searrow 0$, it converges to a cylindrically symmetric solution U to

$$cU_z + U_{zz} + \Delta' U = f(U), \quad |U| \leqslant 1, \quad U_z \geqslant 0 \geqslant U_r \quad \text{on } \mathbb{R}^{n+1}, \quad U(0,0) = \alpha. \tag{2.4}$$

2.2 The "boundary values"

We shall show that solutions to (2.4) has the right boundary value.

Lemma 2.1 *The following holds:*

(1) *Suppose $n = 1$. Then any symmetric (about x) solution U to (2.4) satisfies*

$$\lim_{z\to\pm\infty} U(x,z) = \pm 1 \ \forall\, x \in \mathbb{R}^n, \qquad \lim_{|x|\to\infty} U(x,z) = -1 \ \forall\, z \in \mathbb{R}. \tag{2.5}$$

(2) *Suppose $n > 1$. Let U be a limit, along a sequence $\varepsilon \searrow 0$, of the cylindrically symmetric family $\{U^\varepsilon\}$ of solutions to (2.3). Then U has the boundary value (2.5).*

Step 1. The limit equations.
As $U_z \geqslant 0 \geqslant U_r$ and $|U| \leqslant 1$, there exist

$$\varphi^\pm(x) := \lim_{z\to\pm\infty} U(x,z) \ \ \forall\, x \in \mathbb{R}^n, \qquad \varphi(z) := \lim_{|x|\to\infty} U(x,z) \ \ \forall\, z \in \mathbb{R}.$$

Consequently, $\lim_{z\to\pm\infty}(|U_z| + |U_{zz}|) = 0$ and $\lim_{|x|\to\infty} \Delta' U = 0$, by the boundedness of the $C^3(\mathbb{R}^{n+1})$ norm of U and the interpolation

$$\|\cdot\|_{C^1(D)} \leq 5\|\cdot\|^{1/2}_{C^2(D)}\|\cdot\|^{1/2}_{C^0(D)} \tag{2.6}$$

for any cubic domain D with side length $\geqslant 1$. Thus,

$$\Delta' \varphi^{\pm} - f(\varphi^{\pm}) = 0 \geqslant \varphi_r^{\pm} \quad \text{on } \mathbb{R}^n, \qquad \varphi^+(0) \geqslant \alpha \geqslant \varphi^-(0),$$
$$c\varphi_z + \varphi_{zz} - f(\varphi) = 0 \leqslant \varphi_z \quad \text{on } \mathbb{R}, \qquad \varphi(0) \leqslant \alpha.$$

To complete the proof, we need show that $\varphi^{\pm} \equiv \pm 1$ and $\varphi \equiv -1$.

Step 2. Radially symmetric stationary solutions.

To show the convergence we prepare for the auxiliary solutions for the maximum principle. For definiteness, in the sequel $\zeta \in C^3(\mathbb{R})$ is a fixed function satisfying

$$\zeta = 0 \quad \text{on } \{-1\} \cup [\hat{\alpha}, 1], \qquad \zeta > 0 \text{ in } (-1, \hat{\alpha}), \qquad \int_{-1}^{1} \left\{ \zeta(s) - \sqrt{2F(s)} \right\} ds > 0.$$

For each $\varepsilon > 0$, we define

$$g_\varepsilon(s) = f_\varepsilon(s) - \varepsilon\zeta(s) = f(s) + \varepsilon\sqrt{2F(s)} - \varepsilon\zeta(s) \quad \forall\, s \in [-1, 1].$$

For each sufficiently small positive ε, notice the following:
(i) both wells ± 1 of g_ε are stable, i.e., $g'_\varepsilon(\pm 1) > 0 = g_\varepsilon(\pm 1)$;
(ii) all wells of g_ε in $(-1, 1)$ lies in $(\hat{\alpha}, \alpha)$;
(iii) 1 is the only deepest well of g_ε on $[-1, 1]$, i.e. $\int_1^s g_\varepsilon(u)du > 0$ for all $s \in [-1, 1)$.

Using a standard shooting argument [7, 16, 41] one can show the following:

Lemma 2.2 *For each sufficiently small positive ε, there exists a unique solution w^ε to*

$$\text{(2.7)} \quad \frac{n}{\rho} w_\rho^\varepsilon + w_{\rho\rho}^\varepsilon - g_\varepsilon(w^\varepsilon) = 0 > w_\rho^\varepsilon \text{ in } (0, \infty), \quad w_\rho^\varepsilon(0) = 0, \quad w^\varepsilon(\infty) = -1.$$

The solution satisfies $w^\varepsilon(0) < 1 = \lim_{\varepsilon \searrow 0} w^\varepsilon(0)$.

These solutions will be used as subsolutions to establish the boundary values of U obtained from a limit process.

Step 3. The $z \to \infty$ limit.

Consider the case where $n = 1$. Integrating $\varphi_x^+\{\varphi_{xx}^+ - f(\varphi^+)\} = 0$ over $[0, \infty)$ and using $\varphi_x^+(0) = 0$ gives $F(\varphi^+(\infty)) = F(\varphi^+(0))$. Since $\varphi^+(0) \geqslant \alpha$, the definition of α in (2.1) implies that $\varphi^+(\infty) \in [-1, \hat{\alpha}] \cup [\alpha, 1]$. As $f(\varphi^+(\infty)) = 0$, we can only have either $\varphi^+(\infty) = -1$ or $\varphi^+(\infty) = 1$. The former case cannot happen, since F is a balanced potential with its deepest well at ± 1 so that $\psi \equiv -1$ is the only solution to

$$\psi_{xx} - f(\psi) = 0 \geqslant \psi_x \quad \text{on } [0, \infty), \ \psi_x(0) = 0, \ \psi(\infty) = -1.$$

Thus $\varphi^+(\infty) = 1$. Consequently, since $\varphi_x^+ \leqslant 0$ on $[0, \infty)$, $\varphi^+ \equiv 1$.

Next consider the case when $n > 1$. We write $\varphi^+(x)$ as $\varphi^+(r)$ where $r = |x|$.

Suppose $\varphi^+ \not\equiv 1$. Since $\varphi_r^+(0) = 0$, we must have $\alpha \leqslant \varphi^+(0) < 1$. Set $\beta := \varphi^+(\infty)$. Then $f(\beta) = 0$. Integrating

$$\varphi_r^+ \left[\varphi_{rr}^+ + \frac{n-1}{r} \varphi_r^+ - f(\varphi^+) \right] = 0$$

over $r \in [0,\infty)$, and using $\varphi_r^+(0)=0$, we obtain

$$F(\beta) - F(\varphi^+(0)) = \int_0^\infty \frac{n-1}{r}(\varphi_r^+)^2 > 0.$$

This implies that $\beta \in (\hat{\alpha}, \alpha)$.

Next, consider the solution w^ε of (2.7). Since $\lim_{\varepsilon \searrow 0} w^\varepsilon(0) = 1$, there exists $\varepsilon_0 > 0$ such that $w^{\varepsilon_0}(0) > \varphi^+(0)$. Also, since $w^{\varepsilon_0}(\infty) = -1$, there exists $R_0 > 0$ such that $w^{\varepsilon_0}(R_0) = \hat{\alpha}$. Set

$$\delta := \tfrac{1}{3}\min\{w^{\varepsilon_0}(0) - \varphi^+(0),\ \beta - \hat{\alpha}\} > 0.$$

Now, since $\lim_{z\to\infty} U(\cdot, z) = \varphi^+(|\cdot|)$ locally uniformly, there exists $z_0 \in \mathbb{R}$ such that $|U(x,z) - \varphi^+(|x|)| < \delta$ for all $|x| \leqslant R_0$ and $z \geqslant z_0$. Also, by the assumption, along a sequence $\varepsilon \searrow 0$, $U^\varepsilon \to U$ uniformly on any compact subset of $\mathbb{R}^{n+1}$. There exists $\varepsilon \in (0, \varepsilon_0)$ such that $|\, U^\varepsilon(x,z) - U(x,z) \,| \leqslant \delta$ for all $|z - z_0| + |x| \leqslant 2R_0$. Hence

$$|\, U^\varepsilon(x,z) - \varphi^+(|x|) \,| \leqslant 2\delta \quad \text{if } z_0 \leqslant z \leqslant z_0 + R_0, |x| \leqslant R_0.$$

We shall compare $U^\varepsilon((0,z)+\cdot)$ with $w^{\varepsilon_0}(|\cdot|)$ on $B(R_0) := \{y \in \mathbb{R}^{n+1} \mid |y| < R_0\}$. Since $\lim_{z\to\pm\infty} U^\varepsilon(\cdot, z) = \pm 1$ locally uniformly, we can define z^* by

$$z^* := \min\{z \in \mathbb{R} \ \mid\ U^\varepsilon((0,z)+y) \geqslant w^{\varepsilon_0}(|y|) \quad \forall y \in \bar{B}(R_0)\ \}.$$

Upon noting that for every $z \leqslant z_0$, $U^\varepsilon(0,z) \leqslant U^\varepsilon(0,z_0) \leqslant \varphi^+(0) + 2\delta < w^{\varepsilon_0}(0)$, we see that $z^* > z_0$.

Let $y_0 \in \bar{B}(R_0)$ be a point such that

$$0 = U^\varepsilon((0,z^*)+y_0) - w^{\varepsilon_0}(|y_0|) = \min_{y\in\bar{B}(R_0)}\Big\{U^\varepsilon((0,z^*)+y) - w^{\varepsilon_0}(|y|)\Big\}.$$

Since $U^\varepsilon(x,z)$ is monotonic in z and in $|x|$,

$$w^{\varepsilon_0}(|y_0|) = U^\varepsilon((0,z^*)+y_0) \geqslant U^\varepsilon((0,z_0)+y_0) \geqslant \varphi^+(|y_0|) - 2\delta \geqslant \beta - 2\delta > \hat{\alpha} = w^{\varepsilon_0}(R_0).$$

It follows that y_0 is an interior point of $\bar{B}(R_0)$. Consequently, $(\partial_{zz}+\Delta')U^\varepsilon((0,z^*)+y_0) \geqslant (\partial_{zz}+\Delta')w^{\varepsilon_0}(|y_0|)$. Also, as $w^{\varepsilon_0}(|y_0|) > \hat{\alpha}$, we have $\zeta(w^{\varepsilon_0}(|y_0|)) = 0$. Hence

$$\begin{aligned} 0 &= cU_z^\varepsilon + (\partial_{zz}+\Delta')U^\varepsilon - f_\varepsilon(U^\varepsilon)\Big|_{(0,z^*)+y_0} > 0 + (\partial_{zz}+\Delta')w^{\varepsilon_0} - f_{\varepsilon_0}(w^{\varepsilon_0})\Big|_{y_0} \\ &= (\partial_{zz}+\Delta')w^{\varepsilon_0} - g_{\varepsilon_0}(w^{\varepsilon_0})\Big|_{y_0} = 0, \end{aligned}$$

which is impossible. This impossibility shows that $\varphi^+ \equiv 1$.

Step 4. Cases for $z \to -\infty$ and $|x| \to \infty$.

By the similar manners, we can show $\varphi^- \equiv -1$ and $\varphi \equiv -1$. So, we omit the detail of the proof.

Finally, the monotonicity property $U_z > 0$ on $\mathbb{R}^{n+1}$ and $U_r < 0$ for all $r = |x| > 0$ follows from the strong maximum principle.

2.3 Planar Waves

In studying the asymptotic behavior of the interface, a limiting procedure leads to the following, for $\Psi = \Psi(\xi, z)$, $\xi \in \mathbb{R}$, $z \in \mathbb{R}$:

$$(2.8)\ c\Psi_z + \Psi_{zz} + \Psi_{\xi\xi} = f(\Psi), \quad |\Psi| \leqslant 1, \quad \Psi_z \geqslant 0 \geqslant \Psi_\xi \quad \text{on } \mathbb{R}^2, \quad \Psi(0,0) = \alpha.$$

Lemma 2.3 *Assume* (A) *and* $c > 0$. *Then* $\Psi(\xi, z) = \Phi(-\xi)$, $(\xi, z) \in \mathbb{R}^2$, *is the only solution to (2.8).*

This result implies that $\lim_{z\to\infty} \|U_z(\cdot, z)\|_{L^\infty(\mathbb{R}^n)} = 0$ and that the interface is asymptotically vertical.

2.4 The behavior of the interfaces

In this subsection we give the heuristic understanding of the profile of the level sets $\{U(x,z) = \alpha\} = \{|x| = R(z)\}$. It is well-known that the interface (level set) of solutions of (1.2) evolves, in an appropriate space and time scale, according to the motion by mean curvature flow; see [2, 11, 19, 21, 35] and references therein. For a traveling wave solution of (1.3), after shrinking the space by a factor of $R(\hat{z})$, the interface near $\mathbb{R}^n \times \{\hat{z}\}$ is asymptotically, as $\hat{z} \to \infty$, a circular cylinder $\mathbb{S}(1) \times \mathbb{R}$ where $\mathbb{S}(r)$ represents the sphere in $\mathbb{R}^n$ with radius r and center origin. As a hypersurface in $\mathbb{R}^{n+1}$, $\mathbb{S}(1) \times \mathbb{R}$ has a sum of all principal curvatures equal to $n-1$. Thus, when $n > 1$, the interface moves, in a certain scaled space-time, with a normal velocity equal to $n-1$. Translating into the original space-time, this motion should represent a constant vertical velocity c motion. In the moving coordinates, this renders to the approximation equation $cR' \sim (n-1)/R$, from which the asymptotic behavior $cR(z)^2/2 \approx (n-1)z$ for the interface follows.

In the two dimensional case ($n = 1$), the scaled interface is asymptotically two lines $\{\pm 1\} \times \mathbb{R}$, for which the curvature effect is negligible. To discover the dynamics, we compare (1.2) with its one space dimensional version $u_t = \varepsilon^2 u_{\xi\xi} - f(u)$ ($\varepsilon = 1/R(\hat{z}), \xi = x/R(\hat{z})$). It has been discovered more than a decade ago by Carr and Pego [10], Fusco [24], and Fusco and Hale [25] that for well-developed initial profile in a bounded domain with Neumann or periodic boundary conditions, the speed that two interfaces of distance d approach each other is of order $e^{-2\mu d/\varepsilon}$. Such a result was recently extended with simplified proofs by Chen [13] to arbitrary initial data and on the whole real line (see also Ei [20]). In particular, if initially there are two interfaces of distance d, the velocity that the two interfaces approach each other is $Ae^{-2\mu d/\varepsilon + o(1)}$, after an initiation which processes an arbitrary initial data into a special wave profile. The time needed for such an initiation is significantly short in comparing to the exponentially slow motion of the interface. If this size of normal velocity should produce a vertical velocity c motion, the shape of interface for solutions of (1.3) should be asymptotically governed by the equation $cR' = Ae^{-2\mu R}$, resulting a hyperbolic cosine curve, as describes in Theorem 1.2.

From another point of view, formally, for large z we have $cR'' = -2\mu Ae^{-2\mu R}R' = o(1)R'$, so the U_{zz} term in (1.3) can be expected to be dropped without causing any significant change (for large z). Then (1.3) becomes $cU_z + U_{xx} = f(U)$. A change of variables $s = z/c$ gives $U_s + U_{xx} = f(U), (s,x) \in \mathbb{R}^2$. A recent result of Chen, Guo, and Ninomiya [14] shows that there is a unique (up to a translation) entire solution having two interfaces located asymptotically on the hyperbolic cosine curve described in Theorem 1.2.

Thus, Theorem 1.2 verifies the following speculation: when $n > 1$, the pure curvature effect contributes to the motion of the interface; when $n = 1$, the curvature effect is insignificant and it is the interaction of the two branches of the interface.

References

[1] G. ALBERTI, L. AMBROSIO, AND X. CABRÉ, *On a long-standing conjecture of E. De Giorgi: old and recent results*, Acta Appl. Math. **65** (2001), pp. 9-33.

[2] S. ALLEN AND J.W. CAHN, *A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening*, Acta. Metall. **27** (1979), pp. 1084-1095.

[3] L. AMBROSIO AND X. CABRÉ, *Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi*, J. Amer. Math. Soc. **13** (2000), pp. 725-739.

[4] D.G. ARONSON AND H.F. WEINBERGER, *Multidimensional nonlinear diffusions arising in population genetics*, Adv. Math. **30** (1978), pp. 33-76.

[5] H. BERESTYCKI, L. CAFFARELLI, AND L. NIRENBERG, *Monotonicity for elliptic equations in unbounded Lipschitz domains*, Comm. Pure Appl. Math. **50** (1997), pp. 1089-1111.

[6] H. BERESTYCKI, F. HAMEL, AND R. MONNEAU, *One-dimensional symmetry of bounded entire solutions of some elliptic equations*, Duke Math. J. **103** (2000), pp. 375-396.

[7] H. BERESTYCKI AND B. LARROUTUROU, *Planar travelling front solutions of reaction-diffusion problems*, preprint.

[8] A. BONNET AND F. HAMEL, *Existence of nonplanar solutions of a simple model of premixed Bunsen flames*, SIAM J. Math. Anal. **31** (1999), pp. 80-118.

[9] L.A. CAFFARELLI AND X. CABRÉ, *Fully Nonlinear Elliptic Equations*, Colloquium Publications, Amer. Math. Soc. **43**, 1995.

[10] J. CARR AND R.L. PEGO, *Invariant manifolds for metastable patterns in $u_t = \varepsilon^2 u_{xx} - f(u)$*, Proc. Roy. Soc. Edinburgh, Sect. **A, 116** (1990), pp. 133-160.

[11] X.F. CHEN, *Generation and propagation of interfaces for reaction-diffusion equations*, J. Diff. Eqns. **96** (1992), pp. 116-141.

[12] X.F. CHEN, *Spectrum for the Allen-Cahn, Cahn-Hilliard, and phase-field equations for generic interfaces*, Comm. Partial Diff. Equations **19** (1994), pp. 1371-1395.

[13] X.F. CHEN, *Generation, propagation, and annihilation of metastable patterns*, J. Diff. Eqns. **206** (2004), pp. 399-437.

[14] X.F. CHEN, J.S. GUO, AND H. NINOMIYA, *Entire solutions of reaction-diffusion equations with balanced bistable nonlinearities*, Proc. Roy. Soc. Edinburgh, Sect. **A**, to appear.

[15] X.F. CHEN, J.S. GUO, F. HAMEL, H. NINOMIYA, AND J.M. ROQUEJOFFRE *Traveling Waves with Paraboloid Like Interfaces for Balanced Bistable Dynamics*, preprint

[16] X.F. CHEN AND M. TANIGUICHI, *Instability of spherical interfaces in a nonlinear free boundary problem*, Adv. Diff. Equations **5** (2000), pp. 747-772.

[17] K. DECKELNICK, C. M. ELLIOTT, AND G. RICHARDSON, *Long time asymptotics for forced curvature flow with applications to the motion of a superconducting vortex*, Nonlinearity **10** (1997), pp. 655-678.

[18] E. DE GIORGI, *Convergnece problems for functionals and operators*, In: Proc. Int. Meeting on Recent Methods in Nonlinear Analysis, Rome, 1978, Pitagora, 1979, pp. 131-188.

[19] P. DE MOTTONI AND M. SCHATZMAN, *Development of interfaces in* $\mathbf{R}^N$, Proc. Roy. Soc. Edinburgh Sect. A **116** (1990), pp. 207-220.

[20] S.-I. EI, *The motion of weakly interaction pulses in reaction-diffusion systems*, J. Dynamics Differential Equations **14** (2002), pp. 85-137.

[21] L.C. EVANS, H.M. SONER, AND P.E. SOUGANIDIS, *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math. **45** (1992), pp. 1097-1123.

[22] P.C. FIFE, Dynamics of Internal Layers and Diffusive Interfaces, CBMS-NSF Regional Conference, Series in Applied Mathematics 53, 1988.

[23] P.C. FIFE AND J.B. MCLEOD, *The approach of solutions of non-linear diffusion equations to traveling front solutions*, Arch. Rational Mech. Anal. **65** (1977), pp. 335-361.

[24] G. FUSCO, *A geometric approach to the dynamics of* $u_t = \varepsilon^2 u_{xx} + f(u)$ *for small* ε, in "Lecture Notes in Physics" (Kirchgassner Ed.), Vol **359**, 1990, pp. 53-73.

[25] G. FUSCO AND J.K. HALE, *Slow-motion manifolds, dormant instability, and singular perturbations*, J. Dynamics Differential Equations **1** (1989), pp. 75-94.

[26] N. GHOUSSOUB AND C. GUI, *On a conjecture of De Giorgi and some related problems*, Math. Ann. **311** (1998), pp. 481-491.

[27] D. GILBARG AND N.S. TRUDINGER, Elliptic Partial Differential Equations of Second Order, Springer Verlag, 1997.

[28] F. HAMEL AND R. MONNEAU, *Solutions of semilinear elliptic equations in* $\mathbb{R}^N$ *with conical-shaped level sets*, Comm. Partial Diff. Equations **25** (2000), pp. 769-819.

[29] F. HAMEL AND R. MONNEAU, *Existence and uniqueness for a free boundary problem arising in combustion theory*, Interfaces Free Boundaries **4** (2002), pp. 167-210.

[30] F. HAMEL, R. MONNEAU, AND J.-M. ROQUEJOFFRE, *Stability of traveling waves in a model for conical flames in two space dimensions*, Ann. Scient. Ec. Norm. Sup. **37** (2004), pp. 469-506.

[31] F. HAMEL, R. MONNEAU, AND J.-M. ROQUEJOFFRE, *Existence and qualitative properties of multidimensional conical bistable fronts*, Disc. Cont. Dyn. Systems **13** (2005), pp. 1069-1096.

[32] F. HAMEL, R. MONNEAU, AND J.-M. ROQUEJOFFRE, *Asymptotic properties and classification of bistable fronts with Lipschitz level sets*, Disc. Cont. Dyn. Systems **14** (2006), pp. 75-92.

[33] F. HAMEL AND N. NADIRASHVILI, *Travelling waves and entire solutions of the Fisher-KPP equation in* $\mathbb{R}^N$, Arch. Rational Mech. Anal. **157** (2001), pp. 91-163.

[34] M. HARAGUS AND A. SCHEEL, *Corner defects in almost planar interface propagation*, Ann. Inst. H. Poincaré, Anal. Non Linéaire, to appear.

[35] T. ILMANEN, *Convergence of the Allen-Cahn equation to Brakke's motion*, J. Diff. Geom. **38** (1993), pp. 417-461.

[36] T. Kawahara and M. Tanaka *Interactions of traveling fronts: An exact solutions of a nonlinear diffusion equations*, Physics Letters. **97A** (1983), 311-314.

[37] H. NINOMIYA AND M. TANIGUCHI, *Traveling curved fronts of a mean curvature flow with constant driving force,* in "Free boundary problems: theory and applications, I" GAKUTO Internat. Ser. Math. Sci. Appl. **13** (2000), pp. 206-221.

[38] H. NINOMIYA AND M. TANIGUCHI, *Stability of traveling curved fronts in a curvature flow with driving force*, Methods and Application of Analysis **8** (2001), pp. 429-450.

[39] H. NINOMIYA AND M. TANIGUCHI, *Existence and global stability of traveling curved fronts in the Allen-Cahn equations,* J. Diff. Eqns. **213** (2005), pp. 204-233.

[40] H. NINOMIYA AND M. TANIGUCHI, *Global stability of traveling curved fronts in the Allen-Cahn equations,* submitted to Disc. Cont. Dyn. Systems.

[41] T. OUYANG AND J. SHI, *Exact Multiplicity of Positive Solutions for a Class of Semilinear Problems,* J. Diff. Eqns. **146** (1998), pp. 121-156.

[42] O. SAVIN, *Phase transitions: regularity of flat level sets*, preprint.

[43] H.M. SONER, *Motion of a set by the curvature of its boundary*, J. Diff. Eqns. **101** (1993), pp. 313-372.