Title	$GL_n(F_q)$のブロックにおける圏同値の組合わせ論的記述,qとの独立性と分解定数について (組合せ論的表現論をめぐる話題)
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$GL_n(F_q)$のブロックにおける圏同値の組合せ論的記述、qとの独立性と分解定数について

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1 Notation

The result of this paper is a joint work with **Akihiko Hida** [1]. Let G be a finite group, and $(K, \mathcal{O}, \mathbf{k})$ be a splitting ℓ-modular system for G. Here $\mathrm{char}(K) = 0, \mathrm{char}(\mathbf{k}) = \ell > 0$. For $R \in \{\mathcal{O}, \mathbf{k}\}$, let $B_0(RG)$ be the principal block of RG.
$\mathfrak{S}_n$ denotes the symmetric group on n letters. $\mathbb{F}_q$ denotes a field with q elements with $\ell \nmid q$. Let natural numbers $e(q)$ and $r(q)$ be as follows:

$$e(q) := \mathbf{Min}\{\ i \in \mathbb{N} \mid q^i \equiv\ 1\ (\ \mathbf{mod}\ \ell\ )\ \},$$

$$r(q) := \mathbf{Max}\{\ r \in \mathbb{N} \mid \ell^r \mid q^{e(q)} - 1\ \}\text{: the } \ell\text{-part of } q^{e(q)} - 1.$$

Let A and B be blocks ideals. "$A \sim_M B$" means that A is Morita (Puig) equivalent to B. "$A \sim_d B$" means that A is derived (splendid Rickard) equivalent to B (see [34],[35]).

We use results on representation theory of finite general linear groups in non-defining characteristic due to Fong-Srinivasan and Dipper-James (see [14], [15], [9],[10],[11],[12], [13],[19]).

2 Motivations

We wish to prove the following conjectures:

Conjecture 2.1 (Broué). [2], [3], [4] *Let B be an ℓ-block ideal of G with abelian defect group D. Then B and its Brauer correspondent in $\mathcal{N}_G(D)$ are derived equivalent?*

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[0]The detailed version of this paper will be submitted for "Doctor thesis at Chiba Univ., Japan".

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Conjecture 2.2 (James). [20] *Suppose that char*$(\mathbf{k}) = \ell > n$ *and* $e(q) = e$. *Let* ζ *be a primitive e-th root of unity in* $\mathbb{C}$. *Then, the decomposition matrix of Dipper-James Schur algebra* $S_\zeta(n,r)_\mathbb{C}$ *over* $\mathbb{C}$ *is equal to that of Dipper-James Schur algebra* $S_{\overline{q}}(n,r)_\mathbf{k}$ *over* $\mathbf{k}$ *?*

Let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbb{F}_q$ with a Frobenius map F. We assume that the centre of $\mathbf{G}$ is connected. Let ℓ be a prime number with $\ell \nmid q$.

Lusztig series

The following is so-called Lusztig series:

$$\mathcal{E}(\mathbf{G}^F,\{s\}) := \bigcup_{(\mathbf{T},\theta)} \{\chi \in \widehat{\mathbf{G}}^F \mid \langle \chi, R^{\mathbf{G}}_{\mathbf{T}}(\theta)\rangle \neq 0\}.$$

Here, the above pair $(\mathbf{T},\theta)$ runs $s_1 \in \{s\}$ and $\theta \in \widehat{\mathbf{T}}^F \leftrightarrow s_1 \in \mathbf{T}^{*F}$, and $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ is a generalized Deligne-Lusztig character.

Its modular version is given as follows:

For a semisimple ℓ'-element $s \in \mathbf{G}^{*F^*}$, let

$$\mathcal{E}_\ell(\mathbf{G}^F,\{s\}) := \bigcup_t \mathcal{E}(\mathbf{G}^F,\{st\}), \quad t \in (C_{\mathbf{G}^*}(s)^{F^*})_\ell.$$

Theorem 2.3 (Broué-Michel). [5] *Each set* $\mathcal{E}_\ell(\mathbf{G}^F,\{s\})$ *is a union of* ℓ*-block of* $\mathbf{G}^F$.

Definition 2.1. *An* ℓ*-block* B *as an algebra is unipotent, if there exists* $\chi \in \mathcal{E}_\ell(\mathbf{G}^F,\{1\})$ *such that* χ *belongs to* B. *In particular,* $B_0(\mathcal{O}\mathbf{G}^F)$ *is unipotent.*

Theorem 2.4 (Bonnafé-Rouquier). [1] *Suppose that the centre of* $\mathbf{G}$ *is connected and* $C_{\mathbf{G}^*}(s)^{*F}$ *is a Levi subgroup of* $\mathbf{G}$. *Then*

$$\mathcal{E}_\ell(\mathbf{G}^F,\{s\}) \sim_M \mathcal{E}_\ell(C_{\mathbf{G}^*}(s)^{*F},\{1\})$$

as ℓ*-block ideals. (i.e. If a block* B_s *belongs to* $\mathcal{E}_\ell(\mathbf{G}^F,\{s\})$, *then there exists a unipotent block* B'_1 *of* $C_{\mathbf{G}^*}(s)^{*F}$ *such that* B_s *and* B'_1 *are Morita equivalent.)*

Remark 1. *The Morita equivalence in the above theorem is not a Puig equivalence in general.*

In particular, for finite general linear groups we may concentrate unipotent blocks by Bonnafé-Rouquier theorem.

We want to classify the block ideals of $\mathbf{kG}(\mathbb{F}_q)$ up to Morita equivalence, and recover its structure as algebras from some small subgroups. So, we wish to prove the following conjecture:

Conjecture 2.5. *If $e(q) = e(q'), r(q) = r(q')$ then for any unipotent block ideal B of $\mathbf{G}(\mathbb{F}_q)$ there exists a unipotent block ideal B' of $\mathbf{G}(\mathbb{F}_{q'})$ such that $B \sim_M B'$ by an exact ℓ-permutation (B, B')-bimodule. This equivalence preserves the natural indices of modules.*

In this article we deal the special case concerning these three conjectures for finite general linear groups.

3 Abacus and $[w{:}k]$-pairs

Definition 3.1. *For a k-core τ and a non-negative integer w, let $\Lambda_{k,w,\tau}$ be the set of paritions of $kw + |\tau|$ whose k-core is τ.*

Given partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, define $\beta = (\beta_1, \beta_2, \ldots)$ as follows:

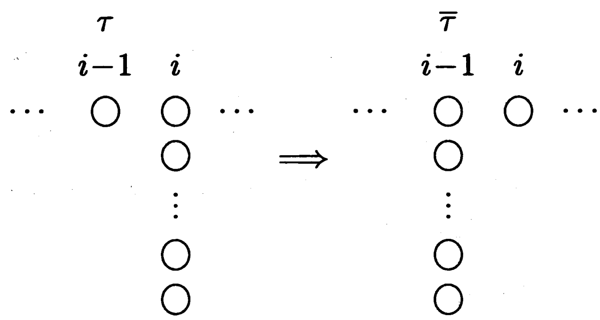
$$\beta_i := r - i + \lambda_i (1 \leq i \leq r).$$

We call this β an r-element β-set for λ.

Definition 3.2 (Scopes). *For a non-negative integer m and an m-core $\tau = (\tau_1, \ldots, \tau_r)$, let Γ be the r-element β-set for τ, and suppose that when Γ is diplayed on an abacus with m-runners there are k more than beads in the i-th column than in the $(i-1)$-th column. Let m-core $\overline{\tau}$ be displayed by an r-element β-set $\overline{\Gamma}$ satisfying*

$$\begin{aligned} \overline{\Gamma}_j &= \Gamma_j \qquad \text{for } j \neq i, i-1 \\ \overline{\Gamma}_i &= \Gamma_{i-1} \\ \overline{\Gamma}_{i-1} &= \Gamma_i, \end{aligned}$$

where Γ_j is the number of beads on the j-th runner in the abacus configulation for Γ. In these situation, we shall say that $\Lambda_{m,w,\tau}$ and $\Lambda_{m,w,\overline{\tau}}$ form a Scopes $[w{:}k]$-pair.



Scopes proved the following:

Theorem 3.1 (Scopes). *[37] If $\Lambda_{p,w,\tau}$ and $\Lambda_{p,w,\nu}$ form a $[w : k]$-pair with $k \geq w$, then p-blocks $B^{w,\tau}$ and $B^{w,\nu}$ of symmetric groups are Morita equivalent.*

By Jost we also know the following:

Theorem 3.2 (Jost). *[23] If $\Lambda_{e,w,\tau}$ and $\Lambda_{e,w,\nu}$ form a $[w : k]$-pair with $k \geq w$, then unipotent ℓ-blocks $B_{w,\tau}$ and $B_{w,\nu}$ are Morita equivalent.*

Example 1. *If B is a unipotent block of $GL_n(q)$ with e-weight 2, then one of the following holds:*

1. $B \cong B_0(\mathbf{k}GL_{2e}(q))$.
2. $(B, \overline{B})$ *forms* $[2\!:\!1]$*-pair for some unipotent block* $\overline{B}$ *of* $\mathbf{k}GL_{n-1}(q)$. *(Actually, these blocks are derived equivalent to its Brauer correpondent of the ℓ-local subgroup. (Hida-Miyachi(1999)) (The method we used is different from J. Chuang's for $\mathfrak{S}_n$)*
3. $B \sim_M B'$ *for some unipotent block* B' *of* $\mathbf{k}GL_m(q)$ *with* $m < n$.

4 A core ρ and results of J. Chuang and R. Kessar

Definition 4.1 (Chuang-Kessar-Rouquier). *[8] Let ρ be the e-core which satisfies the following property : ρ has an abacus configulation in which each runner other than the leftmost one (the 0-th runner) has at least $w-1$ more beads than the runners to its immediate left.*

Chuang and Kessar considered the following setting up:

$e = p > w$.

$r := |\rho|$.

$G := \mathfrak{S}_{pw+r}$.

$B^{w,\rho}$: the p-block of $\mathbf{k}G$ with p-weight w and p-core ρ.

$D :=$ a defect group of $B^{w,\rho}$.

$N := \mathfrak{S}_p \wr \mathfrak{S}_w \supset D$.

$L := \mathfrak{S}_p \times \cdots \times \mathfrak{S}_p \times \mathfrak{S}_r$.

$H := (\mathfrak{S}_p \wr \mathfrak{S}_w) \times \mathfrak{S}_r \supset \mathcal{N}_G(D)$.

$\mathcal{O}Hf$:=the Brauer correspondent of $B^{w,\rho}$ in H.

Let X be the Green correspondent of $B^{w,\rho}$ in $G \times H$ with respect to $(G \times G, \Delta(D), G \times H)$. Chuang and Kessar proved the following:

Theorem 4.1 (Chuang-Kessar). [8] *Suppose that $p > w$. Then, we get an isomorphism*

$$\mathcal{O}Hf \cong \mathrm{End}_G(X_{\mathcal{O}})$$

by checking $\mathrm{rank}_{\mathcal{O}}(\mathrm{End}_G(X)) \leq w! \cdot \mathrm{rank}_{\mathcal{O}}(\mathcal{O}Lf)$. *In particular, $\mathcal{O}Hf$ is Morita equivalent to $B^{w,\rho}$.*

Remark 2. *1. X is exact.*
2. $\mathcal{O}Hf \to \mathrm{End}_G(X)$ is a split $(\mathcal{O}Hf, \mathcal{O}Hf)$-monomorphism.
3. $w!\,\mathrm{rank}_{\mathcal{O}}(\mathcal{O}Lf) = \mathrm{rank}_{\mathcal{O}}(\mathcal{O}Hf)$.
4. By Marcus [27] $\mathcal{O}Hf \sim_d B_0(\mathcal{O}N)$.
5. $(D^\lambda \otimes_{B^{w,\rho}} X)\downarrow_L$ is known, but $D^\lambda \otimes_{B^{w,\rho}} X$ is not known.

5 A theorem of Chuang-Kessar type

We assume that $char(\mathbf{k}) = \ell > w$. Choose a prime power q with $e(q) = e$. Just mimicking Chuang and Kessar's setting up, we consider the following:

$r := |\rho|$.

$G(q) := GL_{ew+r}(q)$.

$B^{w,\rho}(q)$: the unipotent ℓ-block of $\mathbf{k}G(q)$ with e-weight w and e-core ρ.

$D(q) :=$ a defect group of $B^{w,\rho}$.

$N(q) := GL_e(q) \wr \mathfrak{S}_w \supset D(q)$.

$L(q) := GL_e(q) \times \cdots \times GL_e(q) \times GL_r(q)$.

$H_w(q) := (GL_e(q) \wr \mathfrak{S}_w) \times GL_r(q) \supset \mathcal{N}_G(D(q))$.

$\mathcal{O}H_w(q)f_q$:=the Brauer correspondent of $B_{w,\rho}(q)$ in $H_w(q)$.

Once we believe that an anology of Chuang-Kessar theorem holds for finite general linear groups, we can easily prove the following:

Proposition 5.1. **(An anology of Chuang-Kessar theorem)** *Let $X(q)$ be the Green correspondent of $B^{w,\rho}(q)$ in $G(q) \times H_w(q)$ with respect to $(G(q) \times G(q), \Delta(D(q)), G(q) \times H_w(q))$. Then, we get an isomorphism*

$$\mathcal{O}H_w(q)f_q \cong \mathrm{End}_{G(q)}(X_{\mathcal{O}}(q))$$

by checking $\mathrm{rank}_{\mathcal{O}}(\mathrm{End}_{G(q)}(X_{\mathcal{O}}(q))) \leq w! \cdot \mathrm{rank}_{\mathcal{O}}(\mathcal{O}L(q)f_q)$. *In particular, $\mathcal{O}H_w(q)f_q$ is Morita equivalent to $B_{w,\rho}(q)$.*

Remark 3. *One must consider not only unipotent characters but also characters indexed by semisimple ℓ-elements. We can know these characters by [9]. We also need some results by [15] in order to mimic Chuang and Kessar's argument.*

6 Indices of $B_0(GL_e(q) \wr \mathfrak{S}_w)$-modules

In this section we reformulate indices of the simple $B_0(GL_e(q) \wr \mathfrak{S}_w)$-modules to fit that of $B_{w,\rho}(q)$ via the equivalence in Proposition 5.1. For $i = 1, 2, \ldots, e$ let $\nu_i = (i, 1^{e-i}) \vdash e$. The principal block $B_0(\mathbf{k}GL_e(q))$ has e non-isomorphic irreducible modules

$$\{ D_{\mathbf{k},q}(\nu_i) \mid i = 1, 2, \ldots, w\}.$$

Fix $R \in \{K, \mathbf{k}\}$. Let $\boldsymbol{n}$ be an e-tuple non-negative integer of w. i.e. $\sum_{i=1}^{e} \boldsymbol{n}_i = w$. $S_{R,q}(\boldsymbol{n}) := \bigotimes_i (S_{R,q}(\nu_i)^{\otimes \boldsymbol{n}_i})$ is an $R[GL_e(q)^{\times w}]$-module. In particular, $S_{K,q}(\boldsymbol{n})$ is a simple $K[GL_e(q)^{\times w}]$-module. The parabolic subgroup $\mathfrak{S}_{\boldsymbol{n}}$ act on $S_{R,q}(\boldsymbol{n})$. So, $S_{R,q}(\boldsymbol{n})$ is an $R[L_{(e^w)} \rtimes \mathfrak{S}_{\boldsymbol{n}}]$-module. $\mathrm{Ind}_{L_{(e^w)}}^{L_{(e^w)} \rtimes \mathfrak{S}_{\boldsymbol{n}}} S_{R,q}(\boldsymbol{n})$ is decomposed into $\bigoplus_{\boldsymbol{\lambda} \vdash \boldsymbol{n}} (S_{R,q}(\boldsymbol{n}) \otimes_R (\dim_R S_R^{\boldsymbol{\lambda}}) \cdot S_R^{\boldsymbol{\lambda}})$ where S_R^{μ} means the Specht module of $R[\mathfrak{S}_{|\mu|}]$ corresponding to μ, $S_R^{\boldsymbol{\lambda}} = \bigotimes_i S_R^{\boldsymbol{\lambda}_i}$ and $S_{R,q}(\boldsymbol{n}) \otimes_R S_R^{\boldsymbol{\lambda}}$ is the inner tensor product of $R[L_{(e^w)} \rtimes \mathfrak{S}_{\boldsymbol{n}}]$-modules $S_{R,q}(\boldsymbol{n})$ and $S_R^{\boldsymbol{\lambda}}$. Let

$$T_R^{\boldsymbol{\lambda}_i} = \begin{cases} S_R^{\boldsymbol{\lambda}_i} & \text{if } i + e \text{ is even,} \\ S_R^{\boldsymbol{\lambda}'_i} & \text{if } i + e \text{ is odd.} \end{cases}$$

Here, $\boldsymbol{\lambda}'_i$ is the conjugate partition of $\boldsymbol{\lambda}_i$. Let $T_R^{\boldsymbol{\lambda}} = \bigotimes_i T_R^{\boldsymbol{\lambda}_i}$.

For $\boldsymbol{\lambda} \vdash \boldsymbol{n}$ let $U_{R,q}(\boldsymbol{\lambda})$ be $\mathrm{Ind}_{L_{(e^w)} \rtimes \mathfrak{S}_{\boldsymbol{n}}}^{GL_e(q) \wr \mathfrak{S}_w} (S_{R,q}(\boldsymbol{n}) \otimes T_R^{\boldsymbol{\lambda}})$, and let $U_{\mathbf{k},q}(\boldsymbol{\lambda})^{\rho}$ be the $R[H_w(q)]$-module $U_{R,q}(\boldsymbol{\lambda}) \otimes_R S_{R,q}(\rho)$.

Moreover, one can construct modules by using

$$\{ D_{\mathbf{k},q}(\nu_i) \mid i = 1, 2, \ldots, e\}$$

instead of $\mathbf{k}[GL_e(q)]$-modules $\{ S_{\mathbf{k},q}(\nu_i) \mid i = 1, 2, \ldots, e\}$. We denote it by $V_{\mathbf{k},q}(\boldsymbol{\lambda})^{\rho}$.

7 Results

Now we can state our main results of this article as follows:

Theorem 7.1 (Hida-Miyachi). *[18] For any simple $B_{w,\rho}(q)$-module $D_{\mathbf{k},q}(\lambda)$, the Green correspondent $D_{\mathbf{k},q}(\lambda) \otimes_{\mathbf{k}G} X(q)$ of $D_{\mathbf{k},q}(\lambda)$ is independent of q in the following sense:*

Assume that $e(q) = e(q')$ and $r(q) = r(q')$. Let $\mathcal{M}_{q,q'}$ be the canonical $(\mathbf{k}H_w(q)f_q, \mathbf{k}H(q')f_{q'})$-bimodule which induces $\mathbf{k}H_w(q)f_q \sim_M \mathbf{k}H(q')f_{q'}$, due to A. Marcus. Then

$$D_{\mathbf{k},q}(\lambda) \otimes_{B_{w,\rho}(q)} X(q) \otimes_{\mathbf{k}H_w(q)f_q} \mathcal{M}_{q,q'} \otimes_{\mathbf{k}H(q')f_{q'}} X(q')^{\vee} \cong D_{\mathbf{k},q'}(\lambda).$$

Actually, $D_{\mathbf{k},q}(\lambda) \otimes_{B_{w,\rho}(q)} X(q) \cong V_{\mathbf{k},q}(\overline{\lambda})^{\rho}$. Here, $\overline{\lambda}$ is the e-quotient of λ. Moreover, we know the decomposition numbers corresponding to the e-core ρ :

$$d_{\lambda,\mu} = d_{\overline{\lambda},\overline{\mu}} = [U_{\mathbf{k},q}(\overline{\lambda}) : V_{\mathbf{k},q}(\overline{\mu})].$$

(The other parts of $B_{w,\rho}(q)$ can be calculated by Dipper-James theory.)

Remark 4. *First we can determine the Green correspondents of simple $B_{2,\rho}(q)$-modules in $H_2(q)$, finding two trivial source modules of $B_{2,\rho}(q)$, using the decomposition numbers for Hecke algebras of type **A** by [33] and [22], chasing the image of Mullineux-Kleshchev map [29, p.120], the property of Specht modules [19] and induction on $\Lambda_{e,2,\rho}$.*

Next we can determine the Green correspondents of simple $B_{w,\rho}(q)$-modules in $H_w(q)$ using induction on w and some commutative diagrams among $B_{w,\rho}(q)$,$B_0(GL_e(q)) \otimes B_{w-1,\rho}(q)$ and their Brauer correspondents.

In order to prove $B_{w,\rho}(q) \sim_M B_{w,\rho}(q')$ with the property in the above theorem we use [14],[27], and [36].

Corollary 7.2. *[18] If there exist a sequence of e-cores*

$$\rho = \tau^0, \tau^1, \dots, \tau^s$$

such that Λ_{e,w,τ^i} and $\Lambda_{e,w,\tau^{i+1}}$ form a $[w : k_i]$-pair with $k_i \geq w-1$, Broué's conjecture is true for $B_{w,\tau^s}(q)$.

Theorem 7.3 (Hida-Miyachi). *[18] Assume that $e = e(q) = e(q')$ and $r(q) = r(q')$. If there exist a sequence of e-cores*

$$\rho = \tau^0, \tau^1, \dots, \tau^s$$

such that Λ_{e,w,τ^i} and $\Lambda_{e,w,\tau^{i+1}}$ form a $[w : k_i]$-pair with $k_i \geq w-1$, then

$$B_{w,\tau^s}(q) \sim_M B_{w,\tau^s}(q').$$

Here, each $[w : w-1]$-pair is a derived (spelendid) equivalence between two unipotent blocks. Moreover, the above Morita equivalence preserves natural indices (partitions) of modules. (i.e. The simple module $D_{\mathbf{k},q}(\mu)$ (resp. the "Specht"like module $S_{\mathbf{k},q}(\mu)$, the Young module $X_q(\mu)$, PIM $P_q(\mu)$) indexed by a partition μ corresponds to $D_{\mathbf{k},q'}(\mu)$ (resp. $S_{\mathbf{k},q'}(\mu), X_{q'}(\mu)$, $P_{q'}(\mu)$).)

Remark 5. *Just mimicking an argument in [7], constructing a generalization of [38] and using Theorem 7.1, we deduce the above results. (see also [32]).*

8 A conjecture

8.1 The theory of Lascoux-Leclerc-Thibon

Let v be an indeterminate over $\mathbb{Q}$. Let $U_v(\widehat{\mathfrak{sl}_e})$ be the quantized enveloping algebra over $\mathbb{Q}(v)$ corresponding to the Dynkin diagram $A^{(1)}_{e-1}$.

The so-called "Fock space"

$$\mathcal{F}_v = \bigoplus_{\lambda} \mathbb{Q}(v)|\lambda\rangle$$

is the $\mathbb{Q}(v)$-vector space with basis $|\lambda\rangle$ indexed by the set of all partitions. In [24] Lascoux, Leclerc and Thibon introduced an algorithm to compute the *canonical basis* of the basis representation $M_v(\Lambda_0)$ and conjectured that it also compute the decomposition matrix of the Iwahori-Hecke algebras of type **A** at a root of unity over $\mathbb{C}$. This is so called the LLT conjeture.

The LLT conjecture is now a theorem (see, for example, [29, Chap. 6] and the references of Chapter 6).

In [25], Leclerc and Thibon define a canonical basis of the v-deformed Fock space representation $\mathcal{F}_v$ of the affine Lie algebra $\widehat{\mathfrak{gl}}_e$. They conjectured that the entries of the transition matrix between these basis and $\{|\lambda\rangle\}_\lambda$ are also crystalized decomposition numbers of the Dipper-James' Schur algebra for ζ specialized at a primitive e-th root of unity.

This LT conjecture is now a theorem over $\mathbb{Q}(\zeta)$ where ζ is a primitive e-th root of unity over $\mathbb{C}$, due to Varagnolo and Vasserot [39].

Leclerc and Thibon showd that

Theorem 8.1 (Leclerc-Thibon). *There exist bases $\{G(\lambda)\}$ and $\{G^-(\lambda)\}$ of $\mathcal{F}_v$ characterized by :*

1. $\overline{G(\lambda)} = G(\lambda)$, $\overline{G^-(\lambda)} = G^-(\lambda)$
2. $G(\lambda) \equiv |\lambda\rangle \bmod vL$, $\quad G^-(\lambda) \equiv |\lambda\rangle \bmod v^{-1}L^-$.

Here, L (resp. L^-) denotes the $\mathbb{Z}[v]$ (resp. $\mathbb{Z}[v^{-1}]$)-lattice in $\mathcal{F}_v$ with basis $\{|\lambda\rangle\}$.

Let

$$G(\mu) = \sum_\lambda d_{\lambda,\mu}(v)|\lambda\rangle, \quad G^-(\lambda) = \sum_\lambda e_{\lambda,\mu}(v)|\mu\rangle.$$

and $\mathbf{D}_{m,e,0}(v) = [d_{\lambda,\mu}(v)]_{\lambda,\mu\vdash m}$.

Theorem 8.2 (Leclerc-Thibon, Varagnolo-Vasserot). *The matrix $\mathbf{D}_{m,e,0}(1)$ is equal to the decomposition matrix $\mathbf{D}_{m,\zeta,0}$.*

Remark

If both ζ and ζ' are primitive e-th roots of unity in $\mathbb{C}$, then $\mathbf{D}_{m,\zeta,0} = \mathbf{D}_{m,\zeta',0}$. So, we may write $\mathbf{D}_{m,e,0}$ instead of $\mathbf{D}_{m,\zeta,0}$.

8.2 Our hope

Let $l_{\mathbf{k},q}(\lambda)$ be the Loewy length of $S_{\mathbf{k},q}(\lambda)$. For $\ell > w$, we define $rad_{\lambda,\mu}(v) \in \mathbb{N}[v]$ as follows:

$$rad_{\lambda,\mu}(v) = \sum_{k=0}^{l_{\mathbf{k},q}(\lambda)-1} [\mathrm{Rad}^k(U_{\mathbf{k},q}(\overline{\lambda}))/\mathrm{Rad}^{k+1}(U_{\mathbf{k},q}(\overline{\lambda})) : V_{\mathbf{k},q}(\overline{\mu})]v^k$$

Remark 6. *Note that $rad_{\lambda,\mu}(v)$ is given explicitly by some products of Littlewood-Richardson coefficients and v^i. Moreover, an explicit formula for $rad_{\lambda,\mu}(v)$ will be written in [30].*

By the construction of the Loewy series of $S_{\mathbf{k},q}(\lambda)$ for $\lambda \in \Lambda_{e,w,\rho}$, we also know

$$\mathrm{Rad}^i(S_{\mathbf{k},q}(\lambda))/\,\mathrm{Rad}^{i+1}(S_{\mathbf{k},q}(\lambda)) \cong \mathrm{Soc}^{l_{\mathbf{k}}(\lambda)-i}(S_{\mathbf{k},q}(\lambda))/\,\mathrm{Soc}^{l_{\mathbf{k}}(\lambda)-i-1}(S_{\mathbf{k},q}(\lambda)).$$

Theorem 8.3 (Geck). [*16*] *There exists a square lower unitriangular matrix $\mathbf{A}$ such that each entry of $\mathbf{A}$ is non-negative and $\mathbf{D}_{m,\bar{q},\ell} = \mathbf{D}_{m,e,0} \cdot \mathbf{A}$.*

By the above theorem and Theorem 7.1, we deduce

Corollary 8.4. *If $rad_{\lambda,\mu}(v) = 0$, then $d_{\lambda,\mu}(v) = 0$ for any $\lambda, \mu \in \Lambda_{e,w,\rho}$.*

Not only do we want to show that James' conjecture is true, but we want to know an explicit formula for $d_{\lambda,\mu}(v)$ which is now known to be a certain parabolic Kazhdan-Lusztig polynomial.

According to James conjecture and Rouquier-Leclerc-Thibon conjecture [29, 6.33 (see also 6.27)] on Jantzen filtrations over $\mathbb{C}$, we hope the following:

Conjecture 8.5. $d_{\lambda,\mu}(v) = rad_{\lambda,\mu}(v)$ *for any* $\lambda, \mu \in \Lambda_{e,w,\rho}$.

The first announcement of this was stated in the author's lecture "On the unipotent blocks of finite general linear groups" at a conference " Algébres de Hecke affines et groups réductifs (CIRM,Luminy,16-20 octobre 2000)" organized by M. Geck and R. Rouquier.

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