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On the projection which appears in the variational treatment of elasto-plastic torsion problem

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Abstract

In the treatment of variational inequalities, the projection operator P_K from some Hilbert space V onto a certain closed convex subset K plays an important role. But, only for few problems, it is known how to get the explicit form of $P_K u$ for each given $u \in V$. In this article, we consider $K = \{f \in H_0^1(\Omega);\ |\nabla f| \le 1 \text{ a.e.}\}$, which is related to elasto-plastic torsion problems, and propose an iterative method to approximate $P_K u$ for 1 dimensional case $\Omega = (a, b)$. We also show an expansion of it for higher dimensional but radial symmetric cases.

1 Problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary and

$$K := \left\{ f \in H_0^1(\Omega);\ |\nabla f| \le 1 \ \text{ a.e.} \right\}.$$

We will denote by P_K the projection mapping from $H_0^1(\Omega)$ into its convex closed subset K, namely, for $u \in H_0^1(\Omega)$ and $v \in K$,

$$P_K u = v \overset{\text{def}}{\iff} \|u - v\|_{H_0^1(\Omega)} = \inf_{f \in K} \|u - f\|_{H_0^1(\Omega)}.$$

For convenience sake, we take

$$\|u\|_{H_0^1(\Omega)} := \|\nabla u\|_{L^2(\Omega)} = \left\{ \int_\Omega |\nabla u(x)|^2 \, dx \right\}^{1/2},$$

throughout this article. (Note that Ω is bounded.) The problem is to find $v = P_K u \in K$ for each given $u \in H_0^1(\Omega)$.

This projection P_K appears in the variational treatment of elasto-plastic torsion problem. Consider an infinitely long cylindrical elastic-plastic bar of

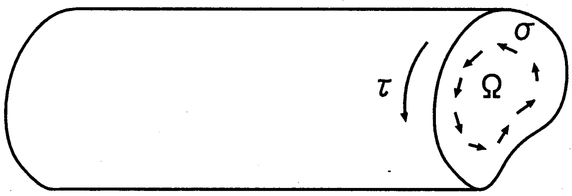

Figure 1: cylindrical elastic-plastic bar of cross section Ω.

cross section Ω to which some torsion momentum (τ denotes the torsion angle per unit length) is applied (Fig. 1). It is known that the stress vector σ in Ω is determined by the minimizer u of

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \tau\int_\Omega v\,dx \qquad (v \in K),$$

namely, $\sigma = \nabla u$ [2, p.42]. This minimizing problem is equivalent to finding $u \in K$ such that

$$u = P_K(u - \rho(Au - l)) \quad \text{for some } \rho > 0,$$

where $A \in \mathcal{L}(V, V)$ and $l \in V$ are defined by

$$\begin{aligned}(Af, g) &= \frac{1}{2}\int_\Omega \nabla f \cdot \nabla g\,dx, \\ (l, f) &= \tau\int_\Omega f\,dx \qquad \Big((\cdot,\cdot) : \text{inner product of } V\Big)\end{aligned}$$

for $f, g \in V := H_0^1(\Omega)$, respectively [2, p.3].

The projection P_K also plays an important role in the error estimates of the corresponding penalized elliptic variational inequalities [5].

2 Rewriting the problem

We introduce a functional $J_u : K \to \mathbb{R}$ for each given $u \in H_0^1(\Omega)$:

$$J_u(f) := \|u - f\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u(x) - \nabla f(x)|^2\,dx. \tag{1}$$

By using it, the problem can be rewritten such as "To find the minimizer v of J_u on K." On this problem, one can easily show:

Proposition 1 *If there exists a solution* $v \in H_0^1(\Omega)$ *to*

$$\nabla v = C(\nabla u) \qquad (a.e.\ in\ \Omega), \tag{2}$$

then v *is the minimizer of* J_u *on* K, *where* $C(z) := \begin{cases} z & (|z| \le 1), \\ z/|z| & (|z| > 1). \end{cases}$

Especially, for 1 dimensional case $\Omega = (a,b) \subset \mathbb{R}$, put

$$v(x) := \int_a^x C(u'(\xi))\, d\xi \qquad (a \le x \le b) \tag{3}$$

for a given function $u \in H_0^1(a,b)$. If this function v ($\in H^1(a,b) \cap C([a,b])$) satisfies that $v(b) = 0$, then v belongs to $H_0^1(a,b)$ and hence $v = P_K u$. An example of this kind: $u(x) = -\frac{3}{10}\cos(\frac{3}{2}\pi x)$ and v defined by (3) for $\Omega = (-1,1)$ are shown in Fig. 2. We also plot their derivatives in Fig. 3. In this case, $P_K u$ and v coincide perfectly (see Fig. 2), and $(P_K u)'$ is only the "cut-off" of u', namely, $(P_K u)' = C(u')$ (see Fig. 3).

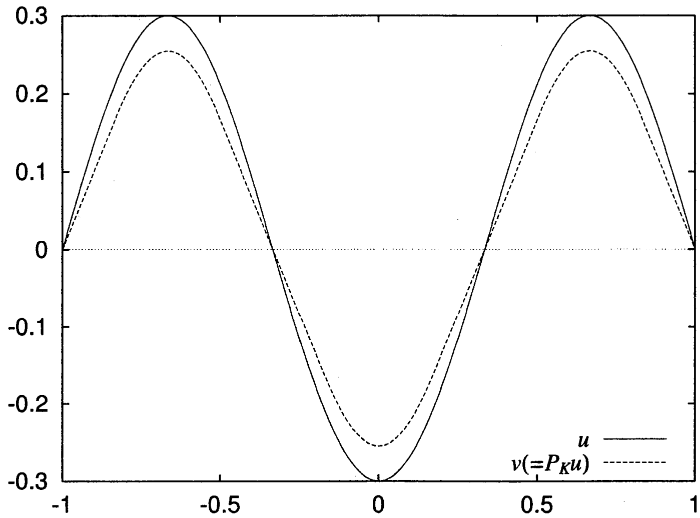

Figure 2: the case $v(b) = 0$; $u(x) = -\frac{3}{10}\cos\frac{3}{2}\pi x$.

In fact, for 1 dimensional case $\Omega = (a,b)$, one can easily show that if the given function u is symmetric (i.e., $u(a+\xi) = u(b-\xi)$ for any ξ), then v defined by (3) satisfies that $v(b) = 0$ and hence $v = P_K u$.

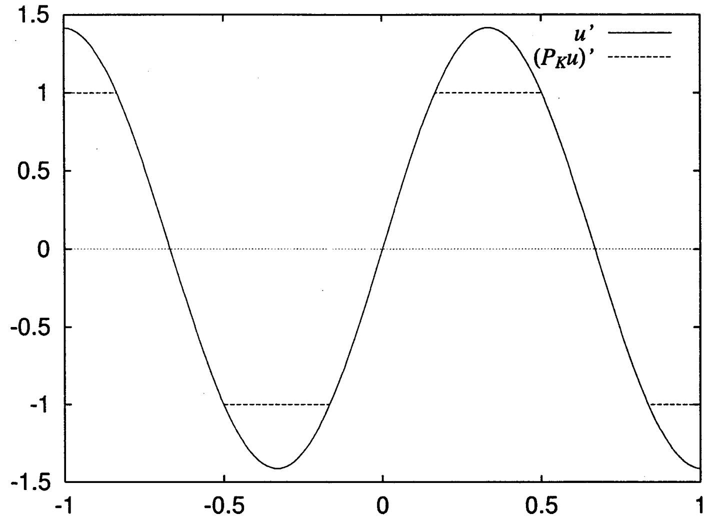

Figure 3: u' and $(P_K u)'$; $u(x) = -\frac{3}{10}\cos\frac{3}{2}\pi x$.

But it is rather special. We will show an example for the case $v(b) \neq 0$: $u(x) = 4(x+1)^2(x+\frac{1}{2})(x-\frac{1}{5})(x-\frac{3}{5})(x-\frac{4}{5})(x-1)$ for $\Omega = (-1, 1)$. The graphs of u, corresponding v and $P_K u$ are shown in Fig. 4. Also the derivatives u' and $(P_K u)'$ are plotted in Fig. 5.

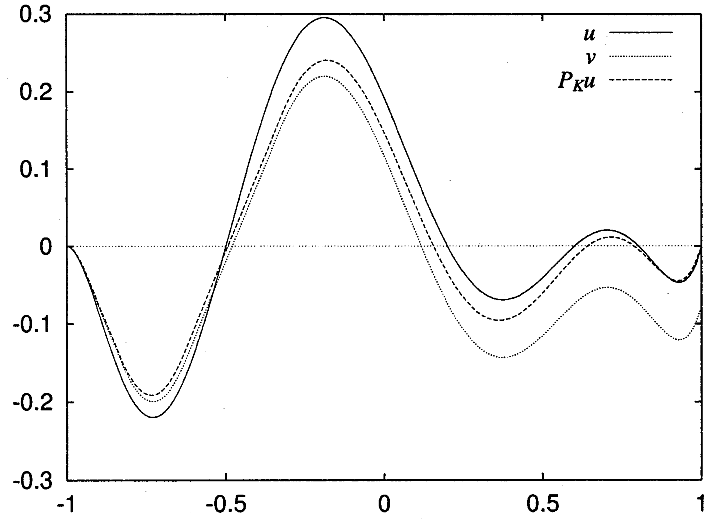

Figure 4: the case $v(b) \neq 0$; $u(x) = 4(x+1)^2(x+\frac{1}{2})(x-\frac{1}{5})(x-\frac{3}{5})(x-\frac{4}{5})(x-1)$.

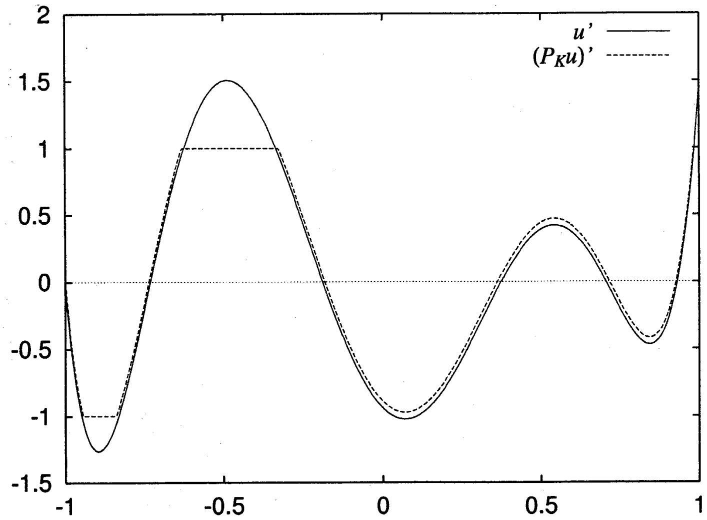

Figure 5: u' and $(P_K u)'$; $u(x) = 4(x+1)^2(x+\frac{1}{2})(x-\frac{1}{5})(x-\frac{3}{5})(x-\frac{4}{5})(x-1)$.

In such a case, it is clear that any primitive function of $C(u')$ can not belong to $H_0^1(\Omega)$ since its values at 2 boundary points are not equal. In other words, (2) has no solution in $H_0^1(\Omega)$, in general.

Then, instead of (2), we consider the following system of equations:

$$\begin{cases} \nabla v = C(\nabla u - \nabla w) & \text{(a.e. in } \Omega), \\ \Delta w = 0 & \text{(weak sense).} \end{cases} \tag{4}$$

It means that at first, we alter u by subtracting the appropriate quantity, namely, a function $w \in H^1(\Omega)$ satisfying $\Delta w = 0$. Then we "cut-off" its gradient and get the primitive function. If the obtained function v belongs to $H_0^1(\Omega)$, then the next theorem assures that $v = P_K u$.

Theorem 1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. If there exists a solution (v, w) in $H_0^1(\Omega) \times H^1(\Omega)$ to the system of equations (4) with a given parameter $u \in H_0^1(\Omega)$, then v belongs to K and minimizes the functional J_u defined by (1).*

(Proof) It is clear that $v \in K$. Hence, it suffices to show that

$$\forall f \in K, \quad J_u(f) - J_u(v) \geq 0.$$

Let denote $\Omega_p := \{x \in \Omega;\ |\nabla(u - w)| > 1\}$ and $\Omega_z := \Omega \backslash \Omega_p$. Fix $f \in K$ and put $\delta := f - v \in H_0^1(\Omega)$. For this δ, we can easily show

$$\nabla\delta \cdot \nabla v = \nabla f \cdot \nabla v - |\nabla v|^2 = \nabla f \cdot \nabla v - 1 \leq 0 \qquad \text{(a.e. in } \Omega_p)$$

since $|\nabla f| \le 1$ and $|\nabla v| = 1$ (a.e. in Ω_p), and hence

$$\nabla\delta \cdot (\nabla u - \nabla w) \le 0 \qquad \text{(a.e. in } \Omega_p).$$

On the other hand, since $\triangle w = 0$ (weak sense in $H^1(\Omega)$) and $\delta \in H_0^1(\Omega)$,

$$\int_\Omega \nabla\delta \cdot \nabla w \, dx = \int_{\Omega_p} \nabla\delta \cdot \nabla w \, dx + \int_{\Omega_z} \nabla\delta \cdot \nabla w \, dx = 0.$$

By using these facts, we get

$$\begin{aligned}
J_u(f) - J_u(v) &= \int_\Omega |\nabla u - \nabla(v+\delta)|^2 \, dx - \int_\Omega |\nabla u - \nabla v|^2 \, dx \\
&= \int_\Omega |\nabla\delta|^2 \, dx - 2\int_\Omega \nabla\delta \cdot (\nabla u - \nabla v) \, dx \\
&= \int_\Omega |\nabla\delta|^2 \, dx - 2\int_{\Omega_p} \nabla\delta \cdot \left(\nabla u - \frac{\nabla u - \nabla w}{|\nabla u - \nabla w|}\right) dx - 2\int_{\Omega_z} \nabla\delta \cdot \nabla w \, dx \\
&= \int_\Omega |\nabla\delta|^2 \, dx + 2\int_{\Omega_p} \nabla\delta \cdot \left(\frac{\nabla u - \nabla w}{|\nabla u - \nabla w|} - \nabla u\right) dx + 2\int_{\Omega_p} \nabla\delta \cdot \nabla w \, dx \\
&= \int_\Omega |\nabla\delta|^2 \, dx + 2\int_{\Omega_p} \left(|\nabla u - \nabla w|^{-1} - 1\right) \nabla\delta \cdot (\nabla u - \nabla w) \, dx \\
&\ge \int_\Omega |\nabla\delta|^2 \, dx \ \ge \ 0.
\end{aligned}$$

□

3 1 dimensional case

Theorem 1 assures that if one could solve the system of equations (4) with a given parameter $u \in H_0^1(\Omega)$, one get the projection $P_K u$. But unfortunately, there may not be any solution to (4) in general, except 1 dimensional case. In fact, when $\Omega = (a,b) \subset \mathbb{R}^1$ $(-\infty < a < b < \infty)$, the equation $w'' = 0$ can be solved such as $w' \equiv$ const. a.e. in (a,b). Hence it is sufficient to solve

$$v' = C(u' - \alpha) \qquad \text{(a.e. in } \Omega) \tag{5}$$

for $v \in H_0^1(a,b)$ and $\alpha \in \mathbb{R}$ instead of (4). And we got an iterative solution to (5), namely, an algorithm to produce the sequences $\{v_k\} \subset H^1(a,b)$ and $\{\alpha_k\} \subset \mathbb{R}$ which approximate v and α, respectively.

Algorithm I *Put* $\alpha_0 := 0$ *and iterate the followings on* $k = 0, 1, 2, \cdots$.

1. *Define* $v_k \in H^1(a,b) \cap C([a,b])$ *by using* α_k *such as*

$$v_k(x) := \int_a^x C(u'(\xi) - \alpha_k)\, d\xi \qquad (a \leq x \leq b).$$

2. *Put* $\delta_k := \dfrac{v_k(b)}{b-a}$ *and* $\alpha_{k+1} := \alpha_k + \delta_k$.

When $v_k \to v$ in $H^1(a,b)$ and $\alpha_k \to \alpha$ in $\mathbb{R}$ as $k \to \infty$, one can expect $v(b) = 0$, i.e., $v \in H_0^1(a,b)$. If it holds, the pair of v and α solves to (4). In fact, these properties are assured by the following theorem.

Theorem 2 *For any* $u \in H_0^1(a,b)$*, each sequence* $\{\alpha_k\}$ *and* $\{v_k\}$ *in Algorithm I converges. Moreover, the limit function of* v_k *belongs to* $H_0^1(a,b)$.

Theorem 2 is the direct result of following 3 lemmas. At first, we will prove the convergence of $\{\alpha_k\}$ by showing the monotonicity and the boundedness of it.

Lemma 1 (monotonicity) *In Algorithm I, if*

$$\alpha_1 = \delta_0 := \frac{1}{b-a}\int_a^b C(u'(\xi))\, d\xi > 0,$$

then the sequence $\{\delta_k\}$ *satisfies that* $0 \leq \delta_{k+1} \leq \delta_k$ $(k = 0, 1, 2, \cdots)$.

(Proof) Fix $k \in \{0, 1, 2, \cdots\}$ and assume $\delta_k \geq 0$. Let denote

$$\Omega_p(f) := \{x \in \Omega;\ f(x) > 1\}, \qquad \Omega_n(f) := \{x \in \Omega;\ f(x) < -1\},$$
$$\Omega_z(f) := \Omega \backslash (\Omega_p(f) \cup \Omega_n(f)),$$

where $\Omega = (a,b)$, and define Ω_{ij} by

$$\Omega_{ij} \ := \ \Omega_i(u' - \alpha_{k+1}) \cap \Omega_j(u' - \alpha_k) \qquad (i, j \in \{p, z, n\}).$$

For brevity, we will use the notations

$$|\Omega_{ij}| := \int_{\Omega_{ij}} dx \quad \text{and} \quad \omega_{ij} := \frac{|\Omega_{ij}|}{|\Omega|} = \frac{1}{|\Omega|}\int_{\Omega_{ij}} dx \qquad (i, j \in \{p, z, n\}).$$

Note that

$$|\Omega| := b - a = \sum_{i,j} |\Omega_{ij}| \quad \text{and} \quad \sum_{i,j} \omega_{ij} = 1 \qquad (i, j \in \{p, z, n\}),$$

and $|\Omega_{pz}| = |\Omega_{pn}| = |\Omega_{zn}| = 0$ since $\alpha_{k+1} = \alpha_k + \delta_k \geq \alpha_k$. By using them, we can write

$$\begin{aligned}
\delta_{k+1} - \delta_k &= \frac{1}{|\Omega|} \sum_{i,j} \int_{\Omega_{ij}} \{C(u' - \alpha_{k+1}) - C(u' - \alpha_k)\}\, dx \\
&= \frac{1}{|\Omega|} \left\{ \int_{\Omega_{zp}} (u' - \alpha_{k+1} - 1)\, dx + \int_{\Omega_{zz}} (-\alpha_{k+1} + \alpha_k)\, dx \right. \\
&\qquad \left. + \int_{\Omega_{np}} (-2)\, dx + \int_{\Omega_{nz}} (-1 - u' + \alpha_k)\, dx \right\} \\
&= \frac{1}{|\Omega|} \int_{\Omega_{zp}} (u' - \alpha_{k+1} - 1)\, dx - \omega_{zz}\delta_k - 2\omega_{np} + \frac{1}{|\Omega|} \int_{\Omega_{nz}} (-1 - u' + \alpha_k)\, dx.
\end{aligned}$$

From the definition of Ω_{zp} and Ω_{nz}, we obtain the following evaluations:

$$\begin{aligned}
-\min\{2, \delta_k\} &\leq u'(x) - \alpha_{k+1} - 1 \leq 0 \qquad (\text{a.e. } x \text{ in } \Omega_{zp}), \\
-\min\{2, \delta_k\} &\leq -1 - u'(x) + \alpha_k \leq 0 \qquad (\text{a.e. } x \text{ in } \Omega_{nz}).
\end{aligned}$$

By the estimates from above, we get the monotone decreasingness of $\{\delta_k\}$:

$$\delta_{k+1} \leq (1 - \omega_{zz})\delta_k - 2\omega_{np} \leq \delta_k.$$

Next, we will show the non-negativeness of $\{\delta_k\}$. By the estimates from below, we get

$$\delta_{k+1} \geq -\min\{2, \delta_k\}\omega_{zp} + (1 - \omega_{zz})\delta_k - 2\omega_{np} - \min\{2, \delta_k\}\omega_{nz}.$$

When $\delta_k \geq 2$, we can deduce from this estimate

$$\delta_{k+1} \geq 2(-\omega_{zp} + 1 - \omega_{zz} - \omega_{np} - \omega_{nz}) \geq 0.$$

In the other hand, when $\delta_k < 2$, we can easily show that $\omega_{np} = 0$, and hence

$$\delta_{k+1} \geq \delta_k(-\omega_{zp} + 1 - \omega_{zz} - \omega_{nz}) \geq 0.$$

□

One can get similar result as Lemma 1 for the case $\delta_0 < 0$.

Corollary 2 *In Algorithm I, if*

$$\alpha_1 = \delta_0 := \frac{1}{b-a}\int_a^b C(u'(\xi))\,d\xi < 0,$$

then the sequence $\{\delta_k\}$ *satisfies that* $0 \geq \delta_{k+1} \geq \delta_k$ $(k = 0, 1, 2, \cdots)$.

It is obvious that $\delta_k = 0$ implies $\delta_{k'} = 0$ for all $k' \in \{k, k+1, k+2, \cdots\}$. Since $\alpha_k = \sum_{j=0}^{k-1} \delta_j$, it is easy to look that $\{\alpha_k\}$ is also monotone and that the sign of α_k is "same" as that of δ_k in the sense considering the sign of 0 to belong to both of plus and minus one. Hence, we get the following.

Corollary 3 *For the sequences* $\{\delta_k\}$ *and* $\{\alpha_k\}$ *generated by Algorithm I, it holds that*

$$\alpha_k > 0 \Rightarrow \delta_k \geq 0 \qquad \textit{and} \qquad \alpha_k < 0 \Rightarrow \delta_k \leq 0 \qquad (k = 0, 1, 2, \cdots).$$

We use this property in the proof of Lemma 4.

Lemma 4 (boundedness) *In Algorithm I, the sequence* $\{\alpha_k\}$ *is bounded such as*

$$|\alpha_k| \leq \left(\frac{2}{b-a}\right)^{1/2} \|u\|_{H_0^1(a,b)} + 1 \qquad (k = 0, 1, 2, \cdots).$$

(Proof) When $u = 0$ in $H_0^1(a,b)$, it is clear that $\alpha_k = 0$ for any $k \in \{0, 1, 2, \cdots\}$. Then, we take $u \neq 0$, namely, $\|u\|_{H_0^1(a,b)} = \|u'\|_{L^2(a,b)} > 0$. And we will show only for the case $\alpha_k > 0$ here. Almost the same proof works for the case $\alpha_k < 0$.

For each fixed $\varepsilon > 0$, assume that

$$\exists k \in \mathbb{N} \quad \text{s.t.} \quad \alpha_k \geq \left(\frac{2+\varepsilon}{b-a}\right)^{1/2} \|u\|_{H_0^1(a,b)} + 1. \qquad (*)$$

Note that $\delta_k \geq 0$ since $\alpha_k > 0$. Putting

$$\Omega_1 := \{x \in \Omega;\ u'(x) - \alpha_k \geq -1\}, \quad \Omega_2 := (a,b)\backslash\Omega_1,$$

we get the inequality

$$\begin{aligned}(b-a)\delta_k &= \int_{\Omega_1} C(u'(\xi) - \alpha_k)\,d\xi + \int_{\Omega_2} C(u'(\xi) - \alpha_k)\,d\xi \\ &\leq \int_{\Omega_1} |C(u'(\xi) - \alpha_k)|\,d\xi - \int_{\Omega_2} d\xi \ \leq\ |\Omega_1| - |\Omega_2|,\end{aligned} \qquad (\dagger)$$

where $|\Omega_i| := \int_{\Omega_i} dx$. Since $|\Omega_2| = (b-a) - |\Omega_1|$, $|\Omega_1| = 0$ implies that $\delta_k < 0$ which contradicts to the assumption $(*)$. Then, we assume $|\Omega_1| > 0$ hereafter. By using $(*)$ and the definition of Ω_1, we can easily show that

$$\xi \in \Omega_1 \;\Rightarrow\; |u'(\xi)|^2 \geq (\alpha_k - 1)^2 \geq \frac{2+\varepsilon}{b-a} \|u'\|^2_{L^2(a,b)}.$$

Hence, it follows that

$$\|u'\|^2_{L^2(a,b)} \geq \int_{\Omega_1} |u'(\xi)|^2 \, d\xi \geq \frac{2+\varepsilon}{b-a} \|u'\|^2_{L^2(a,b)} |\Omega_1|,$$

and then,

$$|\Omega_2| - |\Omega_1| \geq \varepsilon |\Omega_1|.$$

This and (†) lead that $\delta_k < 0$ which contradicts to $(*)$. □

Lemma 1 (Corollary 2) and Lemma 4 show the convergence of $\{\alpha_k\}$ generated by Algorithm I. Then, we will show the convergence of $\{v_k\}$ in $H^1(a,b)$.

Lemma 5 *For $\{\alpha_k\}$ and $\{v_k\}$ generated by Algorithm I, denoting*

$$\alpha := \lim_{k\to\infty} \alpha_k \quad \textit{and} \quad v(x) := \int_a^x C\left(u'(\xi) - \alpha\right) d\xi \quad (a \leq x \leq b),$$

it holds that $v_k \to v$ $(k \to \infty)$ in $H^1(a,b)$ and $v \in H^1_0(a,b)$.

(Proof) It is easy to see that

$$\forall z_1, z_2 \in \mathbb{R}, \quad |C(z_1) - C(z_2)| \leq |z_1 - z_2|.$$

By using this property and the definitions of v and v_k, we get

$$|v'(x) - v_k'(x)| = |C(u'(x) - \alpha) - C(u'(x) - \alpha_k)| \leq |\alpha - \alpha_k| \quad \text{(a.e. in } \Omega).$$

Therefore, we obtain

$$\begin{aligned}
\|v - v_k\|^2_{H^1(\Omega)} &:= \int_a^b |v(x) - v_k(x)|^2 \, dx + \int_a^b |v'(x) - v_k'(x)|^2 \, dx \\
&= \int_a^b \left| \int_a^x (v'(\xi) - v_k'(\xi)) \, d\xi \right|^2 dx + \int_a^b |v'(x) - v_k'(x)|^2 \, dx \\
&\leq \int_a^b |\alpha - \alpha_k|^2 (x-a)^2 \, dx + \int_a^b |\alpha - \alpha_k|^2 \, dx \\
&= |\alpha - \alpha_k|^2 \left(\tfrac{1}{3}(b-a)^3 + (b-a) \right),
\end{aligned}$$

and then the convergence $v_k \to v$ in $H^1(a,b)$. Furthermore, since

$$\begin{aligned} |v(b) - v_k(b)| &= \left| \int_a^b (v'(x) - v_k'(x))\, dx \right| \\ &\le \int_a^b |v'(x) - v_k'(x)|\, dx \le |\alpha - \alpha_k|\,(b-a), \end{aligned}$$

it holds that $v_k(b) \to v(b)$ $(k \to \infty)$. In the other hand,

$$v_k(b) = \delta_k(b-a) = (\alpha_{k+1} - \alpha_k)(b-a)$$

implies $v_k(b) \to 0$, hence we get $v(b) = 0$, namely, $v \in H_0^1(a,b)$. □

4 Radial symmetric case

For higher dimensional cases, the system of equations (4) may not have any solution, in general. But, when both of domain Ω and given function u are radial symmetric, the problem is reducible to 1 dimensional one, and can be solved. In this section, we consider that both Ω and u are radial symmetric.

At first, we mention about the most simple (trivial) case, namely, the domain Ω is spherical one:

$$\Omega = \{x \in \mathbb{R}^N;\ |x| < a\} \qquad \text{with} \quad 0 < a < \infty.$$

In this case, it is obvious that $v = P_K u$ can be obtained such as

$$v(x) := -\int_{|x|}^a C(\tilde{u}'(\rho))\, d\rho \quad (x \in \Omega),$$

where $\tilde{u} : \mathbb{R} \to \mathbb{R}$ is defined by $\tilde{u}(|x|) := u(x)$.

For more interesting case, we consider a ring domain:

$$\Omega = \{x \in \mathbb{R}^N;\ a < |x| < b\} \qquad \text{with} \quad 0 < a < b < \infty. \tag{6}$$

In this case, the system of equations (4) can be written as

$$\begin{cases} v_r = C(u_r - w_r) & \text{(a.e. in } \Omega), \\ w_{rr} + \dfrac{N-1}{r} w_r = 0 & \text{(weak sense)} \end{cases}$$

with $r := |x|$. Since the 2nd equation of this system is solvable such as

$$w_r(x) = \alpha r^{1-N} \qquad (\text{a.e. } x \in \Omega),$$

with arbitrary constant α, it suffices to solve

$$\tilde{v}'(r) = C\left(\tilde{u}'(r) - \alpha r^{1-N}\right) \qquad (\text{a.e. } r \in [a,b]) \tag{7}$$

for $\tilde{v} \in H_0^1(a,b)$ and $\alpha \in \mathbb{R}$. The equation (7) is similar to (5) and we can expand Algorithm I to solve it as followings.

Algorithm II *Put $\alpha_0 := 0$ and iterate the followings for $k = 0, 1, 2, \cdots$.*

1. *Define $v_k(x)$ by using α_k such as*

$$v_k(x) := \int_a^{|x|} C\left(\tilde{u}'(\rho) - \frac{\alpha_k}{\rho^{N-1}}\right) d\rho \qquad (x \in \Omega).$$

2. *Put $\delta_k := \dfrac{a^{N-1}}{b-a} \lim_{|x| \to b} v_k(x)$ and $\alpha_{k+1} := \alpha_k + \delta_k$.*

This algorithm is justified by the next theorem.

Theorem 3 *If Ω is a ring domain such as (6) and $u \in H_0^1(\Omega)$ is radial symmetric one, then each sequence of $\{\alpha_k\}$ and $\{v_k\}$ in Algorithm II converges.*

The sequence $\{\alpha_k\}$ generated by Algorithm II also has the monotonicity and the boundedness, and the convergence of $\{\alpha_k\}$ is direct result of them. Once the convergence of $\{\alpha_k\}$ was shown, one can also show the convergence of $\{v_k\}$. These lemmas written below prove Theorem 3.

Lemma 6 (monotonicity) *In Algorithm II, if*

$$\alpha_1 = \delta_0 := \frac{a^{N-1}}{b-a} \int_a^b C(\tilde{u}'(\rho))\, d\rho > 0,$$

then the sequence $\{\delta_k\}$ satisfies that $0 \le \delta_{k+1} \le \delta_k$ $(k = 0, 1, 2, \cdots)$.

Lemma 7 (boundedness) *In Algorithm II, the sequence $\{\alpha_k\}$ is bounded such as*

$$|\alpha_k| \le b^{N-1} \left(\frac{2}{b-a}\right)^{1/2} \|\tilde{u}'\|_{L^2(a,b)} + 1 \qquad (k = 0, 1, 2, \cdots).$$

Lemma 8 *For $\{\alpha_k\}$ and $\{v_k\}$ generated by Algorithm II, denoting*

$$\alpha := \lim_{k \to \infty} \alpha_k \quad \text{and} \quad v(x) := \int_a^{|x|} C\left(\tilde{u}'(\rho) - \frac{\alpha}{\rho^{N-1}}\right) d\rho \quad (x \in \Omega),$$

then it holds that $v_k \to v$ $(k \to \infty)$ in $H^1(\Omega)$ and $v \in H_0^1(\Omega)$.

The proofs of Lemma 6, 7 and 8 are done by almost same arguments as Lemma 1, 4 and 5, respectively, and we omit them here.

Finally, we will show an example of numerical result of Algorithm II. In Fig. 6, u and $P_K u$ defined in 2 dimensional ring domain Ω such as

$$u(x) = 4(|x|+1)^2(|x|+\frac{1}{2})(|x|-\frac{1}{5})(|x|-\frac{3}{5})(|x|-\frac{4}{5})(|x|-1),$$
$$\Omega = \{x \in \mathbb{R}^2;\ 0.5 \leq |x| \leq 2.5\},$$

are plotted as 3D graphs.

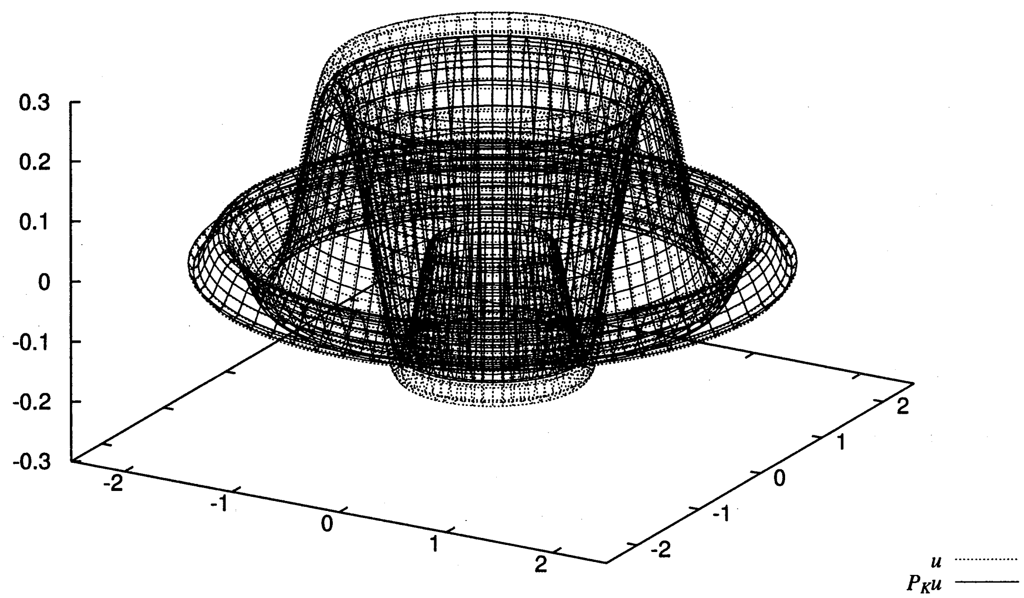

Figure 6: u and $P_K u$ for 2 dimensional ring domain case:
$u(r) = 4(r+1)^2(r+\frac{1}{2})(r-\frac{1}{5})(r-\frac{3}{5})(r-\frac{4}{5})(r-1)$.

In Fig. 7, the same u and $P_K u$ expressed above but for 1, 2 and 3 dimensional domains are plotted as r–u and r–$P_K u$ graphs. One may notice that the difference between the values of u and those of $P_K u$ is rather uniform in 1 dimensional case. But in a higher dimensional case, the difference between the values of u and those of $P_K u$ near the origin is larger than that of them far from the origin.

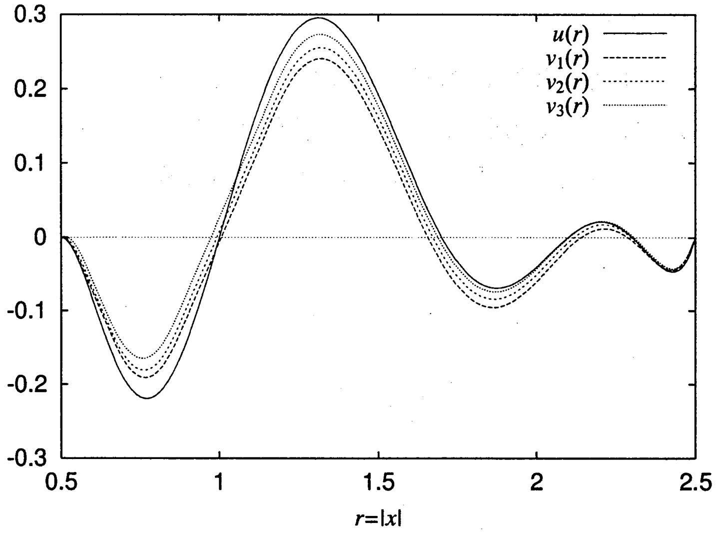

Figure 7: u and $P_K u$ for higher dimensional cases:
$u(r) = 4(r+1)^2(r+\frac{1}{2})(r-\frac{1}{5})(r-\frac{3}{5})(r-\frac{4}{5})(r-1)$;
v_n denotes $P_K u$ for n dimensional case.

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