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Exact WKB analysis of the harmonic oscillator and its Fourier transform

— An example of interplay between exact WKB analysis and Fourier analysis —

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1 Introduction

Although the exact WKB analysis was successful for the global analysis of solutions of second-order linear ordinary differential equations with a large parameter, its generalization to higher-order equations has not been accomplished yet: The local aspect of the theory such as the connection formula at a simple turning point is established in a satisfactory manner (cf. [AKT1]), while the global aspect is not fully understood; the biggest problem is to give a complete description of the Stokes geometry (i.e., geometry of Stokes curves) for higher-order equations, which is, in fact, really difficult due to the necessity of introducing "new Stokes curves" (cf. [BNR], [AKT1], [AKT2]).

In the case of ordinary differential equations of Laplace type, that is, equations whose coefficients are all linear functions, the Fourier-Laplace transformation gives us an integral representation of solutions and the so-called "steepest descent method" for the integral representation provides a useful tool for the determination of new Stokes curves. (Cf. [T2]. See also [U1], [U2]. Note that this point of view is closely related also to the theory of hyperasymptotics of integrals ([BH], [H] etc.).) We want to generalize such an approach via integral representations to equations with arbitrary polynomial coefficients, since the assumption "of Laplace type" is quite restrictive. In this note, as a starting point of our trial for the generalization, we discuss the well-known harmonic oscillators from this viewpoint.

2 Preliminaries — WKB solutions & their Borel transform

The equation we want to discuss in this note is the harmonic oscillator

$$(1) \qquad P\psi = \left(\frac{d^2}{dx^2} - \eta^2 x^2 + \eta\lambda\right)\psi = 0$$

with a parameter λ. Here and in what follows η denotes a large parameter. For (1) there exist the following formal (asymptotic) solutions called WKB solutions:

$$(2) \qquad \psi_\pm = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int^x S_{\mathrm{odd}} dx\right), \quad S_{\mathrm{odd}} = \eta x - \frac{\lambda}{2x} + \cdots .$$

In particular, let us denote by $\psi^\dagger_\pm$ the WKB solutions normalized in the following way:

$$(3) \qquad \begin{aligned} \psi^\dagger_\pm &= \frac{1}{\sqrt{S_{\mathrm{odd}}}} (\eta^{1/2} x)^{\mp\lambda/2} \exp \pm \left(\eta \frac{x^2}{2} + \int_\infty^x \left(S_{\mathrm{odd}} - \eta x + \frac{\lambda}{2x}\right) dx\right) \\ &= e^{\pm \eta x^2/2} \sum_{n=0}^\infty \frac{\psi_{\pm,n}}{x^{2n+(1\pm\lambda)/2}} \eta^{-\left(\frac{1}{2} \pm \frac{\lambda}{4} + n\right)}, \end{aligned}$$

where $\psi_{\pm,n}$ are constants independent of x and η.

As is well-known, WKB solutions does not converge. In the exact WKB analysis, to give an analytic meaning to them, we employ the Borel resummation technique. That is, we regard the Borel sum

$$(4) \qquad \Psi^\dagger_\pm = \int_{\mp x^2/2}^\infty \exp(-\eta y) \psi^\dagger_{\pm,B}(x,y) dy$$

as an analytic substitute of them. (In this note a formal series (WKB solution) is written by a small letter and its Borel sum is denoted by the corresponding capital letter.) Here the path of integration is assumed to be parallel to the positive real axis and $\psi^\dagger_{\pm,B}(x,y)$ denotes the Borel transform of $\psi^\dagger_\pm$ which is, by definition,

$$(5) \qquad \psi^\dagger_{\pm,B}(x,y) = \sum_{n=0}^\infty \frac{\psi_{\pm,n}}{x^{2n+(1\pm\lambda)/2} \Gamma(\frac{1}{2} \pm \frac{\lambda}{4} + n)} \left(y \pm \frac{x^2}{2}\right)^{-\frac{1}{2} \pm \frac{\lambda}{4} + n}.$$

The explicit form of $\psi^\dagger_{\pm,B}(x,y)$ is described in terms of Gauss' hypergeometric functions in the following way:

Lemma 1 *Letting s denote $y/x^2+1/2$, we have*

$$
(6)\qquad \begin{cases} \psi^{\dagger}_{+,B}(x,y) &= \dfrac{x^{-3/2}}{\Gamma(\frac{1}{2}+\frac{\lambda}{4})} s^{-1/2+\lambda/4} F(\dfrac{1}{4}+\dfrac{\lambda}{4},\dfrac{3}{4}+\dfrac{\lambda}{4},\dfrac{1}{2}+\dfrac{\lambda}{4};s), \\ \psi^{\dagger}_{-,B}(x,y) &= \dfrac{x^{-3/2}}{\Gamma(\frac{1}{2}-\frac{\lambda}{4})} (s-1)^{-1/2-\lambda/4} F(\dfrac{1}{4}-\dfrac{\lambda}{4},\dfrac{3}{4}-\dfrac{\lambda}{4},\dfrac{1}{2}-\dfrac{\lambda}{4};1-s). \end{cases}
$$

The expression (6) follows from the following two facts; (i) $\psi^{\dagger}_{\pm,B}(x,y)$ have particular homogeneity, i.e., $x^{3/2}\psi^{\dagger}_{\pm,B}(x,y)$ are functions of one variable y/x^2 (or, equivalently, of s), (ii) $\psi^{\dagger}_{\pm,B}(x,y)$ satisfy the Borel transform of equation (1), i.e., $(\partial^2/\partial x^2 - x^2\partial^2/\partial y^2 + \lambda\partial/\partial y)\psi^{\dagger}_{\pm,B}(x,y)=0$. For the details of discussion see [T1, p. 293].

Remark 1. In a similar manner we can compute the explicit form of the Borel transform of WKB solutions $\psi^{\ddagger}_{\pm} \overset{\text{def}}{=} \eta^{-1/2}\psi^{\dagger}_{\pm}$ as follows:

$$
(7)\qquad \begin{cases} \psi^{\ddagger}_{+,B}(x,y) &= \dfrac{x^{-1/2}}{\Gamma(1+\frac{\lambda}{4})} s^{\lambda/4} F(\dfrac{1}{4}+\dfrac{\lambda}{4},\dfrac{3}{4}+\dfrac{\lambda}{4},1+\dfrac{\lambda}{4};s), \\ \psi^{\ddagger}_{-,B}(x,y) &= \dfrac{x^{-1/2}}{\Gamma(1-\frac{\lambda}{4})} (s-1)^{-\lambda/4} F(\dfrac{1}{4}-\dfrac{\lambda}{4},\dfrac{3}{4}-\dfrac{\lambda}{4},1-\dfrac{\lambda}{4};1-s), \end{cases}
$$

where $s=y/x^2+1/2$. This formula will be used in §4 and §5.

In the case of equation (1) $x=0$ is a unique (double) turning point and $\Im x^2=0$ (i.e., the real and imaginary axes) describes the Stokes curves of (1). As a matter of fact, Lemma 1 implies that Borel sums of $\psi^{\dagger}_{\pm}$ are well-defined except on the real and imaginary axes and on each Stokes curve they satisfy the so-called connection formula (cf. [T1, Proposition 5]). In the exact WKB theoretic treatment of the harmonic oscillator Borel resummed WKB solutions thus defined play a central role. The main question we want to discuss in this note is:

> QUESTION: How are the Borel resummed WKB solutions transformed by Fourier-Laplace transformation with respect to the independent variable x?

3 Steepest descent method for the Fourier transform

The image of the harmonic oscillator (1) under the Fourier-Laplace transformation (with a large parameter)

$$
(8)\qquad \hat{\psi}(\xi) = \int \exp(-\eta x\xi)\psi(x)dx, \quad \psi(x) = \int \exp(\eta x\xi)\hat{\psi}(\xi)dx
$$

is again a harmonic oscillator in the ξ-variable

$$\hat{P}\hat{\psi} = -\left(\frac{d^2}{d\xi^2} - \eta^2\xi^2 - \eta\lambda\right)\hat{\psi} = 0. \tag{9}$$

Hence the inverse transform of WKB solutions of (9), in particular,

$$\int \exp(\eta x\xi)\hat{\psi}_{\pm}^{\dagger}(\xi,\eta)d\xi = \int \exp(\eta x\xi)\cdot\exp(\pm\eta\frac{\xi^2}{2})\cdot(\text{amplitude})\ d\xi, \tag{10}$$

becomes a solution of the harmonic oscillator (1) in the x-variable. What we want to clarify is the relationship between (10) and WKB solutions of (1).

If the amplitude part of the integrand were an ordinary function of ξ, (10) should be an integral containing a large parameter with the phase function

$$\varphi_{\pm} = x\xi \pm \frac{\xi^2}{2}, \tag{11}$$

and it should be possible to apply the so-called steepest descent method to obtain its asymptotic expansion for the large parameter η. Note that, in the case of the "integral representation" (10), the saddle points of $\varphi_{\pm}$ are $\xi = \mp x$ and the steepest (descent) paths of $\Re\varphi_{\pm}$ passing through $\xi = \mp x$ are given by $\Im(\xi \pm x)^2 = 0$ respectively (cf. Figure 1, where the steepest descent paths are drawn by thick lines).

It might thus be expected that the asymptotic expansion of the inverse Fourier-Laplace transform of WKB solutions of (9) defined by the integral (10) along one of such steepest descent paths should be an asymptotic solution of (1) and hence should become (possibly a linear combination of) WKB solutions of (1). However, the amplitude part of (10) is a formal power series of η^{-1} and, analytically speaking, it is necessary to consider its Borel sum instead of the formal power series expansion. In the subsequent sections we discuss the inverse Fourier-Laplace transform of the Borel sum of WKB solutions of (9) and compare it with the Borel resummed WKB solutions of the original equation (1).

4 The inverse transform of the Borel resummed WKB solutions (I) — local theory

In the preceding section we have observed that there are two saddle points $\xi = \mp x$ of the phase function $\varphi_{\pm}$. Since there is no essential difference between them, we only consider $\xi = x$, which is a saddle point of φ_{-}, in what follows. For the sake of specification we also assume that both $\Re x$ and $\Im x$ are positive (as is shown in Figure 1). Let us denote the steepest descent path passing through x by Γ. Then the object of our discussion is

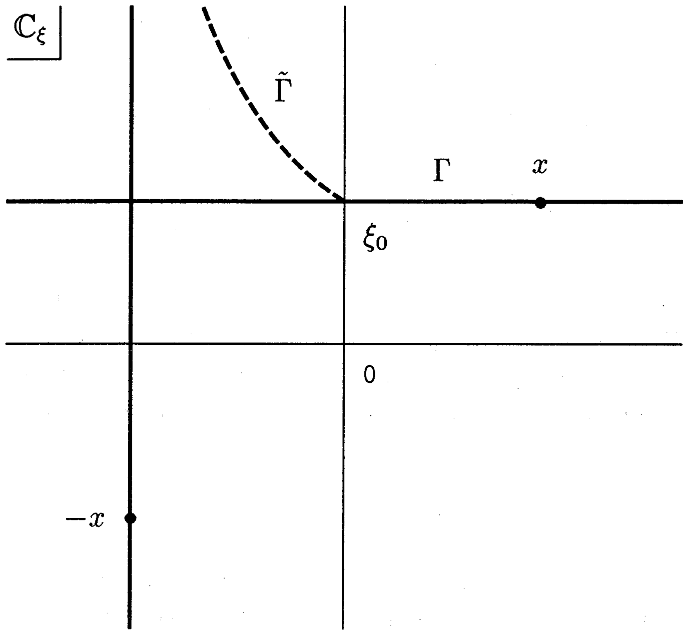

Figure 1: Saddle points and steepest descent paths.

the inverse Fourier-Laplace transform of the Borel resummed WKB solutions which is defined as follows:

$$\int_\Gamma \exp(\eta x\xi)\hat{\Psi}_-^\dagger(\xi,\eta)d\xi = \int_\Gamma \exp(\eta x\xi)\left(\int_{\xi^2/2}^\infty \exp(-\eta z)\hat{\psi}_{-,B}^\dagger(\xi,z)dz\right)d\xi. \tag{12}$$

Employing the following change of variables of integration

$$z - x\xi \longmapsto y, \quad \xi \longmapsto \xi, \tag{13}$$

we find that (12) is equal to

$$\int \exp(-\eta y)\left(\int \hat{\psi}_{-,B}^\dagger(\xi, y + x\xi)d\xi\right)dy. \tag{14}$$

Here let us specify the domain of integration of (14). If we introduce new variables of integration defined by

$$\xi = x + u \, (u \in \mathbb{R}), \qquad z = \frac{1}{2}\xi^2 + v \, (v \geq 0), \tag{15}$$

then the domain of integration of the original integral (12) is represented by

(16) $$\{(u,v) \in \mathbb{R}^2 | v \geq 0\}$$

in this variable (u,v) (cf. Figure 2). On the other hand, the variable y defined by (13)

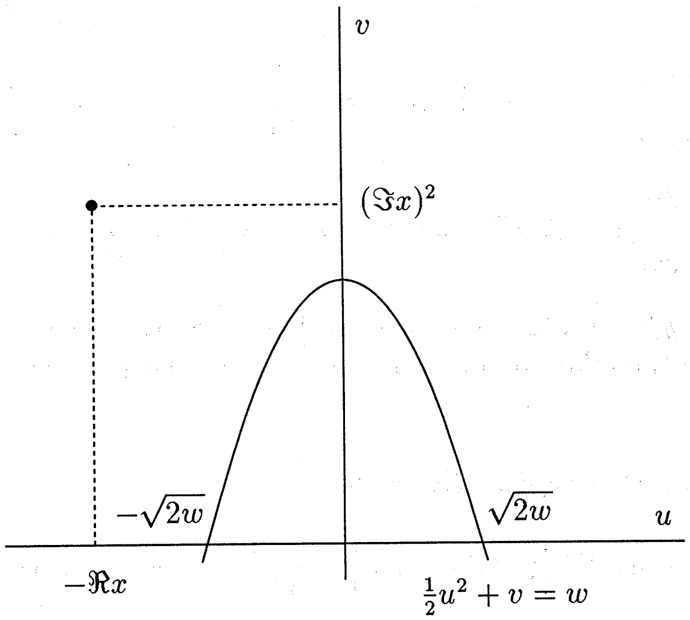

Figure 2: The domain of integration.

becomes

(17) $$y = z - x\xi = \frac{1}{2}(x+u)^2 + v - x(x+u) = -\frac{1}{2}x^2 + \frac{1}{2}u^2 + v.$$

Hence, denoting $u^2/2 + v$ by w, we obtain the following specific expression of (14):

(18) $$\int_{y=-\frac{1}{2}x^2+w, w\geq 0} \exp(-\eta y) \left(\int_{|\xi - x| \leq \sqrt{2w}} \hat{\psi}^{\dagger}_{-,B}(\xi, y + x\xi) d\xi \right) dy.$$

Note that, if it may be possible to view $\int_{|\xi-x|\leq\sqrt{2w}} \hat{\psi}^{\dagger}_{-,B}(\xi, y+x\xi)d\xi$ as a "Borel transform", the integral (18) is of the form of Borel resummed WKB solutions of equation (1). This, in fact, is true as we can prove the following:

Proposition 1 *If w is positive and sufficiently small, we have*

(19) $$\int_{|\xi-x|\leq\sqrt{2w}} \hat{\psi}^{\dagger}_{-,B}(\xi, y + x\xi) d\xi = \sqrt{2\pi} \psi^{\ddagger}_{+,B}(x,y).$$

Proof. It follows from Lemma 1 that the left-hand side of (19) is equal to

$$\frac{1}{\Gamma(\frac{1}{2}+\frac{\lambda}{4})}\int_{|\xi-x|\le\sqrt{2w}}\xi^{-2/3}$$
$$\times\left[(s-1)^{-\frac{1}{2}+\frac{\lambda}{4}}F(\frac{1}{4}+\frac{\lambda}{4},\frac{3}{4}+\frac{\lambda}{4},\frac{1}{2}+\frac{\lambda}{4};1-s)\right]\Bigg|_{s=\frac{y+x\xi}{\xi^2}+\frac{1}{2}}d\xi$$
$$=\frac{1}{\Gamma(\frac{1}{2}+\frac{\lambda}{4})}\int_{-\sqrt{2w}}^{\sqrt{2w}}(x+u)^{-2/3}\left[(-t)^{-\frac{1}{2}+\frac{\lambda}{4}}F(\frac{1}{4}+\frac{\lambda}{4},\frac{3}{4}+\frac{\lambda}{4},\frac{1}{2}+\frac{\lambda}{4};t)\right]\Bigg|_{t=\frac{u^2-2w}{2(x+u)^2}}du$$
$$=\frac{x^{-1/2}}{\Gamma(\frac{1}{2}+\frac{\lambda}{4})}\int_{-\sqrt{2a}}^{\sqrt{2a}}(1+u)^{-2/3}\left[(-t)^{-\frac{1}{2}+\frac{\lambda}{4}}F(\frac{1}{4}+\frac{\lambda}{4},\frac{3}{4}+\frac{\lambda}{4},\frac{1}{2}+\frac{\lambda}{4};t)\right]\Bigg|_{t=\frac{u^2-2a}{2(1+u)^2}}du,$$

where $a=w/x^2=y/x^2+1/2$. (We have used the scaling $u\longmapsto xu$ in obtaining the last formula.) Then the relation (19) is an immediate consequence of (7) and Lemma 2 below. Q.E.D.

Lemma 2 *When $a>0$ is sufficiently small,*

(20)
$$\int_{-\sqrt{2a}}^{\sqrt{2a}}(1+u)^{-2/3}\left[(-t)^{-\frac{1}{2}+\frac{\lambda}{4}}F(\frac{1}{4}+\frac{\lambda}{4},\frac{3}{4}+\frac{\lambda}{4},\frac{1}{2}+\frac{\lambda}{4};t)\right]\Bigg|_{t=\frac{u^2-2a}{2(1+u)^2}}du$$
$$=\sqrt{2\pi}\,\frac{\Gamma(\frac{1}{2}+\frac{\lambda}{4})}{\Gamma(1+\frac{\lambda}{4})}a^{\frac{\lambda}{4}}F(\frac{1}{4}+\frac{\lambda}{4},\frac{3}{4}+\frac{\lambda}{4},1+\frac{\lambda}{4};a).$$

Proof of Lemma 2. Let α, β and γ respectively denote $\frac{1}{4}+\frac{\lambda}{4}$, $\frac{3}{4}+\frac{\lambda}{4}$ and $\frac{1}{2}+\frac{\lambda}{4}$. Using the power series expansion of hypergeometric functions

(21)
$$F(\alpha,\beta,\gamma;t)=\sum_{j\ge0}\frac{(\alpha)_j(\beta)_j}{(\gamma)_j j!}t^j$$

(where $(\alpha)_j=\alpha(\alpha+1)\cdots(\alpha+j-1)=\Gamma(\alpha+j)/\Gamma(\alpha)$ etc.), we can rewrite the left-hand side of (20), denoted by L.H.S. in this proof, as

$$\text{L.H.S.}=\sum_{j\ge0}\frac{(\alpha)_j(\beta)_j}{(\gamma)_j j!}(-1)^j2^{\frac{1}{2}-\frac{\lambda}{4}-j}\int_{-\sqrt{2a}}^{\sqrt{2a}}(1+u)^{-(\frac{1}{2}+\frac{\lambda}{2}+2j)}(2a-u^2)^{-\frac{1}{2}+\frac{\lambda}{4}+j}du.$$

Note that the power series expansion (21) is uniformly convergent in the domain of integration if a is sufficiently small. Furthermore, expanding $(1+u)^{-(1/2+\lambda/2+2j)}$ into the binomial series (which is also uniformly convergent), we find that

$$\text{L.H.S.}=\sum_{j,k\ge0}\frac{(\alpha)_j(\beta)_j}{(\gamma)_j j!}\frac{(\frac{1}{2}+\frac{\lambda}{2}+2j)_k}{k!}(-1)^{j+k}2^{\frac{1}{2}-\frac{\lambda}{4}-j}H_{jk},$$

where

$$\begin{aligned}
H_{jk} &= \int_{-\sqrt{2a}}^{\sqrt{2a}} u^k(2a-u^2)^{-\frac{1}{2}+\frac{\lambda}{4}+j}du \\
&= (2a)^{\frac{\lambda}{4}+j+\frac{k}{2}} \int_{-1}^{1} u^k(1-u^2)^{-\frac{1}{2}+\frac{\lambda}{4}+j}du \\
&= \begin{cases} 0 & \text{(when } k \text{ is odd)}, \\ (2a)^{\frac{\lambda}{4}+j+\frac{k}{2}} B\left(\frac{1+k}{2}, \frac{1}{2}+\frac{\lambda}{4}+j\right) & \text{(when } k \text{ is even)}. \end{cases}
\end{aligned}$$

(Here $B(p,q)$ denotes the beta function.) We thus obtain

$$\text{L.H.S.} = \sum_{j,l\geq 0} \frac{(\alpha)_j(\beta)_j}{(\gamma)_j j!} \frac{(\frac{1}{2}+\frac{\lambda}{2}+2j)_{2l}}{(2l)!} (-1)^{j+2l} 2^{\frac{1}{2}+l} a^{\frac{\lambda}{4}+j+l} B\left(\frac{1}{2}+l, \frac{1}{2}+\frac{\lambda}{4}+j\right).$$

Let us recall here well-known formulas for the beta function and the Γ-function

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad \Gamma(2z) = \frac{2^{2z}}{2\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right). \tag{22}$$

Making use of these formulas, we can compute L.H.S. in the following way:

$$\begin{aligned}
\text{L.H.S.} &= \sum_{j,l\geq 0} \frac{\Gamma(\frac{1}{4}+\frac{\lambda}{4}+j)\Gamma(\frac{3}{4}+\frac{\lambda}{4}+j)\Gamma(\frac{1}{2}+\frac{\lambda}{4})\Gamma(\frac{1}{2}+\frac{\lambda}{2}+2j+2l)}{\Gamma(\frac{1}{4}+\frac{\lambda}{4})\Gamma(\frac{3}{4}+\frac{\lambda}{4})\Gamma(\frac{1}{2}+\frac{\lambda}{4}+j)\Gamma(\frac{1}{2}+\frac{\lambda}{2}+2j)} \\
&\quad \times \frac{1}{\Gamma(1+j)\Gamma(1+2l)} \frac{\Gamma(\frac{1}{2}+l)\Gamma(\frac{1}{2}+\frac{\lambda}{4}+j)}{\Gamma(1+\frac{\lambda}{4}+j+l)} (-1)^{j+2l} 2^{\frac{1}{2}+l} a^{\frac{\lambda}{4}+j+l} \\
&= \sqrt{2\pi} \sum_{j,l\geq 0} \frac{\Gamma(\frac{1}{4}+\frac{\lambda}{4}+j+l)\Gamma(\frac{3}{4}+\frac{\lambda}{4}+j+l)\Gamma(\frac{1}{2}+\frac{\lambda}{4})}{\Gamma(\frac{1}{4}+\frac{\lambda}{4})\Gamma(\frac{3}{4}+\frac{\lambda}{4})\Gamma(1+\frac{\lambda}{4}+j+l)} \frac{1}{j!l!} (-1)^{j+2l} 2^l a^{\frac{\lambda}{4}+j+l} \\
&= \sqrt{2\pi} \frac{\Gamma(\frac{1}{2}+\frac{\lambda}{4})}{\Gamma(1+\frac{\lambda}{4})} \sum_{n=0}^{\infty} \frac{(\frac{1}{4}+\frac{\lambda}{4})_n(\frac{3}{4}+\frac{\lambda}{4})_n}{(1+\frac{\lambda}{4})_n n!} (-1)^n a^{\frac{\lambda}{4}+n} \sum_{j+l=n} \frac{n!}{j!l!}(-2)^l \\
&= \sqrt{2\pi} \frac{\Gamma(\frac{1}{2}+\frac{\lambda}{4})}{\Gamma(1+\frac{\lambda}{4})} a^{\frac{\lambda}{4}} \sum_{n=0}^{\infty} \frac{(\frac{1}{4}+\frac{\lambda}{4})_n(\frac{3}{4}+\frac{\lambda}{4})_n}{(1+\frac{\lambda}{4})_n n!} a^n \\
&= \sqrt{2\pi} \frac{\Gamma(\frac{1}{2}+\frac{\lambda}{4})}{\Gamma(1+\frac{\lambda}{4})} a^{\frac{\lambda}{4}} F\left(\frac{1}{4}+\frac{\lambda}{4}, \frac{3}{4}+\frac{\lambda}{4}, 1+\frac{\lambda}{4}; a\right).
\end{aligned}$$

This completes the proof. Q.E.D.

Proposition 1 implies that $\int_{|\xi-x|\leq\sqrt{2w}} \hat{\psi}^{\dagger}_{-,B}(\xi, y+x\xi)d\xi$ is the Borel transform of the WKB solution $\psi^{\dagger}_{+}$ of equation (1) at least locally (i.e., in a neighborhood of $y=-x^2/2$), and hence it is plausible to guess that the inverse Fourier-Laplace transform of a Borel resummed WKB solution of (9) integrated along a steepest descent path should become a Borel resummed WKB solution of the original equation (1). However, Proposition 1 does not hold globally. To obtain Borel resummed WKB solutions of equation (1) we need to consider some "modified" inverse Fourier-Laplace transform.

5 The inverse transform of the Borel resummed WKB solutions (II) — global theory

The global difficulty of the problem originates from the following simple observation:

Lemma 3 *The integrand* $\hat{\psi}^{\dagger}_{-,B}(\xi, y+x\xi) = \hat{\psi}^{\dagger}_{-,B}(\xi, (x+u)^2/2+v)$ *of the left-hand side of (19) has a singular point (branch point) at* $(u,v) = (-\Re x, (\Im x)^2)$ *(cf. Figure 2).*

This lemma readily follows from the explicit description of $\hat{\psi}^{\dagger}_{-,B}$ in terms of hypergeometric functions (Lemma 1). The existence of such a branch point is closely related to the fact that the steepest descent path Γ intersects the positive imaginary axis, a Stokes curve of (9), at a point $\xi_0 = i\Im x$ (cf. Figure 1).

Lemma 3 suggests that some difficulty may arise when $w \geq w_0 \stackrel{\text{def}}{=} (\Re x)^2/2 + (\Im x)^2$. Figure 3 actually indicates what phenomenon occurs when $w \geq w_0$ with the integral

$$\int_{|\xi-x|\leq\sqrt{2w}} \hat{\psi}^{\dagger}_{-,B}(\xi, y+x\xi)d\xi = \int_{-\sqrt{2w}}^{\sqrt{2w}} \hat{\psi}^{\dagger}_{-,B}(\xi, x^2/2+xu+w)du.$$

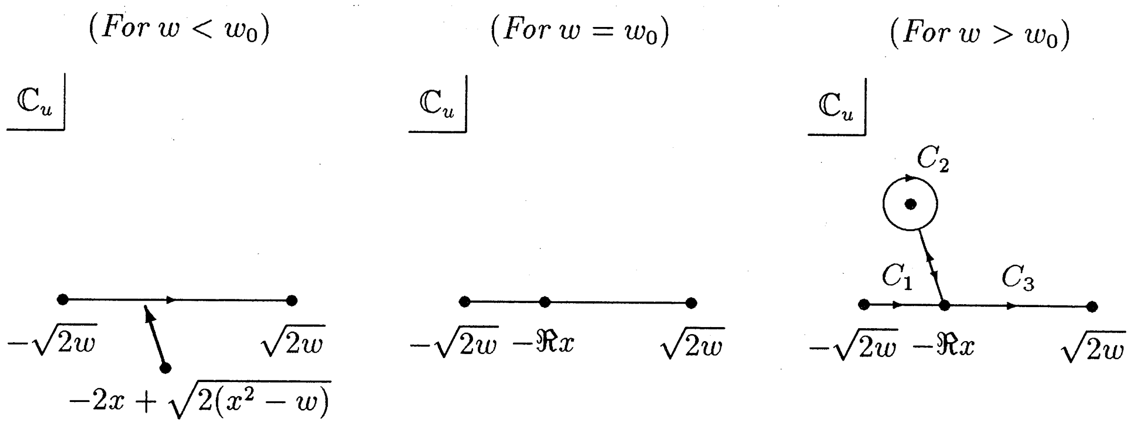

Figure 3: The movement of the brach point and the path of integration.

The corresponding branch point $u_* \overset{\text{def}}{=} -2x + \sqrt{2(x^2 - w)}$ of $\hat{\psi}^\dagger_{-,B}(\xi, y + x\xi)$ crosses the path of integration from below in the complex u-plane. (Here we have chosen the branch of $\sqrt{2(x^2-w)}$ so that it may become $2x - \Re x$ at $w = w_0$.) Hence the integral in question, which is a constant multiple of $\psi^\ddagger_{+,B}(x,y)$ when $w < w_0$ (Proposition 1), is no longer its analytic continuation for $w > w_0$. The above integral contributes only to the integral along the portions C_1 (from $-\sqrt{2w}$ to $-\Re x$) and C_3 (from $-\Re x$ to $\sqrt{2w}$). To obtain the Borel sum of the WKB solution $\psi^\ddagger_+$ of (1) we need the analytic continuation of $\psi^\ddagger_{+,B}(x,y)$. That is, it is necessary to take account of the integral along the portion C_2 also (cf. Figure 3).

Let us now try to compute more convenient form of the integral along C_2. In view of Lemma 1 the integral has the following form:

(23)
$$I_2 = \frac{1}{\Gamma(\frac{1}{2}+\frac{\lambda}{4})} \int_{C_2} \xi^{-2/3} \left[(s-1)^{-\frac{1}{2}+\frac{\lambda}{4}} F\left(\frac{1}{4}+\frac{\lambda}{4}, \frac{3}{4}+\frac{\lambda}{4}, \frac{1}{2}+\frac{\lambda}{4}; 1-s\right)\right]\Bigg|_{s=\frac{y+x\xi}{\xi^2}+\frac{1}{2}} du.$$

Note that the variable s becomes 0 at the branch point u_*. This means that the integrand of I_2 has a singularity at u_*. The behavior of the integrand there can be figured out by the following classical formula for hypergeometric functions:

$$\begin{aligned} F(\alpha,\beta,\gamma;1-s) &= \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} s^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;s) \\ &\quad + \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} F(\alpha,\beta,\alpha+\beta-\gamma+1;s) \end{aligned}$$

(cf. [BMP, §2.10]). Since the second term of the right-hand side is holomorphic at $s = 0$, only the first term contributes to the integral I_2. Making use of this formula together with Kummer's relation

$$(1-s)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta,\gamma;s) = F(\alpha,\beta,\gamma;s)$$

([BMP, Formula (23) in §2.1]), and paying some attention to the determination of the branch of $(s-1)^{-1/2+\lambda/4}$ and of $s^{-1/2-\lambda/4}$, we find

$$\begin{aligned} I_2 &= e^{-i\pi(-\frac{1}{2}+\frac{\lambda}{4})} \left(e^{2i\pi(-\frac{1}{2}-\frac{\lambda}{4})} - 1\right) \frac{\Gamma(\frac{1}{2}+\frac{\lambda}{4})}{\Gamma(\frac{1}{4}+\frac{\lambda}{4})\Gamma(\frac{3}{4}+\frac{\lambda}{4})} \\ &\quad \times \int_{-\Re x}^{u_*} \xi^{-2/3} \left[s^{-\frac{1}{2}-\frac{\lambda}{4}} F\left(\frac{1}{4}-\frac{\lambda}{4}, \frac{3}{4}-\frac{\lambda}{4}, \frac{1}{2}-\frac{\lambda}{4}; s\right)\right]\Bigg|_{s=\frac{y+x\xi}{\xi^2}+\frac{1}{2}} du. \end{aligned}$$

It follows from the well-known formula $\Gamma(1/2+z)\Gamma(1/2-z)=\pi/\cos\pi z$ and (22) that

$$e^{-i\pi(-\frac{1}{2}+\frac{\lambda}{4})}\left(e^{2i\pi(-\frac{1}{2}-\frac{\lambda}{4})}-1\right)\frac{\Gamma(\frac{1}{2}+\frac{\lambda}{4})}{\Gamma(\frac{1}{4}+\frac{\lambda}{4})\Gamma(\frac{3}{4}+\frac{\lambda}{4})}=\frac{-i\sqrt{2\pi}2^{\lambda/2}}{\Gamma(\frac{1}{2}-\frac{\lambda}{4})\Gamma(\frac{1}{2}+\frac{\lambda}{2})}e^{-i\pi\lambda/2}.$$

Hence we finally obtain

$$(24)\qquad I_2 = \frac{-i\sqrt{2\pi}2^{\lambda/2}}{\Gamma(\frac{1}{2}-\frac{\lambda}{4})\Gamma(\frac{1}{2}+\frac{\lambda}{2})}e^{-i\pi\lambda/2} \times\int_{\xi_0}^{\xi_*}\xi^{-2/3}\left[s^{-\frac{1}{2}-\frac{\lambda}{4}}F(\frac{1}{4}-\frac{\lambda}{4},\frac{3}{4}-\frac{\lambda}{4},\frac{1}{2}-\frac{\lambda}{4};s)\right]\Bigg|_{s=\frac{y+x\xi}{\xi^2}+\frac{1}{2}}d\xi$$

where $\xi_*=-x+\sqrt{2(x^2-w)}$.

It is then natural to ask "What is this integral?" The answer is the following: The steepest descent path Γ of $\Re\varphi_-$ intersects a Stokes curve of (9) at a point ξ_0. We now draw the steepest descent path, denoted by $\tilde{\Gamma}$, of $\Re\varphi_+$ from this crossing point ξ_0 (cf. Figure 1, where $\tilde{\Gamma}$ is written by dotted lines) and consider the inverse Fourier-Laplace transform of the Borel resummed WKB solution $\hat{\Psi}_+^\dagger$ integrated along $\tilde{\Gamma}$:

$$(25)\qquad \int_{\tilde{\Gamma}}\exp(\eta x\xi)\hat{\Psi}_+^\dagger(\xi,\eta)d\xi=\int_{\tilde{\Gamma}}\exp(\eta x\xi)\left(\int_{-\xi^2/2}^{\infty}\exp(-\eta z)\hat{\psi}_{+,B}^\dagger(\xi,z)dz\right)d\xi.$$

Similarly to the case of $\hat{\Psi}_-^\dagger$ let us introduce new variables of integration by

$$(26)\qquad x\xi+\frac{1}{2}\xi^2=x\xi_0+\frac{1}{2}\xi_0^2-\tilde{u}\,(\tilde{u}\geq 0),\quad z=-\frac{1}{2}\xi^2+\tilde{v}\,(\tilde{v}\geq 0)$$

and employ the same change of variables (13). Noting that

$$(27)\qquad y=z-x\xi=\tilde{v}+\tilde{u}-(x\xi_0+\frac{1}{2}\xi_0^2)=-\frac{1}{2}x^2+w_0+\tilde{u}+\tilde{v},$$

we then obtain the following expression of (25):

$$(28)\qquad \int_{\substack{y=-\frac{1}{2}x^2+w\\ w=w_0+\tilde{u}+\tilde{v}\geq w_0}}\exp(-\eta y)\left(\int_{w-w_0\geq\tilde{u}\geq 0}\hat{\psi}_{+,B}^\dagger(\xi,y+x\xi)d\xi\right)dy.$$

The endpoints $w-w_0$ and 0 of the inner integral respectively correspond to ξ_* and ξ_0 in the ξ-variable. Hence, in view of Lemma 1 again, the inner integral of (28) can be rewritten as

$$(29)\qquad \frac{1}{\Gamma(\frac{1}{2}-\frac{\lambda}{4})}\int_{\xi_*}^{\xi_0}\xi^{-2/3}\left[s^{-\frac{1}{2}-\frac{\lambda}{4}}F(\frac{1}{4}-\frac{\lambda}{4},\frac{3}{4}-\frac{\lambda}{4},\frac{1}{2}-\frac{\lambda}{4};s)\right]\Bigg|_{s=\frac{y+x\xi}{\xi^2}+\frac{1}{2}}d\xi.$$

This is nothing but a constant multiple of I_2. We have thus verified

Proposition 2 *When $w > w_0$, the following relation holds:*

$$\sqrt{2\pi}\psi^{\ddagger}_{+,B}(x,y) = \int_{|\xi - x| \leq \sqrt{2w}} \hat{\psi}^{\dagger}_{-,B}(\xi, y + x\xi)d\xi + I_2, \tag{30}$$

where

$$I_2 = \frac{i\sqrt{2\pi}2^{\lambda/2}}{\Gamma(\frac{1}{2}+\frac{\lambda}{2})}e^{-i\pi\lambda/2}\int_{w-w_0 \geq \tilde{u} \geq 0} \hat{\psi}^{\dagger}_{+,B}(\xi, y + x\xi)d\xi. \tag{31}$$

Corollary 1 *The inverse Fourier-Laplace transform of (a linear combination of) Borel resummed WKB solutions defined by*

$$\int_{\Gamma} \exp(\eta x\xi)\hat{\Psi}^{\dagger}_{-}(\xi,\eta)d\xi + \frac{i\sqrt{2\pi}2^{\lambda/2}}{\Gamma(\frac{1}{2}+\frac{\lambda}{2})}e^{-i\pi\lambda/2}\int_{\tilde{\Gamma}} \exp(\eta x\xi)\hat{\Psi}^{\dagger}_{+}(\xi,\eta)d\xi \tag{32}$$

coincides with $\sqrt{2\pi}\Psi^{\ddagger}_{+}(x,\eta)$, the Borel sum of a WKB solution $\sqrt{2\pi}\psi^{\ddagger}_{+}$ of the original equation (1).

Remark 2. The constant $i\sqrt{2\pi}2^{\lambda/2}e^{-i\pi\lambda/2}/\Gamma(1/2+\lambda/2)$ before the second term of (32) is exactly equal to that appearing in the connection formula for $\hat{\Psi}^{\dagger}_{-}$ on the positive imaginary axis (cf. [T1, Proposition 5]).

Summing up, we can state the conclusion of this note as follows:

> CONCLUSION: To discuss the Fourier-Laplace transform of Borel resummed WKB solutions of the harmonic oscillator, the usual steepest descent method is insufficient. However, if we take into account a "bifurcated" steepest descent path (like $\tilde{\Gamma}$) besides usual steepest descent paths, then the (inverse) Fourier-Laplace transform of Borel resummed WKB solutions actually gives us the Borel sum of a suitable WKB solution.

This suggests that the conclusion should hold for general ordinary differential equations with arbitrary polynomial coefficients. Generalization of this result to more general equations shall be discussed in our forthcoming papers.

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