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Author(s)	Jimbo, Shuji; Hashiguchi, Kosaburo; Yamamoto, Osamu
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# On the Shapes of Vertex Subsets of Hypercubes That Minimize Their Boundary

Faculty of Engineering, Shuji Jimbo\*(神保 秀司)  
Okayama University Kosaburo Hashiguchi (橋口 攻三郎)

Ryobi Systems, Inc. Osamu Yamamoto (山本 治)

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## Abstract

A problem of partitioning the vertex-set of a hypercube into two parts is considered. The sizes of the two parts are given and the size of the boundary between the two parts has to be minimized. The boundary of the two parts is the set of the endvertices of edges that join a vertex in one part and one in the other part. The minimum size of the boundary is already known. It is also known that, for every integer  $k$ , the size of the boundary is minimized by partitioning the vertex-set of a hypercube into the set of the first  $k$  vertices in squashed order and the rest. In this paper, the shape of such a partition is investigated. First, it is shown that such a partition of a hypercube is implemented by cutting the hypercube with a particular kind of hyperplane. Second, for every integer  $l$  that is a sum of consecutive binomial coefficients  $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{m}$ , where  $n$  is the dimension of the hypercube, if the size of one of the two parts obtained by the partition is exactly  $l$ , then the shape of the part is uniquely determined.

**Keywords:** graph theory, hypercube, parallel computation, network, BANDWIDTH, extremal set theory.

## 1 Introduction

When we divide a computer network into two subnetworks, some loss of performance results. The number of nodes of each subnetwork is given, and the loss has to be minimized. We can measure the loss in two different types of measurement: (a) the number of computers that have a neighbor in the other subnetwork, and (b) the number of links that join a computer in one subnetwork and one in the other. Hypercubes have been investigated intensively as network topologies with which concurrent computation is implemented. If

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\*E-mail: jimbo@kiso.it.okayama-u.ac.jp

the network topology is a hypercube, then measuring the loss in measurement (a) corresponds to Problem A below, while measuring the loss in measurement (b) corresponds to Problem B.

Problem A: partition the vertex-set of a hypercube into two parts, one of which consists of  $p$  vertices and the other of  $q$  vertices, under the constraint that the size of the boundary between the two parts, which we call the vertex-cost of the partition, is minimized. The boundary of the two parts is the set of the endvertices of edges that join a vertex in one part and one in the other part.

Problem B: partition the vertex-set of a hypercube into two parts, one of which consists of  $p$  vertices and the other of  $q$  vertices, under the constraint that the number of edges that join a vertex in one part and one in the other, which we call the edge-cost of the partition, is minimized.

We present answers to those problems below. Let  $Q_n$  denote the hypercube of dimension  $n$ . For a vertex  $x = (x_1, x_2, \dots, x_n)$  of  $Q_n$ , we write  $w(x) = \sum_{i=1}^n x_i$  and  $r_n(x) = \sum_{i=1}^n 2^{i-1} x_i$ .

Answer A: Let  $A_n(p)$  be the first  $p$  vertices in the linear ordering in which  $x$  precedes  $y$  if either  $w(x) < w(y)$  or both  $w(x) = w(y)$  and  $r_n(x) > r_n(y)$ . This order is called squashed order. By partitioning the vertex-set of  $Q_n$  into  $A_n(p)$  and the rest, the vertex-cost is minimized.

Essentially the same problem as Problem B was investigated and solved by Nakano, et al[NMT90].

Answer B (by Nakano, et al.): Let  $B_n(p)$  be the first  $p$  vertices in the linear ordering in which  $x$  precedes  $y$  if  $r_n(x) < r_n(y)$ . By partitioning the vertex-set of  $Q_n$  into  $B_n(p)$  and the rest, the edge-cost is minimized.

Although the answers above have been shown, it is not known what shapes the fragments of partitions form in answers to the problems above. In this paper, the shapes of such partitions are investigated. First, it is shown that such a partition of a hypercube is implemented by cutting the hypercube with a particular kind of hyperplanes. Second, for every integer  $l$  that is a sum of consecutive binomial coefficients  $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{m}$ , where  $n$  is the dimension of the hypercube, if the size of one of the two parts obtained by the partition is exactly  $l$ , then the shape of the part is uniquely determined.

The definition of vertex-cost originated from our previous research on a binary code. Let  $n$  be a positive integer. Assume that we want to construct a binary block code consisting of all  $n$ -bit words, where each codeword represents an integer from 1 to  $2^n$ , that is, the code can be considered as a permutation of the vertex-set of the  $n$ -dimensional hypercube. If we have to minimize the maximum error of a decoded value arising from exactly one error bit in the corresponding codeword, then what code do we need to construct? An answer to this question corresponds to a solution to an instance of BANDWIDTH, an NP-complete problem, when the  $n$ -dimensional hypercube is given to the instance as a problem parameter. One of the answers is the code corresponding to the permutation induced by squashed order.

## 2 Definitions

The size of a finite set  $S$ , namely, the number of elements of  $S$ , is denoted by  $|S|$ . If  $G$  is a simple undirected graph, then  $V(G)$  and  $E(G)$  denote the vertex-set and edge-set of  $G$ , respectively.

Let  $n$  be a positive integer. The hypercube of dimension  $n$ , or  $n$ -cube, is an undirected graph, denoted by  $Q_n$ . The vertex-set of  $Q_n$  is  $\{0, 1\}^n$ , and the edge-set  $\{((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \mid |x_i - y_i| = 1)\}$ . Let  $x = (x_1, x_2, \dots, x_n)$  be a vertex of  $Q_n$ . The number of 1's in the components of  $x$ , namely,  $\sum_{i=1}^n x_i$ , is said to be the weight of  $x$ , denoted by  $w(x)$ . The value  $\sum_{i=1}^n 2^{i-1} x_i$  is said to be the lexicographical rank of  $x$ , denoted by  $r_n(x)$ .

Let  $G$  be an undirected graph, and  $S$  a subset of  $V(G)$ . A vertex in  $S$  that is adjacent to no vertices except ones in  $S$  is said to be an interior vertex of  $S$ . The set of all interior vertices of  $S$  is said to be the interior of  $S$ , denoted by  $\mathcal{I}_G(S)$ , namely,  $\mathcal{I}_G(S) = \{v \in S \mid (\forall u \in V(G) \setminus S)(uv \notin E(G))\}$ . We may omit the subscript  $G$  when there will be no confusion.

The vertex-cost and edge-cost of a partition of the vertex-set of  $G$  into two disjoint sets,  $S$  and  $V(G) \setminus S$ , are defined to be

$$C_n^V(S) = |\{u \in S \mid (\exists v \in V(G) \setminus S)(uv \in E(G))\}| + |\{v \in V(G) \setminus S \mid (\exists u \in S)(uv \in E(G))\}|$$

and

$$C_n^E(S) = |\{e \in E(G) \mid (\exists u \in S)(\exists v \in V(G) \setminus S)(e = uv)\}|,$$

respectively. The vertex-cost of a partition is the number of vertices whose degree the partition lessens, while the edge-cost of a partition is the number of edges that the partition removes. Each of those costs might indicate the magnitude of the loss arising from a partition. We, therefore, prefer those costs as small as possible. By definition,  $C_n^V(S)$  can be expressed as  $|V(G)| - |\mathcal{I}(S)| - |\mathcal{I}(V(G) \setminus S)|$ .

## 3 Fundamental Theorem

Let  $e_n(k)$  denote the maximum size of the interior of a  $k$ -element subset of  $V(Q_n)$ . Sequence  $f_0, f_1, f_2, \dots, f_n, \dots$  of non-negative integer valued functions is defined recursively as  $f_n(0) = 0$ ,  $f_n(2^n) = 2^n$ , and, for an integer  $k$  with  $1 \leq k \leq 2^n - 1$ ,

$$f_n(k) = \max_{\max\{0, k-2^{n-1}\} \leq i \leq k/2} (\min\{f_{n-1}(i), k-i\} + \min\{f_{n-1}(k-i), i\}).$$

The domain of  $f_n$  is  $\{0, 1, 2, \dots, 2^n\}$  for every non-negative integer  $n$ .

**Proposition 1** ([Kru63][Kat66][Fra95]) *Let  $m$  be a positive integer. Then, any positive integer  $k$  is uniquely expressed as*

$$k = \binom{a_m}{m} + \binom{a_{m-1}}{m-1} + \dots + \binom{a_t}{t}, \quad (1)$$

where  $t, a_m, a_{m-1}, \dots$ , and  $a_t$  are integers with  $a_m > a_{m-1} > \dots > a_t \geq t \geq 1$ .

Expression (1) is said to be the  $m$ -binomial coefficient expression of  $k$ , and  $a_m$  in the expression the first parameter of the expression. It follows readily that  $\binom{a_m}{m} \leq k < \binom{a_m+1}{m}$ .

Let  $k$  and  $m$  be positive integers, and  $k = \binom{a_m}{m} + \binom{a_{m-1}}{m-1} + \cdots + \binom{a_t}{t}$  the  $m$ -binomial coefficient expression of  $k$ . The Kruskal-Katona function with base  $m$  is defined to be the function whose value at  $k$  is

$$K_m(k) = \binom{a_m}{m-1} + \binom{a_{m-1}}{m-2} + \cdots + \binom{a_t}{t-1}.$$

We also define  $K_m(k)$  for  $k = 0$  or  $m = 0$  as  $K_m(0) = K_0(k) = 0$ .

Linear order relation  $\leq_S$  on  $V(Q_n)$  is defined as follows:  $u \leq_S v$  if and only if either  $w(u) < w(v)$  or both  $w(u) = w(v)$  and  $r_n(u) \geq r_n(v)$ . We call the linear order relation  $\leq_S$  squashed order relation. Let  $v$  be the  $k$ -th smallest vertex of  $Q_n$  in squashed order. We call  $k$  the rank of  $v$  in squashed order, and express  $k$  as  $\delta_n(v)$ . The  $k$ -th smallest vertex of  $Q_n$  in squashed order, therefore, can be expressed as  $\delta_n^{-1}(k)$ , while the  $k$ -th smallest vertex of  $Q_n$  in increasing order of lexicographical rank can be expressed as  $r_n^{-1}(k)$ . Furthermore,  $A_n(k)$  denotes the set of the  $k$  smallest vertices of  $Q_n$  in squashed order, namely,  $A_n(k) = \{\delta_n^{-1}(1), \delta_n^{-1}(2), \dots, \delta_n^{-1}(k)\}$ .

For positive integers  $n$  and  $k$  with  $k \leq 2^n$ ,  $\mu_n(k)$  and  $\xi_n(k)$  are defined to be the integers such that  $k = \left(\sum_{j=0}^{\mu_n(k)} \binom{n}{j}\right) - \xi_n(k)$  and  $0 \leq \xi_n(k) < \binom{n}{\mu_n(k)}$ . Value  $I_n(k)$  is defined to be  $k - \binom{n}{\mu_n(k)} + \xi_n(k) - K_{\mu_n(k)}(\xi_n(k))$ . Furthermore,  $I_n(0)$  is defined to be 0 for all positive integers  $n$ , and  $I_0(k) = k$  for  $k = 0$  and  $k = 1$ . By definition, we have  $I_n(2^n) = 2^n$  for all non-negative integers  $n$ . The following proposition follows from a part of Kruskal-Katona theorem[YJH98].

**Proposition 2** *Let  $n$  be a positive integer and  $k$  a non-negative integer with  $k \leq 2^n$ . Then,  $|\mathcal{I}(A_n(k))| = I_n(k)$  and*

$$\mathcal{I}(A_n(k)) = \{\delta_n^{-1}(1), \delta_n^{-1}(2), \dots, \delta_n^{-1}(I_n(k))\}$$

*hold.*

**Proposition 3 ([YJH98])** *For any positive integer  $n$  and any non-negative integer  $k$  with  $k \leq 2^n$ , expression*

$$|\mathcal{I}(A_n(k))| = I_n(k) \leq e_n(k) \leq f_n(k)$$

*holds.*

**Proof.** It follows easily from the definition of  $e_n(k)$  that  $|\mathcal{I}(A_n(k))| = I_n(k) \leq e_n(k)$  holds. In what follows, we shall prove that  $e_n(k) \leq f_n(k)$  holds. We will prove it by induction on  $n$ . It is clear that the expression holds for  $n = 1$  and all integers  $k$  with  $0 \leq k \leq 2^n = 2$ . Let  $N$  be an integer greater than 1, and assume that the inequality holds for all integers  $n < N$  and all integers  $k$  with  $0 \leq k \leq 2^n$ . Since if  $k = 0$  or  $k = 2^N$  then  $e_N(k) = f_N(k)$  holds, we may assume that  $0 < k < 2^N$ . Let  $S$  be a subset of  $V(Q_N)$  such that  $|S| = k$  and  $|\mathcal{I}(S)| = e_N(|S|)$ . We divide  $V(Q_N)$  into two disjoint subsets  $V_0$  and  $V_1$ , where the first component of each vertex in  $V_0$  is 0, and that of each

vertex in  $V_1$  is 1. We also divide  $S$  into  $S_0 = S \cap V_0$  and  $S_1 = S \cap V_1$ . Let  $i$  denote  $|S_0|$ , hence we have  $|S_1| = k - i$ . Since  $|V_0| = |V_1| = |V(Q_N)|/2 = 2^{N-1}$ , we have  $i \leq 2^{N-1}$  and  $k - i \leq 2^{N-1}$ , hence  $\max\{0, k - 2^{N-1}\} \leq i \leq k/2$ . Let  $S'_0$  denote the sets of the vertices  $(x_1, x_2, \dots, x_N) \in S_0$  such that  $(1 - x_1, x_2, x_3, \dots, x_N) \in S$ . Furthermore, let  $S''_0$  denote the set of the vertices  $(x_1, x_2, \dots, x_N) \in S_0$  such that, for each  $j \in \{2, 3, \dots, N\}$ ,  $(x_1, x_2, \dots, x_{j-1}, 1 - x_j, x_{j+1}, \dots, x_N) \in S$ . Then, we have  $\mathcal{I}(S) \cap S_0 \subseteq S'_0 \cap S''_0$ , hence  $|\mathcal{I}(S) \cap S_0| \leq \min\{|S'_0|, |S''_0|\}$ . Furthermore, we have

$$\begin{aligned} |S'_0| &= |\{(1, x_2, x_3, \dots, x_N) \in S_1 \mid (0, x_2, x_3, \dots, x_N) \in S_0\}|, \\ \{(1, x_2, x_3, \dots, x_N) \in S_1 \mid (0, x_2, x_3, \dots, x_N) \in S_0\} &\subseteq S_1, \quad \text{and} \\ \{(x_2, x_3, \dots, x_N) \in V(Q_{N-1}) \mid (0, x_2, x_3, \dots, x_N) \in S''_0\} \\ &\subseteq \mathcal{I}(\{(x_2, x_3, \dots, x_N) \in V(Q_{N-1}) \mid (0, x_2, x_3, \dots, x_N) \in S_0\}). \end{aligned}$$

By induction hypothesis, we therefore have

$$|\mathcal{I}(S) \cap S_0| \leq \min\{e_{N-1}(i), k - i\} \leq \min\{f_{N-1}(i), k - i\}.$$

Furthermore, by defining  $S'_1$  and  $S''_1$  like  $S'_0$  and  $S''_0$ , we can also have

$$|\mathcal{I}(S) \cap S_1| \leq \min\{e_{N-1}(k - i), i\} \leq \min\{f_{N-1}(k - i), i\}.$$

Thus, by the definition of functions  $f_0, f_1, f_2, \dots$ , we conclude that

$$\begin{aligned} e_N(k) &= |\mathcal{I}(S)| = |\mathcal{I}(S) \cap S_0| + |\mathcal{I}(S) \cap S_1| \\ &\leq \min\{f_{N-1}(i), k - i\} + \min\{f_{N-1}(k - i), i\} \leq f_N(k), \end{aligned}$$

completing the proof.  $\square$

In fact, the three values,  $I_n(k)$ ,  $e_n(k)$ ,  $f_n(k)$ , in Proposition 3 are the same, namely,  $I_n(k) = e_n(k) = f_n(k)$ . This fact follows easily from the following theorem, which we have already proved[JH99].

**Theorem 4** *For any positive integers  $n$  and  $k$  with  $1 \leq k \leq 2^n - 1$ , inequality*

$$I_n(k) \geq \max_{\max\{0, k - 2^{n-1}\} \leq i \leq k/2} (\min\{I_{n-1}(i), k - i\} + \min\{I_{n-1}(k - i), i\})$$

*holds.*

## 4 Shapes of the Partitions That Minimize the Vertex-cost

For positive integer  $n$  and  $k$  with  $k < 2^n$ ,  $B_n(k)$  denotes the  $k$ -element subset of  $V(Q_n)$  defined as  $B_n(k) = \{x = (x_1, x_2, \dots, x_n) \in \{0, 1\}^n \mid r_n(x) = \sum_{i=1}^n 2^{i-1} x_i \leq k - 1\}$ . On one hand, it is known that the edge-cost of partitioning  $V(Q_n)$  into  $B_n(k)$  and the rest is the smallest among all the edge-costs of partitioning  $V(Q_n)$  into a  $k$ -element subset and the rest[NMT90]. On the other hand, the vertex-cost of partitioning  $V(Q_n)$  into  $A_n(k)$  and the rest is the smallest among all the vertex-costs of partitioning  $V(Q_n)$  as above. The following theorem asserts the latter.

**Theorem 5** *Let  $n$  and  $k$  be positive integers with  $k < n$ . The minimum vertex-cost of a partition of  $V(Q_n)$  into a  $k$ -element subset and the rest is  $C_n^V(A_n(k))$ .*

**Proof.** By the definition of cost  $C_n^V$ , Proposition 3, and Theorem 4, we have  $C_n^V(A_n(k)) = |V(Q_n)| - |\mathcal{I}(A_n(k))| - |\mathcal{I}(V(Q_n) \setminus A_n(k))| = 2^n - e_n(k) - |\mathcal{I}(V(Q_n) \setminus A_n(k))|$ . Since the mapping  $x = (x_1, x_2, \dots, x_n) \in V_n \mapsto (1 - x_1, 1 - x_2, \dots, 1 - x_n)$  is an automorphism of  $Q_n$  and takes  $V(Q_n) \setminus A_n(k)$  to  $A_n(2^n - k)$ , we conclude that  $|\mathcal{I}(V(Q_n) \setminus A_n(k))| = e(2^n - k)$ . This completes the proof.  $\square$

We have shown that the  $k$ -element sets  $B_n(k)$  and  $A_n(k)$  can be both obtained as a fragment by cutting  $Q_n$  with a hyperplane. A hyperplane that does not pass the origin is determined by the foot  $f = (f_1, f_2, \dots, f_n) \neq \mathbf{0}$  of the perpendicular to it from the origin. We write the hyperplane determined by  $f \neq \mathbf{0}$  as  $H(f)$ . Furthermore, the two half-spaces obtained by cutting the whole space with  $H(f)$  are denoted by  $H_-(f)$  and  $H_+(f)$ , where both  $H_-(f)$  and  $H_+(f)$  include  $H(f)$ , and  $H_-(f)$  includes the origin while  $H_+(f)$  does not. For vectors  $x = (x_1, x_2, \dots, x_n)$  and  $y = (y_1, y_2, \dots, y_n)$  in  $n$  dimensional Euclidean space  $\mathbb{R}^n$ ,  $\langle x, y \rangle$  denotes the inner product of  $x$  and  $y$ , namely,  $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ . Expression  $\sqrt{\langle x, x \rangle}$ , therefore, expresses the norm of  $x$ , denoted by  $\|x\|$ . By definition, we have  $H(f) = \{x \in \mathbb{R}^n \mid \langle x, f \rangle = \|f\|^2\}$  and  $H_-(f) = \{x \in \mathbb{R}^n \mid \langle x, f \rangle \leq \|f\|^2\}$ .

**Theorem 6** *Let  $n$  be a positive integer and  $k$  an integer with  $0 \leq k \leq 2^n$ . Let  $\alpha$  and  $\beta$  denote the vectors in  $\mathbb{R}^n$  whose  $i$ -th components are  $\alpha_i = 1 - (1/2)^{n+1-i}$  and  $\beta_i = (1/2)^{n+1-i}$ , respectively, for each  $i \in \{1, 2, \dots, n\}$ . Furthermore, let  $f_\alpha$  and  $f_\beta$  denote*

$$\frac{\langle \delta_n^{-1}(k), \alpha \rangle}{\|\alpha\|^2} \alpha \quad \text{and} \quad \frac{\langle r_n^{-1}(k), \beta \rangle}{\|\beta\|^2} \beta,$$

respectively. Then, it holds that  $A_n(k) = V(Q_n) \cap H_-(f_\alpha(k))$  and  $B_n(k) = V(Q_n) \cap H_-(f_\beta(k))$ .

**Proof.** We easily see that  $\delta_n^{-1}(k)$  and  $r_n^{-1}(k)$  are on  $H(f_\alpha)$  and  $H(f_\beta)$ , respectively. Therefore, the rest of the proof is to show that, for any positive integer  $l$  with  $l < k$ , both

$$\langle \delta_n^{-1}(l), \alpha \rangle \leq \langle \delta_n^{-1}(k), \alpha \rangle \quad (2)$$

and

$$\langle r_n^{-1}(l), \beta \rangle \leq \langle r_n^{-1}(k), \beta \rangle \quad (3)$$

hold. For each  $j \in \{1, 2, \dots, n\}$ , let  $x_j^l$  and  $x_j^k$  be the  $j$ -th components of  $\delta_n^{-1}(l)$  and  $\delta_n^{-1}(k)$ , respectively.

First, we shall prove inequality (3). There is a  $j \in \{1, 2, \dots, n\}$  such that  $x_j^l = 0$ ,  $x_j^k = 1$ , and,  $x_{j+1}^l = x_{j+1}^k$ ,  $x_{j+2}^l = x_{j+2}^k$ ,  $\dots$ ,  $x_n^l = x_n^k$ . We therefore have

$$\begin{aligned} \langle r_n^{-1}(l), \beta \rangle &\leq \sum_{h=1}^{j-1} (1/2)^{n+1-h} + \sum_{h=j+1}^n x_h^k (1/2)^{n+1-h} \\ &< (1/2)^{n+1-j} + \sum_{h=j+1}^n x_h^k (1/2)^{n+1-h} \\ &\leq \sum_{h=1}^n x_h^k (1/2)^{n+1-h} = \langle r_n^{-1}(k), \beta \rangle. \end{aligned}$$

Second, we shall prove inequality (2). To that end, we consider the following two cases: (a)  $w(\delta_n^{-1}(l)) < w(\delta_n^{-1}(k)) = m$ , and (b)  $w(\delta_n^{-1}(l)) = w(\delta_n^{-1}(k)) = m$  and  $r_n(\delta_n^{-1}(l)) > r_n(\delta_n^{-1}(k))$ . If condition (a) holds, then we have

$$\begin{aligned} \langle \delta_n^{-1}(l), \alpha \rangle &\leq m - 1 - \sum_{j=1}^{m-1} (1/2)^{n+1-j} < m - 1 \\ &< m - \sum_{j=n+1-m}^n (1/2)^{n+1-j} \leq \langle \delta_n^{-1}(k), \alpha \rangle. \end{aligned}$$

We therefore have inequality (2). If condition (b) holds, then there is a  $j \in \{1, 2, \dots, n\}$  such that  $x_j^l = 1$ ,  $x_j^k = 0$ , and,  $x_{j+1}^l = x_{j+1}^k$ ,  $x_{j+2}^l = x_{j+2}^k$ ,  $\dots$ ,  $x_n^l = x_n^k$ . Then, it follows that

$$\begin{aligned} \langle \delta_n^{-1}(l), \alpha \rangle &\leq \langle \delta_n^{-1}(k), \alpha \rangle - (1/2)^{n+1-j} + \sum_{h=1}^{j-1} (1/2)^{n+1-h} \\ &< \langle \delta_n^{-1}(k), \alpha \rangle. \end{aligned}$$

Thus, we also have inequality (2).  $\square$

When we want to partition a hypercube at as low a cost as possible, cutting the hypercube with a hyperplane is intuitively a cogent method. So far, we have defined two kinds of cost of partitioning a hypercube into two parts, the vertex-cost and edge-cost. These costs have turned out to be minimized by cutting a hypercube with a hyperplane, as we expect. Furthermore, for the same dimension and cost, all of the hyperplanes used above are of the same direction. However, there is another kind of partition that minimizes the vertex-cost for specific pairs of the sizes of the two parts. In fact, the vertex-cost is minimized by cutting a hypercube with two hyperplanes, so that one of the two subgraphs of  $Q_n$  induced by such a cutting of  $V(Q_n)$  is disconnected.

**Example:** Let  $A$  denote  $\{(0, 0, 0), (0, 0, 1), (0, 1, 0)\}$ , and  $B$   $\{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$ . Then, we have

$$C_3^V(A) = C_3^V(B) = 7.$$

The subgraph of  $Q_3$  induced by  $A$  and the one induced by  $V(Q_3) \setminus A$  are both connected, while the subgraph of  $Q_3$  induced by  $V(Q_3) \setminus B$  is disconnected.

In spite of the example above, we can show that if a specific value is designated as the size of a fragment of a partition, then the shapes of the partitions that minimize the vertex-cost are congruent one another.

**Theorem 7** *Let  $n$  be a positive integer and  $m$  an integer with  $0 \leq m < n$ . Let  $k$  denote  $\sum_{i=0}^m \binom{n}{i}$ . Let  $S$  be a subset of  $V(Q_n)$  such that  $|S| = k$  and  $C_n^V(S) = C_n^V(A_n(k))$ . Then, there is a vector  $x = (x_1, x_2, \dots, x_n) \in \{0, 1\}^n$  such that  $A_n(k) = \{(x_1 \oplus v_1, x_2 \oplus v_2, \dots, x_n \oplus v_n) \mid (v_1, v_2, \dots, v_n) \in S\}$ , where  $a \oplus b$  denotes  $(a + b) \bmod 2$ .*



Each component  $x_i$  of  $x$  in Theorem 7 can be determined as follows: if the sum of the  $i$ -th components of the vectors in  $S$  is greater than  $k/2$ , then set  $x_i$  to be 1, otherwise set it to be 0. The details of the proof of Theorem 7 are omitted. We give a rough sketch of the proof below.

In our previous report[JH99], we presented several lemmas to prove Theorem 4 above. We shall strengthen one of those lemmas.

**Lemma 8** *Let  $n$ ,  $m$ ,  $x$ , and  $y$  be positive integers with  $m \leq n$  and  $x + y = \binom{n}{m}$ . Then,*

$$K_m(x) + K_m(y) > \binom{n}{m-1}$$

*holds.*

Let  $n$  and  $m$  be positive integers with  $m \leq n$ . Let  $k$  denote  $\sum_{i=0}^m \binom{n}{i}$ . By this lemma, we can determine the integer  $k_1$  in  $\{\max\{0, k - 2^{n-1}\}, \max\{0, k - 2^{n-1}\} + 1, \dots, k/2\}$  such that

$$\begin{aligned} & \min\{I_{n-1}(k_1), k - k_1\} + \min\{I_{n-1}(k - k_1), k_1\} \\ &= \max_{\max\{0, k - 2^{n-1}\} \leq i \leq k/2} (\min\{I_{n-1}(i), k - i\} + \min\{I_{n-1}(k - i), i\}). \end{aligned}$$

Integer  $k_1$  turns out to be  $\sum_{i=0}^{m-1} \binom{n-1}{i} = I_{n-1}(k - k_1)$ . It follows, therefore, that  $k - k_1 = \sum_{i=0}^m \binom{n-1}{i}$ . Let  $k_0$  denote  $k - k_1$ . We shall prove Theorem 7 by induction on  $n$ . Of course, the assertion of Theorem 7 holds for every  $n = 1, 2$ , and  $3$ . Let  $S$  be a  $k$ -element subset of  $V(Q_n)$  with  $|\mathcal{I}(S)| = I_n(k)$ . Defining  $V_0, V_1, S_0, S_1, S'_0$ , and  $S'_1$  as in the proof of Proposition 3, that is,

$$V_0 = \{(x_1, x_2, \dots, x_n) \in V(Q_n) \mid x_1 = 0\}, \quad V_1 = \{(x_1, x_2, \dots, x_n) \in V(Q_n) \mid x_1 = 1\},$$

$$S_0 = S \cap V_0, \quad S_1 = S \cap V_1,$$

$$S'_0 = \{(x_1, x_2, \dots, x_n) \in V(Q_n) \mid (x_1, x_2, \dots, x_n) \in \mathcal{I}(S_0) \text{ and } (1 - x_1, x_2, \dots, x_n) \in S\},$$

and

$$S'_1 = \{(x_1, x_2, \dots, x_n) \in V(Q_n) \mid (x_1, x_2, \dots, x_n) \in \mathcal{I}(S_1) \text{ and } (1 - x_1, x_2, \dots, x_n) \in S\},$$

by definition, we have either  $|S_0| = k_0$  and  $|\mathcal{I}(S_0)| = k_1 = |S_1|$  or  $|S_1| = k_0$  and  $|\mathcal{I}(S_1)| = k_1 = |S_0|$ . First, we assume that the former holds. By induction hypothesis, we have the fact that  $\text{cdr}(S_0)$ , the set of  $(n-1)$ -dimensional vectors obtained by removing the first components from all of the vectors in  $S_0$ , is congruent with  $A_{n-1}(k_0)$ , that is, there is a vector  $a = (a_2, a_3, \dots, a_n) \in \{0, 1\}^{n-1}$  such that  $A_{n-1}(k_0) = \text{cdr}(S_0) \oplus a = \{(x_2 \oplus a_2, x_3 \oplus a_3, \dots, x_n \oplus a_n) \mid (x_2, x_3, \dots, x_n) \in \text{cdr}(S_0)\}$ . Let  $\text{cons}(0, a)$  denote  $(0, a_2, a_3, \dots, a_n)$ . Since  $A_{n-1}(k_0)$  consists of all the  $(n-1)$ -dimensional 0/1-vectors that have at most  $m$  1's, and  $\mathcal{I}(A_{n-1}(k_0))$  consists of all the  $(n-1)$ -dimensional 0/1-vectors that have at most  $m-1$  1's, it follows that  $S \oplus \text{cons}(0, a)$  consists all of the  $n$ -dimensional 0/1-vectors that have at most  $m$  1's, hence  $S \oplus \text{cons}(0, a)$  is  $A_n(k)$ .

If both  $|S_1| = k_0$  and  $|\mathcal{I}(S_1)| = k_1 = |S_0|$  hold, then we can show similarly that the congruent transformation  $x \mapsto x \oplus \text{cons}(1, a)$  converts  $S$  to  $A_n(k)$ .

## 5 Remarks

We conjecture that, if a partition of  $V(Q_n)$  into two parts minimizes the vertex-cost and the sizes of the two parts are both greater than  $n$ , then one of the two parts is congruent to a unique subset of  $V(Q_n)$  that consists of consecutive vertices from the origin  $(0, 0, \dots, 0)$  in squashed order. Hence, the partition can be obtained by cutting  $Q_n$  with exact one hyperplane.

In the case where a hypercube is cut with a hyperplane, if the direction of the hyperplane is far away from the ones in Theorem 6, then both of the vertex-cost and edge-cost are hardly minimized. However, there might be another type of cost different from both vertex-cost and edge-cost that is minimized by cutting a hypercube with such a hyperplane.

There is a recent piece of research dealing with a cut of a hypercube with a hyperplane, which includes estimation of volume of a fragment obtained by such a cut, though it does not relate directly to the results of this paper[SS97].

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## Correction of an Error in RIMS Kokyuroku 1166

I would appreciate it if you could correct the error at lines 6 – 7 in Page 111 as follows:

... The vertex-set of  $Q_n$  is  $\{0, 1\}^n$ , and the edge-set

$$\left\{ ((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) \mid \sum_{i=1}^n |x_i - y_i| = 1 \right\}.$$

Let  $x = (x_1, x_2, \dots, x_n)$  be a vertex of  $Q_n$ . ...