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# Jørgensen groups of parabolic type

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ABSTRACT. This paper is a report without proofs on Jørgensen groups obtained recently. In this paper we consider Jørgensen groups of parabolic type. In particular, we consider four kinds of one-parameter families of Jørgensen groups. Here a Jørgensen group is a Kleinian group whose Jørgensen number is one.

## 0. Introduction.

It is an important problem to decide whether or not a non-elementary subgroup of the Möbius transformation group, which is denoted by Möb, is discrete. In 1976 Jørgensen [3] gave a necessary condition for a non-elementary Möbius transformation group $G = \langle A, B\rangle$ to be discrete : If $\langle A, B\rangle$ is a non-elementary discrete group, then

$$J(A,B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2| \geq 1.$$

The lower bound 1 is best possible.

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Let $\langle A, B\rangle$ be a marked two-generator subgroup of Möb. We call

$$J(A,B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2|$$

the *Jørgensen number* for $\langle A, B\rangle$. Let $G$ be a two-generator subgroup of Möb. The *Jørgensen number* $J(G)$ for the group $G$ is the infimum of $J(A, B)$ where $A$ and $B$ generate $G$. We call a non-elementary two-generator discrete subgroup $G$ of Möb a *Jørgensen group* if $J(G) = 1$.

With respect to Jørgensen numbers it gives rise to the following problems:

(1) Problem 1 is to find all Jørgensen groups.

(2) Problem 2 is to find the infimum of Jørgensen numbers for some subspaces of the Kleinian space, for example for the Teichmüller space and for the Schottky space.

For Problem 2, Gilman [1] and Sato [7] gave the best lower bound of Jørgensen numbers for purely hyperbolic two-generator groups, and Sato [8], [9] gave the best lower bound of Jørgensen numbers for the classical Schottky space $R\mathbf{S}_2$ of real type of genus two. Namely,

$$\inf\{J(G) \mid G \in R\mathbf{S}_2\} = 4.$$

The family of groups, $P = \{G_\sigma = \langle A, B_{1/\sigma,\sigma}\rangle \mid G_\sigma$ is a discrete group, $\sigma \in \mathbf{C} \setminus \{0\}\}$ contains the Riley slice $RS$ (see Keen and Series [5] for the definition of the Riley slice). If $\langle A, B_{1/\sigma,\sigma}\rangle$ is a group in $P$, then $J(A, B_{1/\sigma,\sigma}) = |\sigma|^2$. It is easily seen that $\inf\{J(G) \mid G \in P\} = 1$, since $J(A, B_{1/\sigma,\sigma}) = 1$ for $\sigma = 1$, that is, in this case the group is the classical modular group. Furthermore we easily see that

$$1 \le \inf\{J(G) \mid G \in RS\} \le 2.$$

As far as the author knows, the value of $\inf\{J(G) \mid G \in RS\}$.

For Problem 1, Jørgensen-Kiikka [3], Jørgensen-Lascurain-Pignataro [4], Sato [10] and Sato-Yamada [12] studied extreme discrete groups, that is, Jørgensen groups.

In particular Jørgensen-Kiikka [3] obtained the following theorem: Let $\langle A, B\rangle$ be a non-elementary discrete group with $J(A,B)=1$, that is, a Jørgensen group. Then $A$ is elliptic of order at least seven or $A$ is parabolic.

In this paper we only consider the case where $A$ is parabolic, that is, Jørgensen groups of parabolic type. Namely, we consider two-generator groups $G_{\mu,\sigma}=\langle A, B_{\mu,\sigma}\rangle$ generated by

$$A=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\mu,\sigma}=B_{ik,\sigma}=\begin{pmatrix} ik\sigma & -k^2\sigma-1/\sigma \\ \sigma & ik\sigma \end{pmatrix},$$

where $k \in \mathbf{R}$ and $\sigma \in \mathbf{C}\setminus\{0\}$.

This paper contains six theorems. Theorems 2, 5 and 6 are new.In §1 we will state some definitions. In §2 theorems will be stated. The proofs of the theorems will appear elsewhere.

## §1. Definitions

In this section we will state some definitons, for example a Jørgensen group. Let Möb denote the set of all Möbius transformations. In this paper we use a Kleinian group in the same meaning as a discrete group. A Kleinian group $G$ is of the *first kind* if the limit set $\Lambda(G)$ of $G$ is ll of the extended complex plane $\hat{\mathbf{C}}$ and it is of the *second kind* otherwise. A subgroup $G$ of Möb is non-elementary grup if $\sharp\Lambda(G)\geq 3$.

In 1976 Jørgensen obtained the following important theorem called Jørgensen's inequality, which gives a necessary condition for a non-elementary Möbius transformtion group $G=\langle A,B\rangle$ to be discrete.

THEOREM A (Jørgensen [2]). *Suppose that the Möbius transformations A and B generate a non-elementary discrete group. Then*

$$J(A,B):=|\mathrm{tr}^2(A)-4|+|\mathrm{tr}(ABA^{-1}B^{-1})-2|\geq 1.$$

*The lower bound* 1 *is best possible.*

DEFINITION 1.1. Let $A$ and $B$ be Möbius transformations. The *Jørgensen number* $J(A,B)$ is

$$J(A,B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2|.$$

DEFINITION 1.2. Let $G$ be a non-elementary two-generator subgroup of Möb. The *Jørgensen number* $J(G)$ for $G$ is defined as follows:

$J(G) := \inf\{J(A,B) \mid A \text{ and } B \text{ generate } G\}.$

DEFINITION 1.3. A non-elementary two-generator subgroup $G$ of Möb is a *Jørgensen group* if $G$ is a discrete group with $J(G) = 1$.

## §2. Theorems.

In this section we will theorems without proofs. Jørgensen-Kiikka [3] obtained the following theorem for Jørgensen groups.

THEOREM B (Jørgensen-Kiikka [3]). *Let* $\langle A, B\rangle$ *be a non-elementary discrete group with* $J(A,B) = 1$, *that is, a Jørgensen group. Then* $A$ *is elliptic of order at least seven or* $A$ *is parabolic.*

In this paper we only consider the case where $A$ is parabolic, that is, Jørgensen groups of parabolic type. Namely, we consider two-generator groups $G_{\mu,\sigma} = \langle A, B_{\mu,\sigma}\rangle$ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\mu,\sigma} = B_{ik,\sigma} = \begin{pmatrix} ik\sigma & -k^2\sigma - 1/\sigma \\ \sigma & ik\sigma \end{pmatrix},$$

where $k \in \mathbf{R}$ and $\sigma \in \mathbf{C} \setminus \{0\}$.

Let $C_1$ and $C_2$ be the following cylinders:

$C_1 = \{(\sigma, ik) \mid |\sigma| = 1,\ k \in \mathbf{R}\}$,

$C_2 = \{(\sigma, ik) \mid |\sigma| = 2,\ k \in \mathbf{R}\}$.

THEOREM 1 (Sato [10])

(i) *For each point inside the cylinder $C_1$, the corresponding group $G_{ik,\sigma}$ is not a Kleinian group.*

(ii) *Let $(\sigma, ik)$ be a point outside of the cylinder $C_2$. If $|k| \geq 1$, then $G_{ik,\sigma}$ is a boundary group of the Schottky space of genus two.*

(iii) Every Jørgensen group of type $G_{ik,\sigma}$ lies on the cylinder $C_1$.

By Theorem 1 we consider two-generator groups $G_{\mu,\sigma} = \langle A, B_{\mu,\sigma}\rangle$ with $\mu = ik$ ($k \in \mathbf{R}$) and $\sigma = -ie^{i\theta}$ ($0 \leq \theta < 2\pi$). For simplicity we set $B_{ik,\theta} := B_{ik,\sigma}$ and $G_{ik,\theta} = \langle A, B_{ik,\sigma}\rangle$ for $\sigma = -ie^{i\theta}$.

LEMMA 2.1. *Let $B_{ik,\theta}$ $(0 \leq \theta \leq \pi/2)$ be as in the above, and let $\bar{B}_{ik,\theta}$ be the complex conjugate of $B_{ik,\theta}$. Then $B_{ik,\pi-\theta} = -\bar{B}_{ik,\theta}^{-1}$.*

We easily see the following by Lemma 2.1.

COROLLARY 2.2. *Let $A$ and $B_{ik,\theta}$ be as in Lemma 3.1. Then $G_{ik,\theta} = \langle A, B_{ik,\theta}\rangle$ is discrete if and only if $G_{ik,\pi-\theta} = \langle A, B_{ik,\pi-\theta}\rangle$ is discrete for* $0 \leq \theta \leq \pi/2$.

LEMMA 2.3. *Let $B_{ik,\theta}$ and $G_{ik,\theta}$ be as in Lemma 3.1. Then $B_{ik,\pi+\theta} = B_{ik,\theta}$ and $G_{ik,\pi+\theta} = G_{ik,\theta}$.*

LEMMA 2.4. *A group $G_{ik,\theta}$ is a Kleinian group and so a Jørgensen group if and only if $G_{-ik,\theta}$ is a Kleinian group and so a Jørgensen group.*

This lemma follows from $B_{-ik,\theta} = B_{ik,\theta}^{-1}$. By Corollary 3.2, Lemmas 3.3 and 3.4, it suffices to consider the case of $(0 \leq \theta \leq \pi/2)$ and $k \geq 0$.

THEOREM 2 (Sato [11]). *Let $G_{ik,\theta} = \langle A, B_{ik,\theta}\rangle$ be the group generated by $A$ and $B_{ik,\theta}$.*

(i) *If $0 < \theta < \pi/6$ or $\pi/3 < \theta < \pi/2$, then $G_{ik,\theta} = \langle A, B_{ik,\theta}\rangle$ is not a Kleinian group for every $k \in \mathbf{R}$.*

(ii) *If $|k| < 1/2$, then $G_{ik,\theta} = \langle A, B_{ik,\theta}\rangle$ is not a Kleinian group for every $\theta$ $(0 \leq \theta < 2\pi)$.*

THEOREM 3 (Sato-Yamada [12]). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_k := B_{ik,1} = \begin{pmatrix} ik & -(1+k^2) \\ 1 & ik \end{pmatrix}$$

and let $G_k = \langle A, B_k\rangle$ be the group generated by $A$ and $B_k$ $(k \in \mathbf{R})$. *Then the following hold.*

(i) *In the case of $|k| > 1$, $G_k$ is a Kleinian group of the second kind, a Jørgensen group and $\Omega(G_k)/G_k$ is a single Riemann surface with signature $(0; 2, 2, 3, 3)$ for each $k$, where $\Omega(G_k)$ denotes the region of discontinuity for $G_k$.*

(ii) *In the case of $|k| = 1$, $G_k$ is a Kleinian group of the second kind, a Jørgensen group and $\Omega(G_k)/G_k$ is a single Riemann surface with signature $(0; 3, 3, \infty)$.*

(iii) *In the case of $\sqrt{3}/2 < |k| < 1$, $G_k$ is a Kleinian group of the second kind, a Jørgensen group and $\Omega(G_k)/G_k$ is a single Riemann surface with signature $(0; 3, 3, q)$ for $k$ with $k^2 = \{1 + \cos(\pi/q)\}/2, q = 4, 5, 6, \cdots$.*

(iv) *In the case of $1/2 \leq |k| \leq \sqrt{3}/2$, $G_k$ is a Kleinian group of the first kind and a Jørgensen group for $|k| = \sqrt{3}/2, \sqrt{2}/2$ or $1/2$. The volumes $V(G_{ik,1})$ of 3-orbifolds for $G_{ik,1}$ are as follows, where $L(\theta)$ is the Lobachevskiĭ function:*

$$L(\theta) = -\int_0^\theta \log|2\sin u|du.$$

(1) $V(G_{i\sqrt{3}/2,1}) = 5L(\pi/3)$.

(2) $V(G_{i\sqrt{2}/2,1}) = 2\{2L(\pi/4) - L(5\pi/12) - L(\pi/12)\}$.

(3) $V(G_{i/2,1}) = 7L(\pi/3)/2 - L(\varphi_0 + \pi/6) + L(\varphi_0 - \pi/6)$,

*where* $\varphi_0 = \sin^{-1}(1/2\sqrt{3})$.

(v) *In the case of* $0 < |k| < 1/2$, $G_k$ *is not a Kleinian group for every* $k$.

(vi) *In the case of* $k = 0$, $G_k$ *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_k)/G_k$ *is a union of two Riemann surfaces with signature* $(0; 2, 3, \infty)$.

REMARK. The group $G_{i/2,1}$ is conjugate to the Picard group in Möb and the group $G_{0,1}$ is the classical modular group.

THEOREM 4 (Sato [10]). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_\theta := B_{\sqrt{3}i/2, -ie^{i\theta}} = \begin{pmatrix} \sqrt{3}e^{i\theta}/2 & i(3e^{i\theta}/4 - e^{-i\theta}) \\ -ie^{i\theta} & \sqrt{3}e^{i\theta}/2 \end{pmatrix}$$

*and let* $G_\theta = \langle A, B_\theta \rangle$ *be the group generated by* $A$ *and* $B_\theta$ $(0 \le \theta \le \pi/2)$. *Then the following hold.*

(i) *In the case of* $\theta = \pi/6$, $G_{\pi/6}$ *has the following properties*:

(1) $G_{\pi/6}$ *is a Kleinian group of the first kind.*

(2) $G_{\pi/6}$ *is a Jørgensen group.*

(3) $V(G_{\pi/6}) = 6L(\pi/3)$, *where* $L(\theta)$ *is the Lobachevskiĭ function*:

$L(\theta) = -\int_0^\theta \log|2\sin u| du$.

(ii) *In the case of* $\theta = \pi/2$, $G_{\pi/2}$ *has the following properties*:

(1) $G_{\pi/2}$ *is a Kleinian group of the first kind.* (2) $G_{\pi/2}$ *is a Jørgensen group.*

(3) $V(G_{\pi/2}) = 2(L(\pi/6) + L(\pi/3))$.

(iii) *In the case of* $\theta = 0$, $G_0$ *has the following properties* :

(1) $G_0$ *is a Kleinian group of the second kind.*

(2) $G_0$ *is a Jørgensen group.*

(3) $\Omega(G_0)/G_0$ *is a Riemann surface with signature* $(0;2,3,\infty)$.

(iv) *If* $0<\theta<\pi/6$ *or* $\pi/3<\theta<\pi/2$, *then* $G_\theta$ *is not a Kleinian group.*

REMARKS. (1) enskip $G_{\pi/6}$ is congugate with the figure -eight knot group.

(2) Maskit [6] shows that the essentially same group as $G_0$ is discrete, that is, he shows that a group conjugate to $G_\pi$ is discrete. Our proof is different from his.

THEOREM **5 (Sato [11]).** *Let*

$$A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \quad \text{and} \quad B_\theta=\begin{pmatrix}0 & -ie^{-i\theta}\\ -ie^{i\theta} & 0\end{pmatrix}$$

*and let* $G_\theta=\langle A,B_\theta\rangle$ *be the group generated by* $A$ *and* $B_\theta$ $(0\le\theta\le\pi/2)$. *Then the following hold.*

(i) *In the case of* $\theta=0, G_0$ *has the following properties:*

(1) $G_0$ *is a Kleinian group of the second kind.*

(2) $G_{\pi/2}$ *is a Jørgensen group.*

(3) $\Omega(G_0)/G_0$ *is a single Riemann surface with signature* $(0;2,3,\infty)$.

(ii) *In the case of* $\theta=\pi/2$, $G_{\pi/2}$ *has the following properties:*

(1) $G_{\pi/2}$ *is a Kleinian group of the second kind.*

(2) $G_0$ *is a Jørgensen group.*

(3) $\Omega(G_{\pi/2})/G_{\pi/2}$ *is a union of two Riemann surfaces with signature* $(0;2,3,\infty)$.

(iii) *If* $0<\theta<\pi/6, \pi/6<\theta<\pi/4, \pi/4<\theta<\pi/3$ *or* $\pi/3<\theta<\pi/2$, *then* $G_\theta$ *is not a Kleinian group and so not a Jørgensen group.*

THEOREM **6 (Sato [11]).** *Let*

$$A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \quad \text{and} \quad B_k:=B_{ik,-i}=\begin{pmatrix}k & i(k^2-1)\\ -i & k\end{pmatrix}$$

*and let* $G_k = \langle A, B_k \rangle$ *be the group generated by* $A$ *and* $B_k$ $(k \in \mathbf{R})$. *Then the following hold.*

(i) *In the case of* $|k| > 1$, $G_k$ *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_k)/G_k$ *is two Riemann surfaces with signatures* $(0; 2, 2, 2, 3)$ *and* $(0; , 2, 3, \infty)$ *for each* $k$, *where* $\Omega(G_k)$ *denotes the region of discontinuity for* $G_k$.

(ii) *In the case of* $|k| = 1$, $G_k$ *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_k)/G_k$ *is two Riemann surfaces with signature* $(0; 2, 3, \infty)$.

(iii) *In the case of* $\sqrt{3}/2 < |k| < 1$, $G_k$ *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_k)/G_k$ *is two Riemann surfaces with signatures* $(0; 2, 3, q)$ *and* $(0; 2, 3, \infty)$ *for* $k$ *with* $k^2 = \{1 + \cos(\pi/q)\}/2, q = 4, 5, 6, \cdots$.

(iv) *In the case of* $1/2 \leq |k| \leq \sqrt{3}/2$, $G_k$ *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_k)/G_k$ *is a Riemann surface with signature* $(0; 2, 3, \infty)$ *for* $|k| = \sqrt{3}/2$.

(v) *In the case of* $0 < |k| < 1/2$, $G_k$ *is not a Kleinian group and not a Jørgensen group for every* $k$.

(vi) *In the case of* $k = 0$, $G_k$ *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_k)/G_k$ *is a Riemann surface with signature* $(0; 2, 3, \infty)$.

CORRECTION The part (from lines 23 through 31 on page 2 in the Introduction in the previous paper [10]) contains mistake, which gives no effect the paper. It should be changed as follows.

The family of groups, $P = \{G_\sigma = \langle A, B_{1/\sigma,\sigma} \rangle \mid G_\sigma$ is a discrete group, $\sigma \in \mathbf{C} \setminus \{0\}\}$ contains the Riley slice $RS$ (see Keen and Series [14] for the definition of the Riley slice). If $\langle A, B_{1/\sigma,\sigma} \rangle$ is a group in $P$, then $J(A, B_{1/\sigma,\sigma}) = |\sigma|^2$. As far as the author knows, it is unknown whether or not $J(A, B_{1/\sigma,\sigma})$ achieves the infimum over the whole group $P$. It is easily seen that $\inf\{J(G) \mid G \in P\} = 1$, since

$J(A, B_{1/\sigma,\sigma}) = 1$ for $\sigma = 1$, that is, in this case the group is the classical modular group. Furthermore we easily see that

$$1 \leq \inf\{J(G) \mid G \in RS\} \leq 2.$$

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