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## Local Langlands correspondence について (紹介)

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局所 Langlands 予想が、標数 0 の場合にも (一般の次数  $n$  に対して) 最近 M. Harris と R. Taylor [HT] によって証明され、G. Henniart のより簡単な証明 [He] も出ましたので、正確に何が証明されたのか、という所を中心に紹介させていただきます。尚、ここでは紹介し切れませんが、[HT] を見ると過去の日本人の仕事 - 特に志村五郎、井草準一、本田平 (故人)、藤原一宏各氏によるもの - も重要なところで使われていることがわかります。

$F$  を標数 0 の局所体 ( $\mathbb{Q}_p$  の有限次拡大) とするとき、主要結果は、大まかに云うと、1 対 1 対応

$Gal(\bar{F}/F)$  の  $n$  次複素表現  $\leftrightarrow GL_n(F)$  の既約表現 (一般に無限次元)

が存在する、という事ですが、左辺 (Galois side) で  $Gal(\bar{F}/F)$  は Weil 群  $W_F$  に置換え、その表現は "Φ-semisimple なもの" とし、右辺 (Automorphic side) では smooth な表現としなくてはなりません。それらの定義から復習します。その前に主な文献を列挙しましょう。

### 1. Main references

《“決定的”文献》

[HT] M. Harris, R. Taylor, "On the geometry and cohomology of some simple Shimura varieties" (Preprint, 1998, 99) (99, Aug 30 version を参照しました)

[He] G. Henniart, "Une preuve simple des conjectures des Langlands pour  $GL(n)$  sur un corps  $p$ -adique", To appear in Inv. Math.

《基本的文献》

(Weil(-Deligne) 群については)

[De] P. Deligne, "Les constantes des équations fonctionnelles des fonctions  $L$ " In: "Modular functions of one variable" II, SLN 349.

[Ta] J. Tate, "Number Theoretic Background" AMS Proc. Symp. Pure Math. 33-2 (1979).

(Automorphic representations については) AMS Proc. Symp. Pure Math. 33-1 の Cartier 等の Survey, 及び

[JPS] H. Jacquet, I. I. Piatetskii-Shapiro, J. Shalika, “Rankin-Selberg convolutions”, Amer. J. Math. **105** (1983).

[Z] A. V. Zelevinskii, “Induced representations of reductive  $p$ -adic groups II” Ann. Sci. ENS **13** (1980), 等.

( $n = 2$  源泉)

[JL] H. Jacquet, R. P. Langlands, “Automorphic forms on  $GL(2)$ ” SLN 114.

(比較的最近の部分的結果)

[He 1] G. Henniart, “La conjecture de Langlands locale numérique pour  $GL(n)$ ” Ann. Sci. ENS **21** (1988).

[He 2] G. Henniart, “Caractérisation de la correspondance de Langlands locale par les facteurs  $\varepsilon$  de paires”, Inv. Math. **113** (1993).

[Ha 1] M. Harris, “The local Langlands conjecture for  $GL(n)$  over a  $p$ -adic field,  $n < p$ ”, Inv. Math. **134** (1998).

(Char  $p > 0$  の場合の証明は)

[LRS] G. Laumon, M. Rapoport, U. Stuhler, “ $\mathcal{D}$ -elliptic sheaves and the Langlands correspondence”, Inv. Math. **113** (1993).

## 2. Main definitions

《Galois side》

$p$  : a prime number,  $[F : \mathbb{Q}_p] < \infty$  (a commutative field),  $\bar{F}$ : an algebraic closure of  $F$ ,  $q = \#$  (the residue field of  $F$ ).

- $W_F$  (the Weil group of  $F$ ) :=  $\{w \in Gal(\bar{F}/F); w \text{ acts as } * \rightarrow *^{|w|} \text{ on the residue field, with some } |w| \in \mathbb{Z}\}$   
 $= \Phi^{\mathbb{Z}} \rtimes I_F$  (locally compact),

$I_F$  : the inertia group (open compact),

$\Phi$ : a geometric Frobenius element, i.e.,  $\Phi \in W_F$  s.t.  $|\Phi| = q^{-1}$ .

Remark.  $W_F$  is a dense subgroup of  $\text{Gal}(\bar{F}/F)$ , but we consider  $W_F$  as a topological group in such a way that  $I_F$  is open (and hence  $W_F/I_F(\cong \mathbb{Z})$  is discrete).

- The abelianization  $W_F^{ab}$  of  $W_F$  is canonically isomorphic with the multiplicative group  $F^\times$ , via local classfield theory;

$$W_F^{ab} \xrightarrow{\sim} F^\times.$$

The induced homomorphism  $W_F \rightarrow F^\times$  maps  $\Phi$  to a prime element of  $F^\times$ , and  $|w|$  on  $W_F$  corresponds with the standard valuation  $|\cdot|_F$  of  $F^\times$ .

- $\rho : W_F \rightarrow GL_n(\mathbb{C})$  : a continuous representation, i.e.,  $\rho$  is a group homomorphism such that its restriction to  $I_F$  factors through a finite quotient.

$\rho$  : completely reducible  $\leftrightarrow \rho(\Phi)$  semi-simple for one  $\Phi \leftrightarrow$  for all  $\Phi$ .

- Def  $\Phi$ -semi-simple representation of  $W'_F = W_F \rtimes \mathbb{G}_a$  (the Weil-Deligne group) of deg  $n := \{(\rho, N) | \rho : W_F \rightarrow GL_n(\mathbb{C}), \text{continuous completely reducible representation}, N \in M_n(\mathbb{C}), \text{nilpotent}, \rho(w)N\rho(w)^{-1} = |w|N \ (\forall w \in W_F)\}$

Remark (i)  $\oplus, \otimes$  defined; (ii)  $\rho$  : irreducible  $\rightarrow N = 0$ ; (iii) each  $l$ -adic ( $l \neq p$ ) representation  $\rho_l$  of  $W_F$  gives rise to a  $\Phi$ -semi-simple representation  $(\rho, N)$  of  $W'_F$ . ( $N$  is determined by its restriction to  $\mathbb{Z}_l \subset I_F$ , and  $\rho$  is a certain modification of  $\rho_l$ .)

- Examples of  $(\rho, N)$ .

(I) Each continuous homomorphism  $\chi : F^\times \rightarrow \mathbb{C}^\times$  (a “quasi-character” of  $F^\times$ ) can be regarded as a 1-dimensional representation of  $W_F$ , via  $W_F \rightarrow W_F^{ab} \xrightarrow{\sim} F^\times$ . The unramified quasi-characters are those of the form  $\chi_s$  ( $s \in \mathbb{C}$ ) defined by  $\chi_s(a) = |a|_F^s$ .

(II) Given any irreducible  $\rho_0 : W_F \rightarrow GL_{n_0}(\mathbb{C})$  and an integer  $d \geq 1$ ,

the “Steinberg representation”  $St_d(\rho_0) = (\rho, N)$  is by definition:

$$\rho : W_F \ni w \rightarrow \begin{pmatrix} \rho_0(w) & & & 0 \\ & \rho_0(w)|w|^{-1} & & \\ & & \cdots & \\ & & & \rho_0(w)|w|^{-(d-1)} \\ & 0 & & \end{pmatrix} \in GL_{n_0 d}(\mathbb{C}),$$

$$N = \begin{pmatrix} 0 & I_{n_0} & & 0 \\ & 0 & \cdots & \\ & & \cdots & I_{n_0} \\ 0 & & & 0 \end{pmatrix}$$

It is known that a  $\Phi$ -semisimple representation of  $W'_F$  is of the form  $St_d(\rho_0)$  if and only if it is indecomposable, and that  $St_d(\rho_0)$  determines both  $d$  and  $\rho_0$ .

(III) Each  $\Phi$ -semisimple representation of  $W'_F$  can be decomposed uniquely as a direct sum of indecomposable representations.

Thus,  $(\rho, 0)$  with  $\rho$ : irreducible are the most fundamental  $\Phi$ -semi-simple representations of  $W'_F$ .

«Automorphic representation side»

$n \geq 1$ ,  $G_n = GL_n(F)$  (locally compact groups).

• Def A smooth representation  $(\pi, V)$  of  $G_n$  is:

$V$ : a complex vector space (in most cases infinite dimensional),

$\pi : G_n \rightarrow \text{Aut}_{\mathbb{C}} V$ , a group homomorphism s.t. the stabilizer of each  $v \in V$  in  $G_n$  is open.

All representations will be assumed smooth.

• Def  $\pi$ : admissible  $\leftrightarrow$  for each open compact subgroup  $U \subset G_n$ ,  $V^U := \{v \in V | \pi(u)v = v (\forall u \in U)\}$  is finite dimensional.

$\pi$ : irreducible  $\rightarrow$  admissible (in the case of representations of  $G_n$ ), and

$\lambda \in \text{End}_{\mathbb{C}} V$ ,  $\lambda \circ \pi(g) = \pi(g) \circ \lambda (\forall g \in G_n) \Rightarrow \lambda$ : scalar.

esp.  $z \in F^\times \cdot I_n$  (the center of  $G_n$ )  $\Rightarrow \pi(z)$ : scalar.

This defines the central character  $\omega_\pi : F^\times \rightarrow \mathbb{C}^\times$  of  $\pi$ .

- The smooth dual  $(\pi^\vee, V^\vee)$  of  $(\pi, V)$  for  $\pi$ : irreducible.  $V^\vee := \{\text{linear } V \rightarrow \mathbb{C} \text{ which annihilates the unique } G_n\text{-stable complement of } V^U \text{ for some } U\}$ ,

$$\langle \pi(g)v, v^\vee \rangle = \langle v, \pi^\vee(g)^{-1}v^\vee \rangle$$

$$(\pi^\vee)^\vee = \pi.$$

- Def  $\pi$ : an irreducible smooth representation is called supercuspidal, if for any  $v \in V, v^\vee \in V^\vee$ , the support of  $\langle \pi(g)v, v^\vee \rangle$  (a  $\mathbb{C}$ -valued function on  $G_n$ ) is compact modulo the center  $F^\times \cdot I_n$ .

- Induced representations.  $P \subset_{\text{closed subgp}} G = G_n, (\pi_P, U)$ : representation of  $P$ , given.

Then:  $\text{Ind}_{P \uparrow G}(\pi_P, U) = (\pi_G, V)$ : defined by

$$V = \{f : U\text{-valued fctns on } G; f(pg) = \delta_P^{1/2}(p)\pi_P(p)f(g)\}.$$

$$(p \in P, g \in G)$$

$(\pi_G(g)f)(x) = f(xg)$ , where  $\delta_P: G \rightarrow (\mathbb{R}^+)^\times$  is a character defined by the formula:

$$\delta_P(p)^{-1} \times (\text{right inv. Haar measure of } P) = \text{left inv. Haar measure of } P.$$

- Examples.

(I) Each quasi-character  $\chi: F^\times \rightarrow \mathbb{C}^\times$  can be regarded as an irreducible representation of  $GL_1(F)$ . It is supercuspidal. (But the composition of  $\chi$  with the determinant  $\det: GL_n(F) \rightarrow F^\times$  is not supercuspidal if  $n > 1$ .)

(II) Given any irreducible supercuspidal representation  $\rho_0$  of  $GL_{n_0}(F)$  and an integer  $d \geq 1$ , put  $n = n_0d$  and let  $P$  be the parabolic subgroup of  $GL_n(F)$  generated by the block-diagonal matrices  $GL_{n_0}(F) \times \cdots \times GL_{n_0}(F)$  ( $d$  copies) and the upper triangular unipotent matrices. Consider the representation

$$(\rho_0 \otimes |\det|_F^{d-1}) \otimes \cdots \otimes (\rho_0)$$

of

$$GL_{n_0}(F) \times \cdots \times GL_{n_0}(F).$$

Regard this as a representation  $\rho_0^{(d,P)}$  of  $P$  by passage to the quotient, and take  $\text{Ind}_{P \uparrow G} \rho_0^{(d,P)}$ . Then, this has a unique irreducible subrepresentation, called  $St_d(\rho_0)$ . It is known that an irreducible representation of  $G_n$  is of the form  $St_d(\rho_0)$  if and only if it is quasi-square integrable i.e., a tensor product of a quasi-character of  $F^\times$  (regarded as a 1-dimensional representation of  $G_n$  via the determinant) and a square integrable (modulo the center) representation of  $G_n$ . They are also precisely those irreducible representations of  $G_n$  that correspond naturally with some irreducible representation of  $D_n^\times$ , where  $D_n$  is the central division algebra over  $F$  of degree  $n^2$  and the Hasse invariant  $1/n$  (Deligne-Kazhdan-Vigneras).  $St_d(\rho_0)$  determines each of  $d$  and  $\rho_0$  uniquely.

(III) An arbitrary irreducible representation  $\pi$  of  $G_n$  can be expressed uniquely as the “Langlands sum”  $\boxplus$  of representations of the form treated in (II).

The sum  $\rho_1 \boxplus \cdots \boxplus \rho_r$ , where each  $\rho_i$  is of the type treated in (II), is defined by using the induced representation  $\text{Ind}_{P \uparrow G}(\rho_1 \otimes \cdots \otimes \rho_r)$  analogous to the situation in (II). If  $n = n_1 + \cdots + n_r$  and  $\rho_i$  is a representation of  $GL_{n_i}(F)$  ( $1 \leq i \leq r$ ),  $P$  is now the semi-direct product of

$$GL_{n_1}(F) \times \cdots \times GL_{n_r}(F) \subset GL_n(F)$$

and the upper triangular unipotent matrices. But the ordering of  $\rho_1, \dots, \rho_r$  and whether we choose the unique irreducible subrepresentation of  $\text{Ind}_{P \uparrow G}(\rho_1 \otimes \cdots \otimes \rho_r)$  or the quotient representation is a delicate point (whose choice should be reversed if we take as  $P$  the lower block-triangular matrices). I could not find an appropriate explicit reference for this, but the content of [Z] shows that it should be, here, the unique irreducible, quotient of  $\text{Ind}_{P \uparrow G}(\rho_1 \otimes \cdots \otimes \rho_r)$  if the ordering of  $\rho_1, \dots, \rho_r$  is admissible in the following sense. Write  $\rho_i = St_{d_i}(\rho_{i,0})$ . The ordering is admissible if for each  $i < j$ , there does not exist any positive integer  $a$  such that

$$\rho_{j,0} = \rho_{i,0} \otimes |\det|^a, \quad 1 + d_i - d_j \leq a \leq d_i.$$

Admissible orderings exist, and the resulting (unique irreducible) quotient of the induced representation is independent of the choice of such orderings.

### 3. The main result ([HT] [He])

**Theorem** (Harris-Taylor, Henniart)  $\left( \begin{array}{l} n = 1 : \quad \text{classical, } n = 2 : \dots, \text{ Kutzko} \\ n = 3 \quad \quad \text{(almost) Henniart, } n < p: \text{ Harris} \\ \text{char } p > 0 \quad \dots \text{Laumon-Rapoport-Stuhler} \end{array} \right)$

There exists a unique system of bijections  $LG_n (n \geq 1)$  satisfying (1)~(5) below.

$\{\rho' = (\rho, N); \quad \rho : W_F \rightarrow GL_n(\mathbb{C})$  continuous completely reducible representation,  
 $N \in M_n(\mathbb{C})$  nilpotent, s.t.  $\rho(w)N\rho(w)^{-1} = |w|N(\forall w \in W_F).\}$  /  $\simeq$

$\xleftrightarrow{LG_n} \{ \text{smooth } \underline{\text{irreducible}} \text{ representations } \pi \text{ of } GL_n(F) \} / \simeq.$

This system of bijection also satisfies:

$$\begin{aligned} \rho : \text{irreducible} (\rightarrow N = 0) &\leftrightarrow \pi : \text{supercuspidal}, \\ St_d(\rho) &\leftrightarrow St_d(\pi), \\ (\text{i.e., } \rho : \text{indecomposable}) &\leftrightarrow \pi : \text{quasi square-integrable)} \\ \oplus St_d(\rho_i) &\leftrightarrow \boxplus St_d(\pi_i). \end{aligned}$$

(1)  $n = 1$ ,  $LG_1$  : local classfield theory correspondence via  $W_F^{ab} \xrightarrow{\sim} F^\times$ .

(2)  $\rho' \leftrightarrow \pi \Rightarrow \det \rho \leftrightarrow \omega_\pi,$   
 $\quad \quad \quad \parallel$   
 $\quad \quad \quad (\rho, N)$

(3)  $\Rightarrow (\chi \circ ab) \otimes \rho' \leftrightarrow (\chi \circ \det) \otimes \pi,$

(4)  $\Rightarrow (\rho')^\vee \leftrightarrow \pi^\vee,$

(5)  $\rho'_i \leftrightarrow \pi_i \Rightarrow L(\rho'_1 \otimes \rho'_2, s) = L(\pi_1 \times \pi_2, s),$

$$\varepsilon(\rho'_1 \otimes \rho'_2, s, \psi) = \varepsilon(\pi_1 \times \pi_2, s, \psi),$$

( $\psi : F \rightarrow \mathbb{C}^\times$  : a non-trivial additive character).



As for these local  $L$ -functions and local  $\varepsilon$ -factors, cf. e.g. [Ta] for those on the left side, and [JPS] for those on the right side. Each  $L(*, s)$  is a finite product of  $(1 - \chi(\pi)q^{-s})^{-1}$ , where  $\pi$  is a prime element of  $F$  and  $\chi$  runs over a finite number of unramified quasi-characters of  $F^\times$  determined by  $*$ . Each  $\varepsilon(*, s, \phi)$  is a function of  $s$  of the form  $a \cdot \exp(bs)$  ( $a \in \mathbb{C}^\times, b \in \mathbb{C}$ ), where  $b$  is essentially the conductor of  $*$ .

The basic questions:

- (1) What should correspond to  $\otimes$  of Galois representations on the automorphic side?
- (2) What should correspond to  $\text{Ind}_{F' \uparrow F}(\chi)$  (more precisely, from  $W_{F'}$  to  $W_F$ ) on the automorphic side, where  $\chi$  is any quasi-character of  $(F')^\times$ ?

They seem still open.

#### 4. On proofs

Both in [HT] and [He], the proofs are based on constructions of global correspondences. They construct appropriate automorphic representations on  $GL_n$  over some  $CM$ -fields associated with some Shimura varieties, find corresponding global Galois representations, and compare their local factors. In [HT], study of Shimura varieties at primes with bad reductions are used, but in [He], good reductions are sufficient for the purpose. The proof in [He] is simpler, but [HT] contains more information on the global correspondences.

In [HT], the authors start with the big universal  $l$ -adic representation of  $D_F^\times \times GL_n(F) \times W_F$  constructed by Drinfeld, and study its decomposition in greater detail. In [He], by using the Brauer theorem on representations of finite groups, the author reduces the problem to construction of a supercuspidal representation of  $GL_m(F)$  corresponding to those irreducible representations of  $W_F$  that are induced from a (multiplicative) quasi-character of a finite extension of  $F$  (of degree  $m$ ). Special types of Shimura varieties over a  $CM$ -field (in this case, a composite of a totally real field and an imaginary quadratic field, which has  $F$  as a non-archimedean completion),

and cohomology groups of certain  $l$ -adic sheaves on such varieties are used effectively.