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Berezin Transforms and Laplace-Beltrami Operators on Homogeneous Siegel Domains

— commutativity, symmetry of the domain and a Cayley transform —

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1. Preliminaries

Homogeneous Siegel domains are described in terms of normal j-algebras (cf. [15]), of which we are going to give the definition. Let $\mathfrak{g}$ be a split solvable Lie algebra, J a linear operator on $\mathfrak{g}$ with $J^2 = -I$ and ω a linear form on $\mathfrak{g}$. Then the triple $(\mathfrak{g}, J, \omega)$ is called a *normal j-algebra* if

(1.1) $\quad [Jx, Jy] = [x, y] + J[Jx, y] + J[x, Jy] \quad$ (for all $x, y \in \mathfrak{g}$),

(1.2) $\quad \langle x \,|\, y \rangle_\omega := \langle\, [Jx, y], \omega \,\rangle$ defines a J-invariant inner product on $\mathfrak{g}$.

We describe here some basic facts about normal j-algebras following [15] and [17] (see also [16]). Let $(\mathfrak{g}, J, \omega)$ be a normal j-algebra. Let $\mathfrak{n} := [\mathfrak{g}, \mathfrak{g}]$ be the derived algebra of $\mathfrak{g}$, and $\mathfrak{a}$ the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{g}$ relative to the inner product $\langle \cdot \,|\, \cdot \rangle_\omega$. Evidently we have $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$. Moreover, $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{g}$ such that $\mathrm{ad}(\mathfrak{a})$ consists of semisimple operators on $\mathfrak{g}$. For every $\alpha \in \mathfrak{a}^*$ we set

$$\mathfrak{n}_\alpha := \{x \in \mathfrak{n}\,;\; [h, x] = \langle\, h, \alpha \,\rangle x \quad \text{for all } h \in \mathfrak{a}\}.$$

Take all $\alpha \in \mathfrak{a}^*$ such that $\mathfrak{n}_\alpha \neq \{0\}$ and $J\mathfrak{n}_\alpha \subset \mathfrak{a}$, and number them as $\alpha_1, \ldots, \alpha_r$. We have $\dim \mathfrak{a} = r$ and $\dim \mathfrak{n}_{\alpha_k} = 1$ for every k. The number r is called the *rank* of the normal j-algebra $\mathfrak{g}$. We can reorder $\alpha_1, \ldots, \alpha_r$, if necessary, so that all the α such that $\mathfrak{n}_\alpha \neq \{0\}$ (such an α is called a *root* of the normal j-algebra) are of the following form (some roots might be missing):

$$\tfrac{1}{2}(\alpha_m + \alpha_k) \quad (1 \leqq k < m \leqq r), \qquad \tfrac{1}{2}(\alpha_m - \alpha_k) \quad (1 \leqq k < m \leqq r),$$
$$\tfrac{1}{2}\alpha_k \quad (1 \leqq k \leqq r), \qquad \alpha_k \quad (1 \leqq k \leqq r).$$

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We note that if α, β are distinct roots, then $\mathfrak{n}_\alpha$ is orthogonal to $\mathfrak{n}_\beta$. Put

$$\mathfrak{g}(0) := \mathfrak{a} \oplus \sum_{m>k} \mathfrak{n}_{(\alpha_m-\alpha_k)/2}, \qquad \mathfrak{g}(1/2) := \sum_{i=1}^{r} \mathfrak{n}_{\alpha_i/2},$$
$$\mathfrak{g}(1) := \sum_{i=1}^{r} \mathfrak{n}_{\alpha_i} \oplus \sum_{m>k} \mathfrak{n}_{(\alpha_m+\alpha_k)/2}.$$

Understanding $\mathfrak{g}(i) = 0$ for $i > 1$, we have $[\,\mathfrak{g}(i), \mathfrak{g}(j)\,] \subset \mathfrak{g}(i+j)$. Moreover

$$J\mathfrak{n}_{(\alpha_m-\alpha_k)/2} = \mathfrak{n}_{(\alpha_m+\alpha_k)/2} \quad (m > k), \qquad J\mathfrak{n}_{\alpha_i/2} = \mathfrak{n}_{\alpha_i/2} \quad (1 \leqq i \leqq r),$$

so that $J\mathfrak{g}(0) = \mathfrak{g}(1)$ and $J\mathfrak{g}(1/2) = \mathfrak{g}(1/2)$. Taking $E_i \in \mathfrak{n}_{\alpha_i}$ $(i = 1, \dots, r)$ such that $\alpha_k(JE_i) = \delta_{ki}$, we put $H_i := JE_i \in \mathfrak{a}$ and

$$H := H_1 + \cdots + H_r, \qquad E := E_1 + \cdots + E_r. \tag{1.3}$$

We write down here the constants used frequently in this note:

$$\begin{aligned} n_{mk} &:= \dim_{\mathbb{R}} \mathfrak{n}_{(\alpha_m-\alpha_k)/2} = \dim_{\mathbb{R}} \mathfrak{n}_{(\alpha_m+\alpha_k)/2} \quad (1 \leqq k < m \leqq r), \\ b_i &:= \frac{1}{2} \dim_{\mathbb{R}} \mathfrak{n}_{\alpha_i/2} \quad (1 \leqq i \leqq r), \\ d_j &:= 1 + \frac{1}{2}\Big(\sum_{k>j} n_{kj} + \sum_{i<j} n_{ji}\Big) \qquad (1 \leqq j \leqq r). \end{aligned} \tag{1.4}$$

Let $G = \exp \mathfrak{g}$ be the connected and simply connected Lie group corresponding to $\mathfrak{g}$. Note that $\mathfrak{g}(0)$ is a Lie subalgebra of $\mathfrak{g}$. We denote by $G(0)$ the corresponding subgroup $\exp \mathfrak{g}(0)$ of G. The group $G(0)$ acts on $V := \mathfrak{g}(1)$ by adjoint action. Let Ω be the $G(0)$-orbit through E. By [17, Theorem 4.15] Ω is a regular open convex cone in V, and $G(0)$ acts on Ω simply transitively. Being invariant under J, the subspace $\mathfrak{g}(1/2)$ is considered as a *complex* vector space by means of $-J$. We shall write this complex vector space by U. We put $W := V_{\mathbb{C}}$, the complexification of V. The conjugation of W relative to the real form V is written as $w \mapsto w^*$. The real bilinear map Q defined by

$$Q(u, u') := \frac{1}{2}\big([Ju, u'] - i[u, u']\big) \qquad (u, u' \in \mathfrak{g}(1/2))$$

turns out to be a complex sesqui-linear (complex linear in the first variable and antilinear in the second) Hermitian map $U \times U \to W$ which is Ω-positive. This means that

$$Q(u', u) = Q(u, u')^* \ \ (u, u' \in U), \quad Q(u, u) \in \overline{\Omega} \setminus \{0\} \ \text{ for all } u \in U \setminus \{0\}.$$

With these data we define the *Siegel domain* D corresponding to the normal j-algebra $(\mathfrak{g}, J, \omega)$ to be

$$D := \{(u, w) \in U \times W \; ; \; w + w^* - Q(u, u) \in \Omega\}.$$

Note that we take a generalized *right* half plane rather than a more familiar upper half plane.

Consider the Lie subalgebra $\mathfrak{n}_D := \mathfrak{g}(1) + \mathfrak{g}(1/2)$. It is at most 2-step nilpotent. Let $N_D = \exp \mathfrak{n}_D$ be the corresponding connected and simply connected nilpotent Lie group contained in G. We write the elements of N_D by $n(a, b)$ $(a \in \mathfrak{g}(1),\ b \in \mathfrak{g}(1/2))$. The group N_D acts on D by

$$n(a, b) \cdot (u, w) = \big(u + b,\ w + ia + \tfrac{1}{2}Q(b, b) + Q(u, b)\big) \qquad ((u, w) \in D). \tag{1.5}$$

On the other hand, the adjoint action of $G(0)$ on $\mathfrak{g}(1/2)$ commutes with J. This implies that $G(0)$ acts on U complex-linearly. Moreover the adjoint action of $G(0)$ on $V = \mathfrak{g}(1)$ extends complex-linearly to W, so that $G(0)$ acts on D complex-linearly. Hence $G = N_D \rtimes G(0)$ acts on D simply transitively. To see this more explicitly, put $\mathsf{e} := (0, E) \in D$. Then given $z = (u, w) \in D$, we can find a unique $h \in G(0)$ satisfying $hE = \operatorname{Re} w - Q(u, u)/2$. Taking $n = n(\operatorname{Im} w, u) \in N_D$, we see by (1.5) that $z = nh \cdot \mathsf{e}$.

For every $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$ let $\chi_{\mathbf{s}}$ be the one-dimensional representation of $A := \exp \mathfrak{a}$ defined by

$$\chi_{\mathbf{s}}\Big(\exp \sum_k t_k H_k\Big) = \exp\Big(\sum_k s_k t_k\Big) \qquad (t_1, \ldots, t_r \in \mathbb{R}).$$

Let $N := \exp \mathfrak{n}$. It is clear that $G = N \rtimes A$. We extend $\chi_{\mathbf{s}}$ to a one-dimensional representation of G by defining $\chi_{\mathbf{s}}(n) = 1$ for $n \in N$. Let us define functions $\Delta_{\mathbf{s}}$ $(\mathbf{s} \in \mathbb{C}^r)$ on Ω by $\Delta_{\mathbf{s}}(hE) = \chi_{\mathbf{s}}(h)$ $(h \in G(0))$. Evidently it holds that

$$\Delta_{\mathbf{s}}(hx) = \chi_{\mathbf{s}}(h)\Delta_{\mathbf{s}}(x) \qquad (h \in G(0),\ x \in \Omega). \tag{1.6}$$

We know that $\Delta_{\mathbf{s}}$ extends to a holomorphic function on the tube domain $\Omega + iV$ (cf. for example [7, Corollary 2.5]).

For $h \in G(0)$, let $\operatorname{Ad}_{\mathfrak{g}(1)}(h) := (\operatorname{Ad} h)|_{\mathfrak{g}(1)}$. Moreover let $\operatorname{Ad}_U(h)$ stand for the *complex* linear operator on U defined by the adjoint action of $h \in G(0)$ on $\mathfrak{g}(1/2)$, and $\det \operatorname{Ad}_U(h)$ its determinant as a complex linear operator. Then, with $\mathbf{d} := (d_1, \ldots, d_r)$ and $\mathbf{b} := (b_1, \ldots, b_r)$, we have for $h \in G(0)$

$$\det \operatorname{Ad}_{\mathfrak{g}(1)}(h) = \chi_{\mathbf{d}}(h), \qquad |\det \operatorname{Ad}_U(h)|^2 = \chi_{\mathbf{b}}(h). \tag{1.7}$$

By [6, §5] or [18, §II.6], it is known that D has a Bergman kernel κ. If $\mathrm{Hol}(D)$ denotes the Lie group of the holomorphic automorphisms of D, then κ satisfies

$$(1.8)\qquad \kappa(z_1,z_2)=\kappa(g\cdot z_1,\ g\cdot z_2)\det g'(z_1)\overline{\det g'(z_2)}\qquad (g\in \mathrm{Hol}(D),\ z_1,z_2\in D),$$

where $g'(z)$ is the complex Jacobian map of g at $z\in D$. The description of the simple transitive action of G on D together with the property (1.7) and (1.8) shows

$$(1.9)\qquad \kappa(z_1,z_2)=C\cdot\Delta_{-2\mathbf{d}-\mathbf{b}}\big(w_1+w_2^*-Q(u_1,u_2)\big)\qquad (z_j=(u_j,w_j)\in D)$$

with $C=\kappa(\mathsf{e},\mathsf{e})\Delta_{2\mathbf{d}+\mathbf{b}}(2E)>0$. We put $\eta:=\Delta_{-2\mathbf{d}-\mathbf{b}}$ in what follows for simplicity.

2. Cayley transform

Let D_v be the directional derivative in the direction $v\in V$ given by

$$D_vf(x)=\frac{d}{dt}f(x+tv)\Big|_{t=0}.$$

For every $x\in\Omega$ we define $\mathcal{I}(x)\in V^*$ to be $-\nabla\log\eta(x)$, that is,

$$\langle v,\mathcal{I}(x)\rangle=-D_v\log\eta(x)\qquad (v\in V).$$

$\mathcal{I}$ is called the *pseudoinverse map*. By [3, §2], $\mathcal{I}$ gives a diffeomorphism of Ω onto the dual cone Ω^* in V^*, where

$$\Omega^*:=\{\xi\in V^*\ ;\ \langle x,\xi\rangle>0\quad\text{for all } x\in\overline{\Omega}\setminus\{0\}\}.$$

The group $G(0)$ acts also on V^* by the coadjoint action: $h\cdot\xi=\xi\circ h^{-1}$, where $h\in G(0)$ and $\xi\in V^*$. It is easy to show by using (1.6) that $\mathcal{I}$ is $G(0)$-equivariant:

$$\mathcal{I}(hx)=h\cdot\mathcal{I}(x)\qquad (h\in G(0),\ x\in\Omega).$$

In particular, $\mathcal{I}(\lambda x)=\lambda^{-1}\mathcal{I}(x)$ for all $\lambda>0$, and $G(0)$ acts on Ω^* simply transitively. Moreover, $\mathcal{I}$ can be extended to a rational map $W\to W^*$ [4, Satz I.2.3].

In order to find an inverse map of $\mathcal{I}$, we need to dualize the above matters concerning $\mathcal{I}$. First we define $E_1^*,\dots,E_r^*\in V^*$ by

$$\Big\langle\sum_{j=1}^r x_jE_j+\sum_{m>k}X_{mk},\ E_i^*\Big\rangle=x_i\qquad (x_j\in\mathbb{R},\ X_{mk}\in\mathfrak{n}_{(\alpha_m+\alpha_k)/2}),$$

and for every $\mathbf{s}=(s_1,\dots,s_r)\in\mathbb{R}^r$,

$$E_{\mathbf{s}}^*:=s_1E_1^*+\cdots+s_rE_r^*\in V^*.$$

We can show that $\mathcal{I}(E)=E_{2\mathbf{d}+\mathbf{b}}^*$. Next we put $\mathbf{s}^*:=(s_r,\dots,s_1)$ and set

$$\chi_{\mathbf{s}}^*:=\chi_{-\mathbf{s}^*},\qquad \Delta_{\mathbf{s}}^*(h\cdot E_{2\mathbf{d}+\mathbf{b}}^*):=\chi_{\mathbf{s}}^*(h)\quad (h\in G(0)).$$

$\Delta_{\mathbf{s}}^*$ is a function on Ω^* such that $\Delta_{\mathbf{s}}^*(h\cdot\xi) = \chi_{\mathbf{s}}^*(h)\Delta_{\mathbf{s}}^*(\xi)$ for $h \in G(0)$ and $\xi \in V^*$. We define $\eta^* := \Delta^*_{-2\mathbf{d}^*-\mathbf{b}^*}$ and

$$\langle \mathcal{I}^*(\xi), f\rangle := -D_f \log \eta^*(\xi) \qquad (\xi \in \Omega^*,\ f \in V^*).$$

Thus $\mathcal{I}^*(\xi) \in V$ and $\mathcal{I}^*$ gives a diffeomorphism of Ω^* onto Ω. Moreover, $\mathcal{I}^*$ is $G(0)$-equivariant, that is, $\mathcal{I}^*(h\cdot\xi) = h(\mathcal{I}^*(\xi))$ for any $h \in G(0)$. We can prove that $\mathcal{I}^*$ is extended to a rational map $W^* \to W$.

Proposition 2.1. $\mathcal{I}^* = \mathcal{I}^{-1}$.

Theorem 2.2 ([11]). (1) *$\mathcal{I}$ is holomorphic on $\Omega + iV$, and $\mathcal{I}^*$ is holomorphic on $\Omega^* + iV^*$.*
(2) *$\mathcal{I}(\Omega + iV)$ is contained in the holomorphic domain of $\mathcal{I}^*$, and $\mathcal{I}^*(\Omega^* + iV^*)$ is contained in the holomorphic domain of $\mathcal{I}$.*

Remark 2.3. In general we cannot have $\mathcal{I}(\Omega + iV) \subset \Omega^* + iV^*$ if Ω is no longer selfdual. This failure is given by an example where Ω is the Vinberg cone. See [11] for details.

Now considering $E^*_{2\mathbf{d}+\mathbf{b}}$ naturally as an element of W^*, we define

$$C(w) := E^*_{2\mathbf{d}+\mathbf{b}} - 2\mathcal{I}(w+E) \in W^* \qquad (w \in W).$$

It is evident that C is a rational mapping $W \to W^*$ which is holomorphic on $\Omega + iV$. Let $U^\dagger$ denote the space of all antilinear forms on U. We set for $z = (u, w) \in U \times W$

$$\mathcal{C}(z) := \big(2\mathcal{I}(w+E)\circ Q(u,\cdot),\ C(w)\big) \in U^\dagger \times W^*.$$

Clearly $\mathcal{C}$ is a rational map $U \times W \to U^\dagger \times W^*$. It should be noted that if $z = (u, w) \in D$, then we have $w \in \Omega + iV$, so that $\mathcal{C}(z)$ is holomorphic on D. We call $\mathcal{C}$ a *Cayley transform.* This is a slight modification of Penney's [14]. By a verbal translation of Penney's proof [14] we have

Proposition 2.4. *The image $\mathcal{C}(D)$ of D is bounded.*

To give the inverse map of $\mathcal{C}$ explicitly we note first that

$$\langle v_1 \,|\, v_2\rangle_\eta := D_{v_1}D_{v_2}\log\eta(E) \qquad (v_1, v_2 \in V) \tag{2.1}$$

defines an inner product on V (see [3, §2]). Extending this inner product to a complex bilinear form (denoted by the same symbol $\langle\cdot\,|\,\cdot\rangle_\eta$) on $W \times W$, we define $\widetilde{f} \in W$ and $\widehat{w} \in W^*$ for $f \in W^*$ and $w \in W$ respectively by

$$\langle w' \,|\, \widetilde{f}\rangle_\eta = \langle w', f\rangle, \qquad \langle w', \widehat{w}\rangle = \langle w' \,|\, w\rangle_\eta \qquad (w' \in W).$$

Next we put

$$(u_1 \,|\, u_2)_\eta := \langle\, Q(u_1, u_2) \,|\, E \,\rangle_\eta \qquad (u_1, u_2 \in U). \tag{2.2}$$

It is easy to see that this is a Hermitian inner product on U. Now define linear maps $F \mapsto \widetilde{F}$ from $U^\dagger$ to U and $u \mapsto \widehat{u}$ from U to $U^\dagger$ by

$$(\widetilde{F} \,|\, u')_\eta = \langle\, u', F \,\rangle, \qquad \langle\, u', \widehat{u} \,\rangle = (u \,|\, u')_\eta \qquad (u' \in U).$$

Obviously they are inverse to one another. Moreover, for every $w \in W$, let $\varphi(w)$ be the complex linear operator on U determined through

$$(\varphi(w)u_1 \,|\, u_2)_\eta = \langle\, Q(u_1, u_2) \,|\, w \,\rangle_\eta \qquad (u_1, u_2 \in U). \tag{2.3}$$

Clearly $\varphi(E)$ is the identity operator, and it is easy to see that $\varphi(w^*) = \varphi(w)^*$. Let us set

$$B(f) := 2\mathcal{I}^*(E^*_{2\mathrm{d}+\mathrm{b}} - f) - E \in W \qquad (f \in W^*),$$
$$\mathcal{B}(F, f) := \big(\varphi(E - \widetilde{f})^{-1}\widetilde{F},\ B(f)\big) \in U \times W \quad ((F, f) \in U^\dagger \times W^*).$$

It is evident that both B and $\mathcal{B}$ are rational mappings.

Theorem 2.5 ([11]). *$\mathcal{C} : D \to \mathcal{C}(D)$ is biholomorphic and birational with $\mathcal{C}^{-1} = \mathcal{B}$.*

Remark 2.6. Suppose that D is *quasisymmetric* in this remark. This means that Ω is selfdual with respect to the inner product $\langle\, \cdot \,|\, \cdot \,\rangle_\eta$ defined by (2.1). We identify V^* with V and W with W^* by $\langle\, \cdot \,|\, \cdot \,\rangle_\eta$. Then by [1, Proposition 3] the product $\circ$ defined by

$$\langle\, v_1 \circ v_2 \,|\, v_3 \,\rangle_\eta := -\frac{1}{2}\, D_{v_1} D_{v_2} D_{v_3} \log \eta(E) \qquad (v_1, v_2, v_3 \in V)$$

is a Jordan algebra product, so that V is a Euclidean Jordan algebra in the sense of [5]. The identity element is E, and by the above identification we have $\mathcal{I}(x) = x^{-1}$, the Jordan algebra inverse of x. Identifying further $U^\dagger$ with U by means of $(\cdot \,|\, \cdot)_\eta$ in (2.2), we get

$$\mathcal{C}(u, w) = \big(2\,\varphi(w + E)^{-1}u,\ (w - E)(w + E)^{-1}\big).$$

Thus our $\mathcal{C}$ coincides with Dorfmeister's in [2, (2.8)] for quasisymmetric D. We note that the map $w \mapsto \varphi(w)$ with $\varphi(w)$ as in (2.3) is a representation of the complex Jordan algebra $W = V_{\mathbb{C}}$ in the present case (cf. [2, Theorem 2.1]).

3. A characterization of symmetric Siegel domains

By definition, the spaces $\mathfrak{g}(1/2)$ and $V = \mathfrak{g}(1)$ have the real inner product $\langle \cdot \,|\, \cdot \rangle_\omega$ of (1.2). We first export this inner product to V^* canonically by identifying V^* with V by $\langle \cdot \,|\, \cdot \rangle_\omega$. Note that this identification is not quite the same as in Remark 2.6 in general. The real inner product on V^* obtained this way is again denoted by $\langle \cdot \,|\, \cdot \rangle_\omega$, which is extended naturally to a Hermitian inner product $(\cdot \,|\, \cdot)_\omega$ on W^*. On the other hand the complex vector space U has a Hermitian inner product $(\cdot \,|\, \cdot)_\omega$ defined by

$$(3.1)\qquad (u_1 \,|\, u_2)_\omega := 2 \langle Q(u_1, u_2), \omega \rangle = \langle [Ju_1, u_2], \omega \rangle - i \langle [u_1, u_2], \omega \rangle.$$

We note that $\mathrm{Re}\,(u_1 \,|\, u_2)_\omega = \langle u_1 \,|\, u_2 \rangle_\omega$ for $u_1, u_2 \in U$. By a procedure similar to the above we introduce a Hermitian inner product $(\cdot \,|\, \cdot)_\omega$ on $U^\dagger$ by importing the Hermitian inner product (3.1) from U.

Let $\beta \in \mathfrak{g}^*$ be the *Koszul form* given by

$$\langle x, \beta \rangle := \mathrm{tr}\,(\mathrm{ad}\,(Jx) - J \circ (\mathrm{ad}\, x)) \qquad (x \in \mathfrak{g}).$$

It is known by [10] (see also [9, §5]) that $\langle [Jx, y], \beta \rangle$ is (the real part of) the inner product on $\mathfrak{g}$ induced by the Bergman metric of the corresponding Siegel domain D up to a positive multiple. Indeed we can show that $\beta|_{\mathfrak{n}}$ is equal to $E^*_{2\mathbf{d}+\mathbf{b}}$ extended to $\mathfrak{n}$ by zero-extension.

Theorem 3.1 ([12]). *One has* $\|\mathcal{C}(g \cdot \mathrm{e})\|_\omega = \|\mathcal{C}(g^{-1} \cdot \mathrm{e})\|_\omega$ *for all* $g \in G$ *if and only if the following two conditions are satisfied:*
(1) *D is symmetric,*
(2) *$\omega|_{\mathfrak{n}}$ is equal to a positive number multiple of $\beta|_{\mathfrak{n}}$.*

Remark 3.2. Since $\mathcal{C} : D \to \mathcal{C}(D)$ is biholomorphic with $\mathcal{C}(\mathrm{e}) = 0$, we have

$$\|\mathcal{C}(g \cdot \mathrm{e})\|_\omega = \|\mathcal{C}(g^{-1} \cdot \mathrm{e})\|_\omega \quad \text{for all } g \in G$$
$$\iff \|h \cdot 0\|_\omega = \|h^{-1} \cdot 0\|_\omega \quad \text{for all } h \in \mathcal{G} := \mathcal{C} \circ G \circ \mathcal{C}^{-1}.$$

4. Berezin transforms

For simplicity we set

$$\lambda_0 := \max_{1 \leq j \leq r} \frac{b_j + d_j + p_j/2}{b_j + 2d_j},$$

where $p_j := \sum_{k>j} n_{kj}$. Let $\lambda > \lambda_0$. This is the condition for the non-triviality of certain Hilbert spaces $H^2_\lambda(D)$ of holomorphic functions on D (cf. [17] or [7]). Let κ

be the Bergman kernel of D (see (1.9)). The *Berezin kernel* A_λ on D is given by

$$A_\lambda(z_1,z_2) := \left(\frac{|\kappa(z_1,z_2)|^2}{\kappa(z_1,z_1)\,\kappa(z_2,z_2)}\right)^\lambda \qquad (z_1,z_2\in D).$$

We put $a_\lambda(g) := A_\lambda(g\cdot \mathrm{e},\ \mathrm{e})$ $(g\in G)$. Then it is easy to see that $a_\lambda(g) = a_\lambda(g^{-1})$. We know that a_λ is integrable on G with respect to the left Haar measure. Consider the space $L^2(G)$ on G for the left Haar measure. The *Berezin transform* B_λ, when transferred to $L^2(G)$, is given by the convolution operator

$$B_\lambda f(x) := \int_G f(y)a_\lambda(y^{-1}x)\,dy = f * a_\lambda(x) \qquad (f\in L^2(G)).$$

On the other hand, the inner product $\langle\cdot\,|\,\cdot\rangle_\omega$ on $\mathfrak{g}$ defines a left invariant Riemannian metric on G, relative to which we have the Laplace-Beltrami operator $\mathcal{L}_\omega$ on G. In order to express $\mathcal{L}_\omega$ in terms of the elements of the enveloping algebra $U(\mathfrak{g})$, we set for $X\in\mathfrak{g}$

$$Xf(x) := \frac{d}{dt}f((\exp -tX)x)\Big|_{t=0}, \qquad \widetilde{X}f(x) := \frac{d}{dt}f(x\exp tX)\Big|_{t=0}.$$

These are extended to $U(\mathfrak{g})$ by homomorphisms. Though the following lemma holds for any connected Lie group, we write it down here in our situation. See [19, Theorem 1] for a proof.

Lemma 4.1. *Take $\Psi\in\mathfrak{g}$ for which one has $\langle X\,|\,\Psi\rangle_\omega = \operatorname{tr}\operatorname{ad}(X)$ for all $x\in\mathfrak{g}$. Then $\mathcal{L}_\omega = -\widetilde{\Lambda}+\widetilde{\Psi}$, where $\Lambda := X_1^2+\cdots+X_{2N}^2$ with an orthonormal basis $\{X_j\}_{j=1}^{2N}$ of $\mathfrak{g}$ relative to $\langle\cdot\,|\,\cdot\rangle_\omega$.*

We note that $\Psi\in\mathfrak{a}$ in our case.

Theorem 4.2 ([13]). *Let $\lambda>\lambda_0$ be fixed. Then, B_λ commutes with $\mathcal{L}_\omega$ if and only if D is symmetric and $\omega|_{\mathfrak{n}}$ is equal to a positive number multiple of $\beta|_{\mathfrak{n}}$.*

We indicate here how Theorem 4.2 is derived from Theorem 3.1.

(1) B_λ commutes with $\mathcal{L}_\omega$ $\iff$ $(-\widetilde{\Lambda}+\widetilde{\Psi})a_\lambda = (-\Lambda+\Psi)a_\lambda$.

(2) Since $a_\lambda(g) = a_\lambda(g^{-1})$, we have $\widetilde{X}a_\lambda(g) = Xa_\lambda(g^{-1})$ for all $X\in U(\mathfrak{g})$ and $g\in G$.

(3) $(\Lambda-\Psi)a_\lambda(g) = \lambda\, a_\lambda(g)\big(\lambda\,\|\mathcal{C}(g\cdot\mathrm{e})\|_\omega^2 - \langle\,\Psi,\alpha\,\rangle\big)$ for some $\alpha\in\mathfrak{a}^*$.

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