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An Iterative Approach for
Finite Element Analyses of 3-D Eddy Current Problems

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Abstract

An iterative approach is proposed for a finite element approximation of three-dimensional eddy current problems. The scheme is based on an iterative method derived from a perturbation problem of magnetostatic problems. For an appropriate choice of the perturbation parameter, the scheme converges at a small number of iterations. Moreover, a conjugate gradient-type solver is applicable for the complex linear systems arising in each step of the iterative procedure. A "cake" model and TEAM problem 7 are considered as numerical examples.

1 Introduction

We have shown the effectiveness of mixed finite element methods for three-dimensional eddy current problems using the Nedelec element of simplex type and the conventional piecewise linear tetrahedral element; see [1] and [2]. These computations using a direct method to solve the resultant linear system have consumed huge memories, disk spaces, and CPU time. This implies that an iterative solver may be required for larger scale analyses. However, the resultant linear system derived from the mixed method is indefinite, which causes difficulties in choosing the iterative solver, and the Lagrange multipliers turn out to be zero. This is why it is quite desirable to eliminate these extra quantities.

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We consider the time-harmonic electromagnetic field, and adopt the magnetic vector potential and the scalar electric potential as unknown complex functions. The magnetic vector potential is discretized by the Nedelec element of simplex type, and the scalar electric potential is discretized by the conventional piecewise linear tetrahedral element. Then, we introduce an iterative method without the Lagrange multipliers, which is an extension of an iterative one derived from a perturbation problem of magnetostatic problems; see [3]. A modified version of the iterative method is also proposed, which is separated into each variable step. By choosing an appropriate perturbation parameter, our scheme seems to converge at a small number of iterations; especially at only the initial step when the parameter is small enough in a simple model. The complex linear system arising in the initial step of the modified iterative procedure can be solved by a conjugate gradient-type solver; for example to show this fact, TEAM problem 7 is considered.

This paper is organized as follows. In Section 2 we give an iterative approach where the Lagrange multipliers are eliminated. In Section 3 we show some numerical results. Finally, concluding remarks are given in Section 4.

2 An iterative A-ϕ formulation

Let Ω be a polyhedral domain with boundary $\partial\Omega$. Assume that the domain Ω consists of two non-overlapping regions, a conducting part R and a non-conducting one S, with the interface Γ between two regions. In this section, for simplicity, assume that the conducting part R is strictly included in Ω.

Using the magnetic vector potential A [Wb/m] and the scalar electric potential ϕ [V] as unknown complex functions, we consider the following three-dimensional eddy current problem derived from the time-harmonic Maxwell equations:

$$
\begin{cases}
\text{rot}(\nu\,\text{rot}\,A) = Jo + i\sigma\omega A - \sigma\,\text{grad}\,\phi & \text{in } R, & \text{(1a)} \\[4pt]
\text{div}(i\sigma\omega A - \sigma\,\text{grad}\,\phi) = 0 & \text{in } R, & \text{(1b)} \\[4pt]
\text{rot}(\nu\,\text{rot}\,A) = Jo & \text{in } S, & \text{(1c)} \\[4pt]
\text{div}\,A = 0 & \text{in } \Omega, & \text{(1d)} \\[4pt]
(A \times n)\big|_R = (A \times n)\big|_S & \text{on } \Gamma, & \text{(1e)} \\[4pt]
(A \cdot n)\big|_R = (A \cdot n)\big|_S & \text{on } \Gamma, & \text{(1f)} \\[4pt]
\{(\nu\,\text{rot}\,A) \times n\}\big|_R = \{(\nu\,\text{rot}\,A) \times n\}\big|_S & \text{on } \Gamma, & \text{(1g)} \\[4pt]
(i\sigma\omega A - \sigma\,\text{grad}\,\phi) \cdot n = 0 & \text{on } \Gamma, & \text{(1h)} \\[4pt]
A \times n = 0 & \text{on } \partial\Omega, & \text{(1i)}
\end{cases}
$$

where Jo denotes an excitation current [A/m^2], ν the magnetic reluctivity [m/H], σ the conductivity [S/m], ω the angular frequency [rad/s], and i the imaginary unit. Throughout this paper, assume that ν is a piecewise positive constant, and that σ is a positive constant.

As usual, let $L^2(\Omega)$ be the space of functions defined in Ω and square summable in Ω with its inner product $(\,\cdot\,,\,\cdot\,)_\Omega$, and let $H^1(\Omega)$ be the space of functions in $L^2(\Omega)$ with

derivatives up to the first order. In the conducting region R, the spaces $L^2(R)$, $H^1(R)$ and the inner product $(\cdot, \cdot)_R$ are defined similarly. Let us define some function spaces as follows: $H(\mathrm{rot}; \Omega) \equiv \{ v \in L^2(\Omega)^3; \ \mathrm{rot}\, v \in L^2(\Omega)^3 \}$, $H_0(\mathrm{rot}; \Omega) \equiv \{ v \in H(\mathrm{rot}; \Omega); \ v \times n = 0 \text{ on } \partial\Omega \}$, $U \equiv H^1(R)$, $V \equiv H_0(\mathrm{rot}; \Omega)$, and $W \equiv \{ q \in H^1(\Omega); q = 0 \text{ on } \partial\Omega \}$.

We assume that the excitation current Jo satisfies $(Jo, \mathrm{grad}\, p^*)_\Omega = 0$ for all $p^* \in W$.

A mixed formulation of (1) is described as follows: find $(A, \phi, p) \in V \times U \times W$ such that, for $(A^*, \phi^*, p^*) \in V \times U \times W$,

$$\begin{cases} (\nu\,\mathrm{rot}\, A, \mathrm{rot}\, A^*)_\Omega - i(\sigma\omega A, A^*)_R + (\sigma\,\mathrm{grad}\,\phi, A^*)_R + (\mathrm{grad}\, p, A^*)_\Omega = (Jo, A^*)_\Omega, & \text{(2a)} \\ (\sigma\,\mathrm{grad}\,\phi, \mathrm{grad}\,\phi^*)_R - i(\sigma\omega A, \mathrm{grad}\,\phi^*)_R = 0, & \text{(2b)} \\ (A, \mathrm{grad}\, p^*)_\Omega = 0. & \text{(2c)} \end{cases}$$

For the mixed formulation (2), let us consider the following perturbation problem: for $(A^*, \phi^*, p^*) \in V \times U \times W$,

$$\begin{cases} (\nu\,\mathrm{rot}\, A, \mathrm{rot}\, A^*)_\Omega - i(\sigma\omega A, A^*)_R \\ \qquad\qquad + (\sigma\,\mathrm{grad}\,\phi, A^*)_R + (\mathrm{grad}\, p, A^*)_\Omega + \varepsilon(A, A^*)_\Omega = (Jo, A^*)_\Omega, & \text{(3a)} \\ (\sigma\,\mathrm{grad}\,\phi, \mathrm{grad}\,\phi^*)_R - i(\sigma\omega A, \mathrm{grad}\,\phi^*)_R = 0, & \text{(3b)} \\ (A, \mathrm{grad}\, p^*)_\Omega = 0. & \text{(3c)} \end{cases}$$

Substituting $\mathrm{grad}\, p$ into A^* in (3), we can find that $\mathrm{grad}\, p$ vanishes. Then, we have a formulation without the Lagrange multipliers p: for $(A^*, \phi^*) \in V \times U$,

$$\begin{cases} (\nu\,\mathrm{rot}\, A, \mathrm{rot}\, A^*)_\Omega - i(\sigma\omega A, A^*)_R + (\sigma\,\mathrm{grad}\,\phi, A^*)_R + \varepsilon(A, A^*)_\Omega = (Jo, A^*)_\Omega, & \text{(4a)} \\ (\sigma\,\mathrm{grad}\,\phi, \mathrm{grad}\,\phi^*)_R - i(\sigma\omega A, \mathrm{grad}\,\phi^*)_R = 0. & \text{(4b)} \end{cases}$$

Now we propose an iterative A-ϕ (iA-ϕ) method:

(*Initial step*) Fixed $\varepsilon > 0$, find $(A, \phi) \in V \times U$ satisfying (4). Then, set $(A^{(0)}, \phi^{(0)}) \equiv (A, \phi)$.

(*Iterative step*) For $k = 1, 2, \ldots$, find $(A^{(k)}, \phi^{(k)})$ recursively such that, for $(A^*, \phi^*) \in V \times U$,

$$\begin{cases} (\nu\,\mathrm{rot}\, A^{(k)}, \mathrm{rot}\, A^*)_\Omega - i(\sigma\omega A^{(k)}, A^*)_R \\ \qquad\qquad + (\sigma\,\mathrm{grad}\,\phi^{(k)}, A^*)_R + \varepsilon(A^{(k)}, A^*)_\Omega = (Jo, A^*)_\Omega + \varepsilon(A^{(k-1)}, A^*)_\Omega, & \text{(5a)} \\ (\sigma\,\mathrm{grad}\,\phi^{(k)}, \mathrm{grad}\,\phi^*)_R - i(\sigma\omega A^{(k)}, \mathrm{grad}\,\phi^*)_R = 0. & \text{(5b)} \end{cases}$$

We also propose a modified iterative A-ϕ (miA-ϕ) method, which is separated into each variable step.

(*Initial step*) Fixed $\varepsilon > 0$, find $A^{(0)} \in V$ such that, for $A^* \in V$,

$$(\nu\,\mathrm{rot}\, A^{(0)}, \mathrm{rot}\, A^*)_\Omega - i(\sigma\omega A^{(0)}, A^*)_R + \varepsilon(A^{(0)}, A^*)_\Omega = (Jo, A^*)_\Omega, \tag{6}$$

and find $\phi^{(0)} \in U$ such that, for $\phi^* \in U$,

$$(\sigma\,\mathrm{grad}\,\phi^{(0)}, \mathrm{grad}\,\phi^*)_R = i(\sigma\omega A^{(0)}, \mathrm{grad}\,\phi^*)_R. \tag{7}$$

(*Iterative step*) For $k = 1, 2, \ldots,$ find $A^{(k)} \in V$ such that, for $A^* \in V,$

$$(\nu \operatorname{rot} A^{(k)}, \operatorname{rot} A^*)_\Omega - i(\sigma \omega A^{(k)}, A^*)_R + \varepsilon(A^{(k)}, A^*)_\Omega$$
$$= (Jo, A^*)_\Omega - (\sigma \operatorname{grad} \phi^{(k-1)}, A^*)_R + \varepsilon(A^{(k-1)}, A^*)_\Omega, \quad (8)$$

and find $\phi^{(k)} \in U$ such that, for $\phi^* \in U,$

$$(\sigma \operatorname{grad} \phi^{(k)}, \operatorname{grad} \phi^*)_R = i(\sigma \omega A^{(k)}, \operatorname{grad} \phi^*)_R. \quad (9)$$

The domain Ω is decomposed into tetrahedrons. The Nedelec element of simplex type is used for the magnetic vector potential, and the conventional piecewise linear tetrahedral element for the scalar electric potential. Let V_h and U_h denote finite element spaces corresponding to V and U, respectively. The finite element approximation using the iA-ϕ method is described as follows:

(*Initial step*) Fixed $\varepsilon > 0$, find $(A_h^{(0)}, \phi_h^{(0)}) \in V_h \times U_h$ such that, for $(A_h^*, \phi_h^*) \in V_h \times U_h,$

$$\begin{cases} (\nu \operatorname{rot} A_h^{(0)}, \operatorname{rot} A_h^*)_\Omega - i(\sigma \omega A_h^{(0)}, A_h^*)_R \\ \qquad\qquad + (\sigma \operatorname{grad} \phi_h^{(0)}, A_h^*)_R + \varepsilon(A_h^{(0)}, A_h^*)_\Omega = (Jo, A_h^*)_\Omega, \quad (10a) \\ (\sigma \operatorname{grad} \phi_h^{(0)}, \operatorname{grad} \phi_h^*)_R - i(\sigma \omega A_h^{(0)}, \operatorname{grad} \phi_h^*)_R = 0. \quad (10b) \end{cases}$$

(*Iterative step*) For $k = 1, 2, \ldots,$ find $(A_h^{(k)}, \phi_h^{(k)})$ recursively such that, for $(A_h^*, \phi_h^*) \in V_h \times U_h,$

$$\begin{cases} (\nu \operatorname{rot} A_h^{(k)}, \operatorname{rot} A_h^*)_\Omega - i(\sigma \omega A_h^{(k)}, A_h^*)_R + (\sigma \operatorname{grad} \phi_h^{(k)}, A_h^*)_R \\ \qquad + \varepsilon(A_h^{(k)}, A_h^*)_\Omega = (Jo, A_h^*)_\Omega + \varepsilon(A_h^{(k-1)}, A_h^*)_\Omega, \quad (11a) \\ (\sigma \operatorname{grad} \phi_h^{(k)}, \operatorname{grad} \phi_h^*)_R - i(\sigma \omega A_h^{(k)}, \operatorname{grad} \phi_h^*)_R = 0. \quad (11b) \end{cases}$$

Moreover, the finite element approximation using the mi$A$-ϕ method is described as follows:

(*Initial step*) Fixed $\varepsilon > 0$, find $A_h^{(0)} \in V_h$ such that, for $A_h^* \in V_h,$

$$(\nu \operatorname{rot} A_h^{(0)}, \operatorname{rot} A_h^*)_\Omega - i(\sigma \omega A_h^{(0)}, A_h^*)_R + \varepsilon(A_h^{(0)}, A_h^*)_\Omega = (Jo, A_h^*)_\Omega, \quad (12)$$

and find $\phi_h^{(0)} \in U_h$ such that, for $\phi_h^* \in U_h,$

$$(\sigma \operatorname{grad} \phi_h^{(0)}, \operatorname{grad} \phi_h^*)_R = i(\sigma \omega A_h^{(0)}, \operatorname{grad} \phi_h^*)_R. \quad (13)$$

(*Iterative step*) For $k = 1, 2, \ldots,$ find $A_h^{(k)} \in V_h$ such that, for $A_h^* \in V_h,$

$$(\nu \operatorname{rot} A_h^{(k)}, \operatorname{rot} A_h^*)_\Omega - i(\sigma \omega A_h^{(k)}, A_h^*)_R + \varepsilon(A_h^{(k)}, A_h^*)_\Omega$$
$$= (Jo, A_h^*)_\Omega - (\sigma \operatorname{grad} \phi_h^{(k-1)}, A_h^*)_R + \varepsilon(A_h^{(k-1)}, A_h^*)_\Omega, \quad (14)$$

and find $\phi_h^{(k)} \in U_h$ such that, for $\phi_h^* \in U_h,$

$$(\sigma \operatorname{grad} \phi_h^{(k)}, \operatorname{grad} \phi_h^*)_R = i(\sigma \omega A_h^{(k)}, \operatorname{grad} \phi_h^*)_R. \quad (15)$$

Remark 1 In both iterative methods, the scalar electric potential ϕ is determined up to an additive constant.

Remark 2 If $\varepsilon = 0$, then a part of the initial step (12) of the mi$A$-ϕ method is reduced into the conventional formulation adopting the condition $\phi = 0$. But, in some cases (for example, in a "cake" model described in the next section), many iterative solvers do not converge for $\varepsilon = 0$.

3 Numerical results

3.1 A "cake" model

Let us consider an infinite solenoidal coil including a conductor with radius 0.1 [m]. By its symmetry, it suffices to consider only a sectoral domain (here, its central angle is 20°) as in Figure 1; so this model is called a "cake" model. This model is a three-dimensional extension of one described in [4]. The magnetic reluctivity ν is $1/(4\pi) \times 10^7$ [m/H], the conductivity σ is 7.7×10^6 [S/m], the frequency f is 60 [Hz] ($\omega = 2\pi f$), and the absolute value of the real (or imaginary) part of the excitation current $|Jo_r|$ (or $|Jo_i|$) is 50 (or 0) [A/m^2]. The perturbation parameter ε is 1.0 and 1.0×10^8.

The domain is decomposed into tetrahedrons; see Figure 1. The number of nodal points, elements, and degrees of freedom (DOF) are 695, 324, and 1772, respectively. Because we use an equivalent real linear system in the "cake" model, DOF is twice as large as that of the complex linear system. Let Γ_1 be the cross-section including the z-axis, and Γ_2 a union of the upper plane, the bottom plane and the surface of the cylinder. The following boundary conditions are imposed on $\partial\Omega$, though these conditions are slightly different from those in (1):

$$
\begin{cases}
A \times n = 0 \text{ on } \Gamma_1, \quad (\nu \operatorname{rot} A) \times n = 0 \text{ on } \Gamma_2, \quad A \cdot n = 0 \text{ on } \Gamma_2, & \text{(16a)} \\
\phi = 0 \text{ on } \Gamma_1 \cap \partial R, \quad \partial\phi/\partial n = 0 \text{ on } \Gamma_2 \cap \partial R, & \text{(16b)}
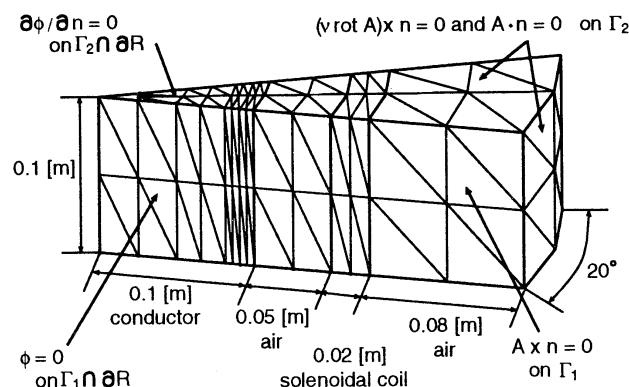\end{cases}
$$

Figure 1: A finite element mesh for the "cake" model.

where ∂R is the boundary of R. Boundary conditions on Γ are described in (1). Computation was performed on Sun UltraSPARC 200MHz with 1 CPU by using the PETSc library (see [5]) for solvers.

We compare the present results by the iA-ϕ and the miA-ϕ methods introduced in Section 2 with the previous ones by the mixed formulation. As the solver, LU decomposition is used. Figures 2 and 3 show the z component of the approximate magnetic field in the conductor versus the radius r along the line with $\theta = 10°$ and $z = 0.05$m. A solid line denotes the previous results, and the broken line denotes the present ones. In the case of $\varepsilon = 1.0$, although only the initial step is executed, the present results agree with the previous ones. Also in the case of $\varepsilon = 1.0 \times 10^8$, the present results agree with the previous ones at a small number of iterations.

3.2 TEAM Problem 7

Let us consider a benchmark problem, TEAM Problem 7. The magnetic reluctivity ν is $1/(4\pi) \times 10^7$ [m/H], the conductivity σ is 3.526×10^7 [S/m], the frequency f is 50 [Hz] ($\omega = 2\pi f$), and the absolute value of the real (or imaginary) part of the excitation current $|Jo_r|$ (or $|Jo_i|$) is 1.0968×10^6 (or 0) [A/m^2]. The perturbation parameter ε is 0.0, 1.0×10^3. Since the domain Ω is large enough, we assume that the normal component of the magnetic flux density vanishes on the whole boundary $\partial \Omega$. Boundary conditions on Γ are described in (1).

Here only the miA-ϕ method is considered. As in Figure 4, the domain Ω is decomposed into tetrahedrons. For simplicity the curve of the coil is disregarded. Two meshes are used; in the coarse case, the number of nodal points, elements and DOF are 7335, 5040 and 6335, respectively; in the fine case, the number of nodal points, elements and DOF are 66286, 47716 and 57631, respectively. To take away the indeterminacy, we impose $\phi = 0$ at the origin. Computation of the TEAM model was performed on an Alpha 600MHz.

Reliable iterative solvers for the resultant linear system are investigated. Unfortunately, as for the mixed A-ϕ formulation, there is no reliable iterative solver which is still effective for larger problems. Here Bi-Conjugate Gradient method (BiCG) is chosen ;see [6]. As the preconditioner, the Jacobi preconditioner is used. Zero vector is chosen as the initial vector of each iterative solver. Each process is stopped as soon as the residual norm $\|M^{-1}(b - Ax)\|/\|M^{-1}b\|$ is reduced by a factor, which is equal to 10^{-7}. Here, A denotes the resultant coefficient matrix, b the resultant given vector, M the preconditioner, and $\|\cdot\|$ the Euclidean norm.

Figure 5 shows the profiles of the residual norm $\|M^{-1}(b - Ax)\|/\|M^{-1}b\|$ versus the number of iterations. Although these profiles are oscillating, the residuals become smaller except for $\varepsilon = 0.0$ using the fine mesh. In the coarse mesh, the convergence for $\varepsilon = 1.0 \times 10^3$ is faster than that for $\varepsilon = 0.0$. We omit to show the results of the approximate magnetic field, but only note that they almost agree with the previous results in [1] or the measured values in [7].

4 Concluding remarks

We have introduced an iterative approach for three-dimensional eddy current problems based on an iterative method without the Lagrange multipliers for magnetostatic ones.

At first we have compared the present results by the iterative schemes with the previous ones by the mixed A-ϕ formulation. Numerical results have shown that both of the present results by the iA-ϕ and by the mi$A$-ϕ method agree with the previous ones at a small number of iterations (at only the initial step, for an appropriate choice of the perturbation parameter). Next we have investigated reliable iterative solvers of the symmetric complex linear system arising in the initial step of the modified iterative method. Except for $\varepsilon = 0.0$ in the fine mesh, we have shown that, by BiCG (with the Jacobi preconditioner), the iterative solver for the resultant linear system converges.

We are planning to compute larger scale problems using an iterative domain decomposition method.

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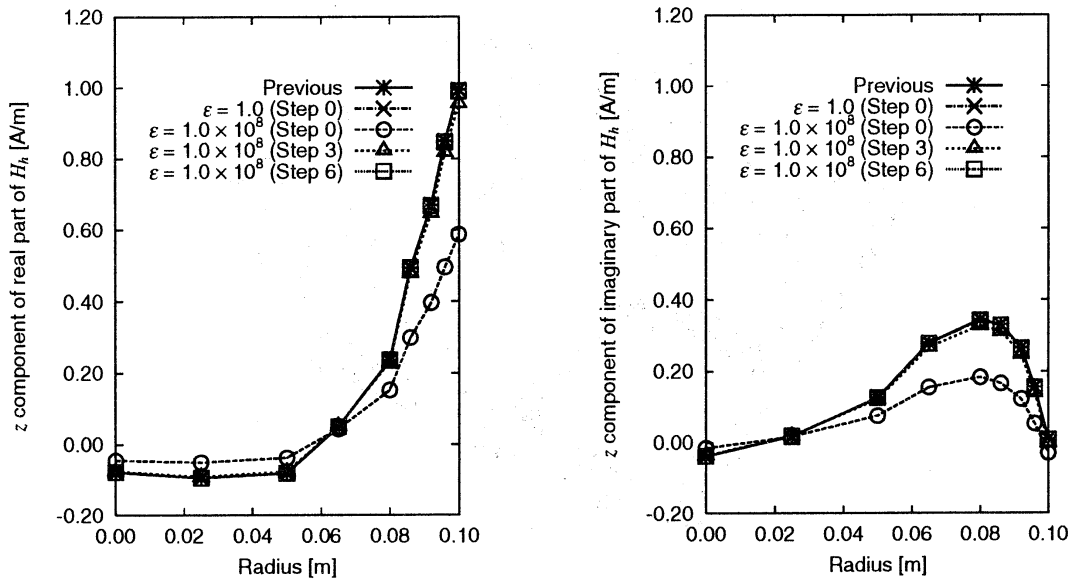
Figure 2: The z component of the approximate magnetic field H_h in the conductor by the iA-ϕ method (left: the real part, right: the imaginary part).
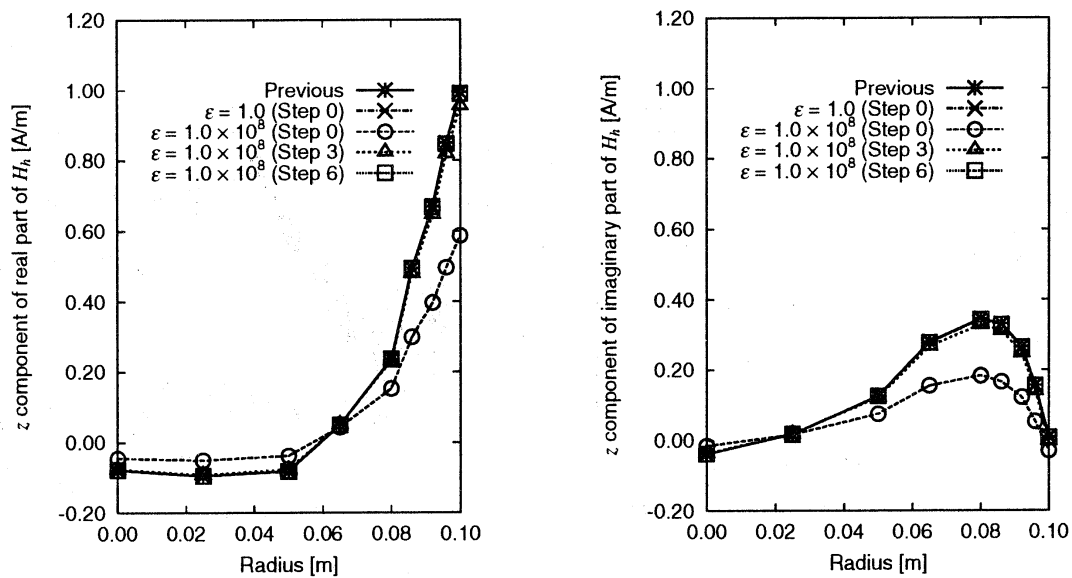

Figure 3: The z component of the approximate magnetic field H_h in the conductor by the mi$A$-ϕ method (left: the real part, right: the imaginary part).
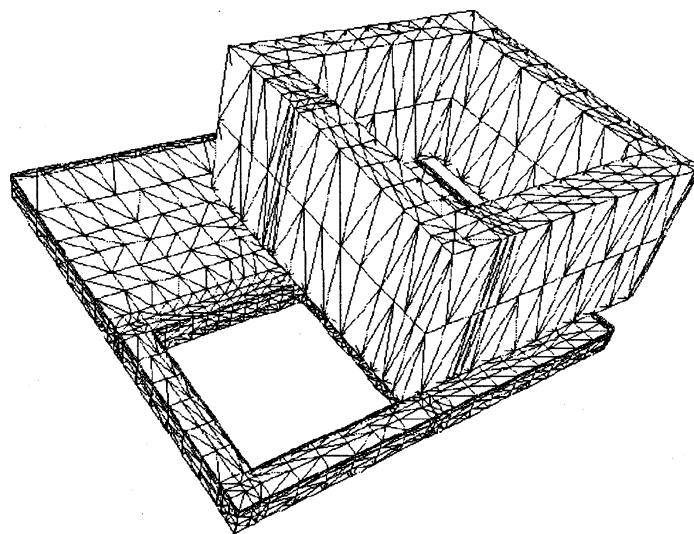
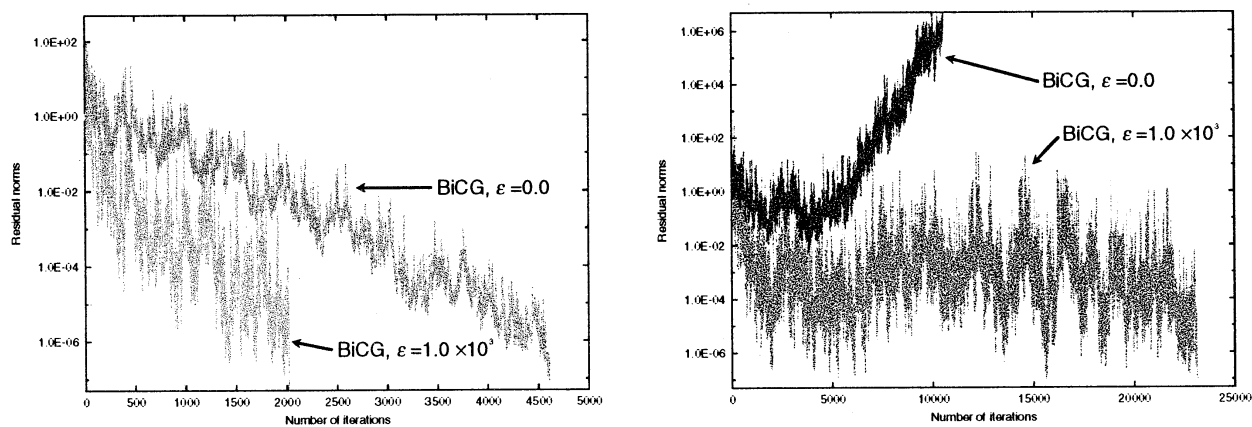Figure 4: A mesh around the coil and the conductor for TEAM problem 7.

Figure 5: The profile of the residual norm $\|M^{-1}(b - Ax)\|/\|M^{-1}b\|$ versus the number of iterations (left: the corse mesh, right: the fine mesh).