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Numerical computations for Ginzburg-Landau equation with a variable coefficient by using the discrete Morse semiflow

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1 Introduction

The following Ginzburg-Landau equation with a variable coefficient in a bounded domain $D \subset \mathbf{R}^2$ subject to Neumann boundary condition are considered:

$$(P) \quad \begin{cases} a(x, y)^{-1} \operatorname{div}(a(x, y) \nabla \Phi) + \lambda(1 - |\Phi|^2) \Phi = 0 & (x, y) \in D, \\ \frac{\partial \Phi}{\partial \nu} = 0 & x \in \partial D, \end{cases}$$

where $a(x, y)$ is a positive smooth function, $\partial/\partial\nu$ denotes the outer normal derivative on the boundary ∂D and $\Phi(x, y)$ is a complex valued function, say $\Phi(x, y) = u(x, y) + iv(x, y)$. $\Phi(x, y)$, which is called order parameter describing a superconducting state, is always identified with the two-component real vector function $(u(x, y), v(x, y))$. This equation (P) is a simplified model to describe a superconducting phenomenon in a thin material with a variable thickness. The thickness of the material with the bottom D is denoted by $a(x, y)$.

For type II super conductors a third state exists, which is known as the mixed state. The mixed state is neither wholly superconducting nor wholly normal but consists of many normal filaments embedded in a superconducting material. These filaments are often known as vortices. Each of these filaments carries with it a quantized amount of magnetic flux and is circled by a vortex of superconducting current; thus these filaments are often known as vortices. From an industrial perspective, it is interesting to know the behavior of vortices, especially their stable condition. If vortices move, electromagnetic induction occurs, causing voltage drop, and therefore loss of energy. Moreover, to apply a superconducting phenomenon (for example "pinning effect"), the position where the vortices appear should be investigated.

Stable solutions of (P) with zero are called vortex solutions. Mathematically, the vortices are considered zero points of $\Phi(x, y)$. For constant $a(x, y)$, there is no stable nonconstant solution to the Ginzburg-Landau equation in any convex domain with Neumann boundary condition [3]. The objective is therefore to investigate the relation between the thickness $a(x, y)$ and the stable vortex solution. Such interest acts as a catalyst for the proposition of

the following numerical method. By applying the discrete Morse semiflow (time discretized functional method) to this problem, numerical experiments are carried out.

2 Mathematical results

(P) is the Euler-Lagrange equation for the energy functional

$$E(\Phi) = \int_D \{|\nabla\Phi|^2 + \frac{\lambda}{2}(1 - |\Phi|^2)^2\} a(x, y) dx dy. \quad (2.1)$$

We call the solution of (P) is stable if it is a local minimizer of (2.1) (cf. [6]).

Let $h > 0$ and let D be a bounded domain in \mathbf{R}^2 with smooth boundary. Let $a(x, y) > 0$ be a bounded function on \bar{D} . Here, smoothness of $a(x, y)$ is not assumed.

Now the sequence of functionals will be defined,

$$\begin{aligned} E_n^h(\Phi) &= \int_D \frac{|\Phi - \Phi_{n-1}^h|^2}{h} a(x, y) dx dy + E(\Phi), \\ \Phi \in K &= W_\psi^{1,2}(D; \mathbf{R}^2) \cap L^4(D; \mathbf{R}^2). \end{aligned} \quad (2.2)$$

If suitable h and λ are chosen, the minimizer is uniquely determined for each functional $E_n^h(\Phi)$.

Lemma 2.1. *For any $m \in \mathbf{N}$, if $1/h > \lambda$ holds, the minimizer of E_m^h is uniquely determined.*

proof) It holds that

$$E_m^h(\Phi) = \int_D \left\{ \left(\frac{1}{h} - \lambda \right) |\Phi|^2 - \frac{2}{h} \Phi \cdot \Phi_{m-1}^h + \frac{1}{h} |\Phi_{m-1}^h|^2 + |\nabla\Phi|^2 + \frac{\lambda}{2}(1 + |\Phi|^4) \right\} a(x, y) dx dy.$$

Then we have, for $1 \leq \theta \leq 1$,

$$\begin{aligned} &(1 - \theta) E_m^h(\Phi) + \theta E_m^h(\Psi) - E_m^h(\Phi + \theta(\Psi - \Phi)) \\ &\geq (1 - \theta) \theta \int_D \left\{ \left(\frac{1}{h} - \lambda \right) |\Phi - \Psi|^2 + |\nabla(\Phi - \Psi)|^2 \right\} a(x, y) dx dy. \end{aligned}$$

If $1/h - \lambda$ is positive, the functional $E_m^h(\Phi)$ is convex. Therefore its minimizer is unique. \square

The sequence of functions $\{\Phi_n^h\}_{n=1}^\infty$ is called discrete Morse semiflow (see [7], [8], [9] and [10]). The boundedness of Φ_m^h for each m is given.

Lemma 2.2. *If $\|\Phi_0\|_\infty \leq 1$, $\|\Phi_m^h\|_\infty \leq 1$ for all $m \in \mathbf{N}$ holds.*

proof) We suppose that our assertion holds for any $n \leq m - 1$. If Φ_m^h is a minimizer of E_m^h and $\{x; |\Phi_m^h| > 0\}$ has a positive Lebesgue measure, the following comparison function

is chosen: $\Psi := \Phi_m^h / |\Phi_m^h|$ in $\{x; |\Phi_m^h| > 1\}$, $:= \Phi_m^h$ in $\{x; |\Phi_m^h| \leq 1\}$. Then $E_m^h(\Psi) < E_m^h(\Phi_m^h)$ is obtained by direct calculation. It contradicts Φ_m^h is a minimizer. \square

Throughout this paper, an initial data Φ_0 satisfies $\|\Phi_0\|_\infty \leq 1$ is supposed.

Lemma 2.3. *It holds that*

$$E(\Phi_M^h) + \sum_{m=1}^M \int_D \frac{|\Phi_m^h - \Phi_{m-1}^h|^2}{h} a(x, y) dx dy \leq E(\Phi_0).$$

proof) Because Φ_m^h is the minimizer of E_m^h , it holds the following inequality,

$$\begin{aligned} E_m^h(\Phi_m^h) &\equiv \int_D \frac{|\Phi_m^h - \Phi_{m-1}^h|^2}{h} a(x, y) dx dy + E(\Phi_m^h) \\ &\leq E_m^h(\Phi_{m-1}^h) = E(\Phi_{m-1}^h). \end{aligned} \quad (2.3)$$

By summing up the both sides of (2.3), Lemma 2.3 can be shown. \square

Now, the existence of the limit function Φ_∞^h of the subsequence $\{\Phi_m^h\}$ will be shown.

Lemma 2.4. *For any subsequence $\{\Phi_{m_j}^h\} \subset \{\Phi_m^h\}$, there exists a subsequence $\{\Phi_{m_{j\nu}}^h\} \subset \{\Phi_{m_j}^h\}$ and a function Φ_∞^h on D such that*

$$\Phi_{m_{j\nu}}^h \rightarrow \Phi_\infty^h \quad \text{weakly in } W^{1,2}, \quad (2.4)$$

$$\Phi_{m_{j\nu}}^h \rightarrow \Phi_\infty^h \quad \text{strongly in } L^2, \quad (2.5)$$

$$\Phi_{m_{j\nu}}^h \rightarrow \Phi_\infty^h \quad \text{weakly in } L^p, \quad \forall p > 1, \quad (2.6)$$

as $\nu \rightarrow \infty$. Moreover, we have

$$|\Phi_\infty^h| \leq 1 \quad \text{a.e. in } D. \quad (2.7)$$

proof) By Lemma 2.3, $\{\Phi_m^h\}$ is weakly compact in $W^{1,2}$. Therefore it holds (2.4) by use of a weak compactness argument and by Rellich's theorem (2.5) is obtained. We readily get (2.6) and (2.7) by Lemma 2.2. \square

Theorem 2.1. *The limit function Φ_∞^h is a minimizer of the functional*

$$E_\infty^h(\Phi) = \int_D \frac{|\Phi - \Phi_\infty^h|^2}{h} a(x, y) dx dy + \int_D \left\{ |\nabla \Phi|^2 + \frac{\lambda}{2} (1 - |\Phi|^2)^2 \right\} a(x, y) dx dy$$

in K , hence, Φ_∞^h satisfies

$$\int_D \nabla \Phi_\infty^h \nabla \phi a(x, y) dx dy - \int_D \lambda \Phi_\infty^h (1 - |\Phi_\infty^h|^2) \phi a(x, y) dx dy = 0$$

for any $\phi \in C_0^\infty(D)$.

proof) We assume that there exists $v \in K$ such that

$$E_\infty^h(\Phi_\infty^h) - E_\infty^h(v) = 3d > 0.$$

It is easy to see

$$\begin{aligned} & |E_{m_j}^h(v) - E_\infty^h(v)| \\ &= \frac{1}{h} \int_D \left\{ 2v \cdot (\Phi_\infty^h - \Phi_{m_j-1}^h) + (|\Phi_{m_j-1}^h|^2 - |\Phi_\infty^h|^2) \right\} a(x, y) dx dy \\ &\leq \frac{1}{h} \|a\|_\infty \cdot \|\Phi_\infty^h - \Phi_{m_j-1}^h\|_{L^2} \left\{ 2\|v\|_{L^2} + \|\Phi_{m_j-1}^h\|_{L^2} + \|\Phi_\infty^h\|_{L^2} \right\}. \end{aligned}$$

Thus, there exists a positive number M such that for all $j \geq M$

$$|E_{m_j}^h(v) - E_\infty^h(v)| \leq d$$

holds.

On the other hand, by Lemma 2.4, it holds that

$$\begin{aligned} \int_D |\nabla \Phi_\infty^h|^2 dx &\leq \liminf_{j \rightarrow \infty} \int_D |\nabla \Phi_{m_j}^h|^2 dx, \\ \int_D \frac{1}{\delta} (1 - |\Phi_\infty^h|^2)^2 dx &\leq \liminf_{j \rightarrow \infty} \int_D \frac{1}{\delta} (1 - |\Phi_{m_j}^h|^2)^2 dx. \end{aligned}$$

Therefore there exists $M \in \mathbb{N}$ such that for $j \geq M$ we have

$$E_\infty^h(\Phi_\infty^h) \leq E_{m_j}^h(\Phi_{m_j}^h) + d.$$

Combining these estimates with a minimality of $E_{m_j}^h(\Phi_{m_j}^h)$, we have

$$\begin{aligned} E_{m_j}^h(\Phi_{m_j}^h) &\leq E_{m_j}^h(v) \\ &\leq E_\infty^h(v) + d \\ &= E_\infty^h(\Phi_\infty^h) - 2d \\ &\leq E_{m_j}^h(\Phi_{m_j}^h) - d. \end{aligned}$$

This is a contradiction. □

3 Numerical results

Here, the following some numerical experiments are introduced. These results are obtained by minimizing method to the functional (2.2).

The numerical scheme used here is the usual finite element method for elliptic variational problems. A minimizer for each step is sought by use of a gradient method (see [9] and [10] for examples). Note that, each minimizer is uniquely determined, if h and δ are chosen suitably by Lemma 2.1. The parameters chosen are $\delta = 1.0 \times 10^{-5}$ and $\lambda = 1/0.05$.

Let $D = \{|x| < 1\}$ and $a(x, y) = a(r)$ be a radially symmetric function. The thickness $a(x, y)$ is defined

$$a(x, y) = a(r) = \begin{cases} 1 & 0.5 < r \leq 1, \\ d & 0 \leq r \leq 0.5. \end{cases}$$

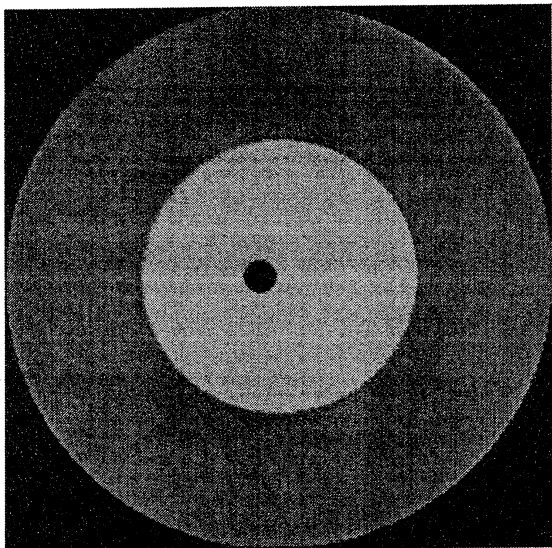
We may consider the d plays an important role in the position of vortex. Numerical computations were tested in the three cases; $d = 0.01$, $d = 0.5$ and $d = 0.4$. All of the cases, the following function is chosen

$$\Phi_0(x, y) = \begin{cases} 0 & \text{if } \rho = 0 \\ ((x + 0.1)/\rho, y/\rho) & \text{otherwise} \end{cases}$$

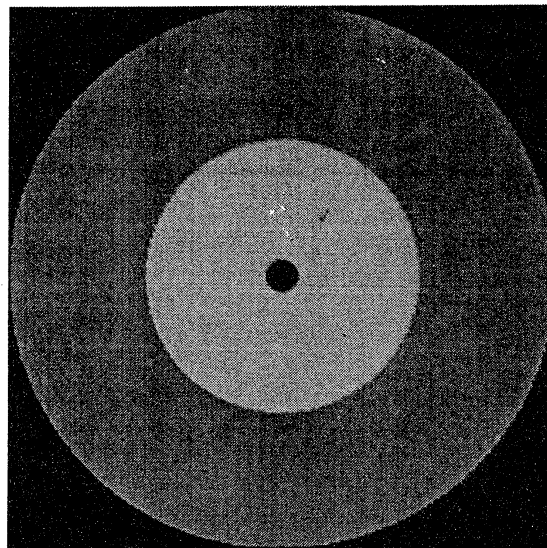
as the initial condition, where $\rho = \sqrt{(x + 0.1)^2 + y^2}$.

Case 1 $d=0.01$

The vortex solution whose vortex is at the center is unstable for constant $a(x, y)$. However, the vortex of Φ_∞ is at the center of the domain. For the result of [5], the vortex solution whose vortex is at the center is known. This fact is ascertained numerically.



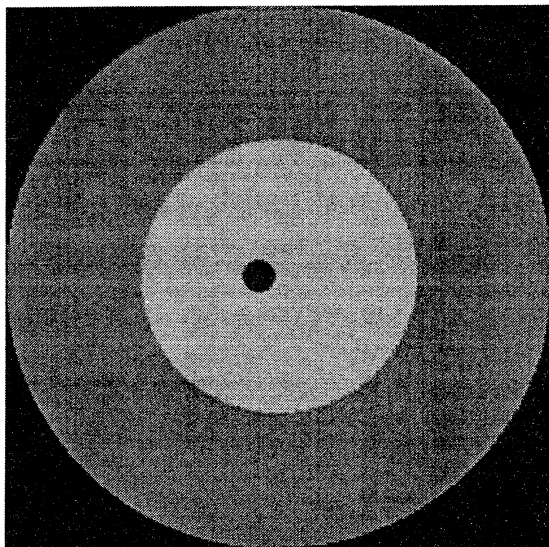
The profile of Φ_0



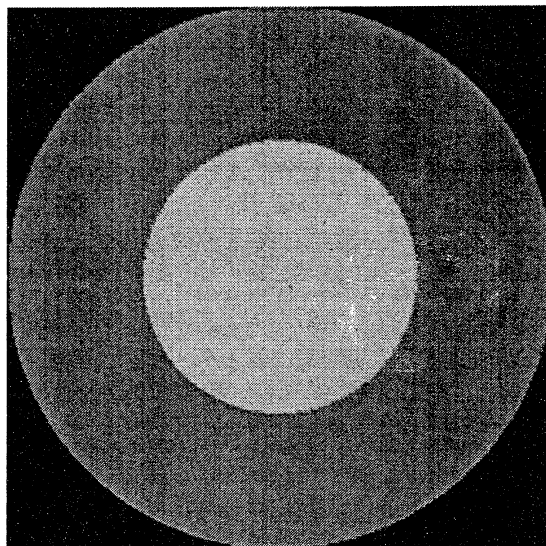
The profile of Φ_∞

Case 2 $d=0.5$

The vortex goes out from the domain. It is the same as $a(x, y)$ is constant.



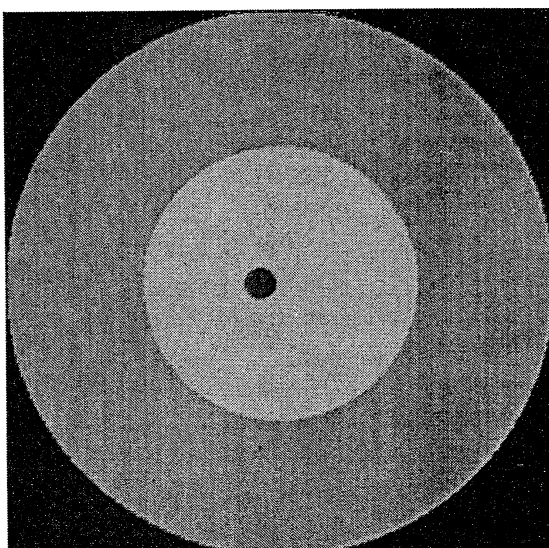
The profile of Φ_0



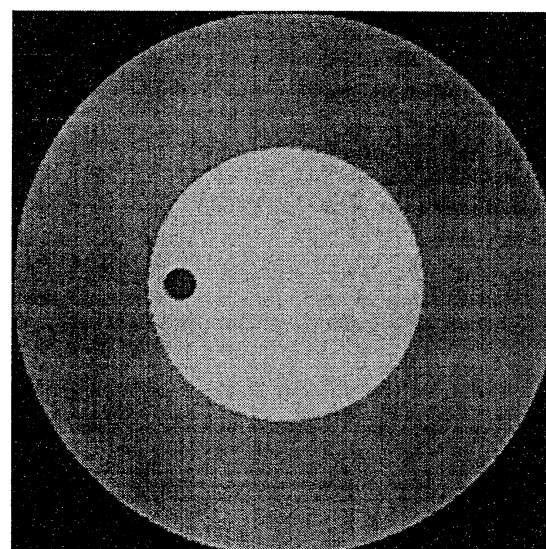
The profile of Φ_∞

Case 3 $d=0.4$

The vortex is trapped in the domain. It can not go over the layer at $r = 0.5$.



The profile of Φ_0



The profile of Φ_∞

4 Conclusion

Here, the Ginzburg-Landau system was treated and its weak solutions were constructed by use of a notion of discrete Morse semiflow. At the same time, numerical computations were also carried out. The numerical scheme used here was the usual finite element method for elliptic variational problems. A minimizer for each step was sought by use of a gradient method. These minimizers were uniquely determined, and located relatively quickly.

Numerical experiments were carried out on a special shape of the domain. The stability of the solution and the position of the vortex were affected by the thickness of the domain. For the result of [5], the vortex solution whose vortex is at the center was known. This fact was ascertained numerically, and our results suggested the existence of another vortex solution exists.

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