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Stable and unstable manifolds of diffeomorphisms with positive entropy

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Abstract

We show that C^2-diffeomorphisms with positive entropy are chaotic in the sense of Li-Yorke. To do so we prove that these maps are $*$-chaotic on the closure of stable manifolds for some points. The notion of "$*$-chaos" was introduced by Kato and it is related to chaos in the sense of Li-Yorke.

1 Introduction

We study chaotic properties of diffeomorphisms with positive entropy. Notions of chaos have been given by Li and Yorke [9], Devaney [2] and others. It is known that if a continuous map of interval has positive entropy, then it is chaotic according to the definition of Li and Yorke (cf. [1]).

In [6] Katok proved the following :let f be a $C^{1+\varepsilon}$-diffeomorphism of a closed surface. If the topological entropy of f is positive, then there exists a hyperbolic set Γ such that the restriction of f into Γ is topologically conjugate to a subshift of finite type with positive entropy. This implies that f is chaotic in the sense of Li-Yorke.

However, Katok's theorem does not hold for the high dimensional case. Indeed, let f be a surface diffeomorphism with positive entropy and let $r : S^1 \to S^1$ be an irrational rotation. Then a product map $f \times r$ has the same positive entropy, but it does not have Γ as above because there are no periodic points of $f \times r$.

In this paper we show the following:

Theorem A *Let f be a C^2-diffeomorphism of a closed C^∞-manifold. If the topological entropy of f is positive, then f is chaotic in the sense of Li-Yorke.*

To my knowledge this theorem gives the most simplest sufficient condition for chaotic phenomena of high dimensional dynamical systems. It remains a question whether Theorem A is true for homeomorphisms. However this question is still unsolved.

Let M be a closed C^∞-manifold and let d be the distance for M induced by a Riemannian metric $\|\cdot\|$ on M. A subset S of M is a *scrambled set* of f if there is a positive number $\tau > 0$ such that for any $x, y \in S$ with $x \neq y$,

1. $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \tau$,
2. $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.

If there is an uncountable scrambled set S of f, then we say that f is *chaotic in the sense of Li-Yorke* . Li and Yorke showed in [9] that if $f\colon [0,1] \to [0,1]$ is a continuous map with a periodic point of period 3, then f is chaotic in this sense. In [9] there was the following one more condition: for any $x \in S$ and any periodic point $p \in M$, $\limsup_{n\to\infty} d(f^n(x), f^n(p)) > 0$. But this condition is unnecessary because a scrambled set contains at most one point which does not satisfy this condition. For the examples and the properties of scrambled sets, the readers may refer to [1], [5], [11], [12], [13], [22], [23] and [24].

Concerning the chaos in the sense of Li-Yorke, Kato introduced the notion of "$*$-chaos" as follows: let F be a closed subset of M. A map $f : M \to M$ is *$*$-chaotic* on F (in the sence of Li-Yorke) if the following conditions are satisfied:

1. there is $\tau > 0$ such that if U and V are any nonempty open subsets of F with $U \cap V = \emptyset$ and N is any natural number, there is a natural number $n \geq N$ such that $d(f^n(x), f^n(y)) > \tau$ for some $x \in U$, $y \in V$, and
2. for any nonempty open subsets U, V of F and any $\varepsilon > 0$ there is a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) < \varepsilon$ for some $x \in U$, $y \in V$.

Such a set F is called a *$*$-chaotic set* . If S is a scrambled set, then the closure of S, $\bar{S}$, is a $*$-chaotic set. In [4] Kato showed that the converse is true.

Lemma 1.1 ([4], Theorem 2.4) *If $f : M \to M$ is continuous and is $*$-chaotic on F, then there F_σ-set $S \subset F$ such that S is a scrambled set of f and $\overline{S} = F$. If F is perfect (i.e. F has no isolated points), we can choose S such that it is a countable union of Cantor sets.*

To obtain Theorem A we need the following theorem.

Theorem B *Let f be a C^2-diffeomorphism of a closed C^∞-manifold M and let μ be an f-invariant ergodic Borel probability measure on M. If the metric entropy of μ is positive, then for μ-almost all $x \in M$ the following hold:*

(a) $\overline{W^s(x)}$ is a perfect $*$-chaotic set, and

(b) $\overline{W^u(x)}$ contains a perfect $*$-chaotic set.

Here $W^s(x)$ and $W^u(x)$ are defined by

$$W^s(x) = \{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^n(x), f^n(y)) < 0\} \text{ and,}$$
$$W^u(x) = \{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) < 0\}$$

respectively.

We notice that for μ-almost all $x \in M$, the above sets $W^s(x)$ and $W^u(x)$ are C^2 immersed manifolds under the assumptions of theorem B. Indeed, let f and μ be as above. For μ-almost all $x \in M$, there exist a splitting of the tangent space $T_xM = \oplus_{i=1}^{s(x)} E_i(x)$ and real numbers $\lambda_1(x) < \lambda_2(x) < \cdots < \lambda_{s(x)}(x)$ such that

(a) the maps $x \mapsto E_i(x)$, $\lambda_i(x)$ and $s(x)$ are Borel measurable, moreover $E_i(f(x)) = D_xf(E_i(x))$ and $\lambda_i(x)$, $s(x)$ are f-invariant $(i = 1, \cdots, s(x))$,

(b) $\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_xf^n(v)\| = \lambda_i(x) \quad (0 \neq v \in E_i(x),\ i = 1, \cdots, s(x))$ and

(c) $\lim_{n\to\pm\infty} \frac{1}{n} \log |\det(D_xf^n)| = \sum_{i=1}^{s(x)} \lambda_i(x) \dim E_i(x)$

([14]). The numbers $\lambda_1(x), \cdots, \lambda_{s(x)}(x)$ are called *Lyapunov exponents* of f at x. Since μ is ergodic, we can put $s = s(x)$, $\lambda_i = \lambda_i(x)$ and $m_i = \dim E_i(x)$ $(i = 1, \cdots, s)$ for μ-almost all $x \in M$.

Let $h_\mu(f)$ denote the metric entropy of f (see [10] for definition). A well-known theorem of Margulis and Ruelle [21] says that entropy is always bounded above by the sum of positive Lyapunov exponents; i.e. $h_\mu(f) \leq \sum_{\lambda_i>0} \lambda_i m_i$. Since f has positive entropy, we have

$$0 < h_\mu(f) \leq \max\{\lambda_i\} = \lambda_s.$$

Therefore, by Pesin's stable manifold theorem ([3], [15], [17]), the set $W^u(x)$ is the image of a C^2 injective immersion of an euclidean space such that $T_xW^u(x) = \oplus_{\lambda_i>0} E_i(x) (\neq \{0\})$ for μ-almost all $x \in M$. $W^u(x)$ is called an *unstable manifold*. Similarly, $W^s(x)$ is a C^2 immersed manifold because $W^s(x)$ is the unstable manifold of f^{-1}, which has positive entropy $h_\mu(f^{-1}) = h_\mu(f) > 0$. $W^s(x)$ is called a *stable manifold*.

Let us see how Theorem A follows from Theorem B. We denote as $h(f)$ the topological entropy of f (see [10] for definition). Then we know that $h(f) = \sup\{h_\mu(f) : \mu \in \mathcal{M}_e(f)\}$ where $\mathcal{M}_e(f)$ is the set of all f-invariant ergodic Borel probability measures (cf [20]). Thus, if $h(f) > 0$, then we can choose $\mu \in \mathcal{M}_e(f)$ with $h_\mu(f) > 0$. Therefore, by Theorem B and Lemma 1.1, f is chaotic in the sense of Li-Yorke.

Remark that (a) of Theorem B does not hold for unstable manifolds in general. For example, the Smale horseshoe has unstable manifolds which intersect a stable manifold of a fixed point (cf.[18]). Since all points in the stable manifold converge to the fixed point, they do not satisfy the first condition of $*$-chaos.

Now we shall give a sufficient condition that f is $*$-chaotic on $\overline{W^u(x)}$ as follows.

Theorem C *If μ is an ergodic SRB measure, then both $\overline{W^s(x)}$ and $\overline{W^u(x)}$ are perfect $*$-chaotic sets for μ-almost all $x \in M$.*

If ξ is a measurable decomposition of M, then a family $\{\mu_x^\xi | x \in M\}$ of Borel probability measures exists, and it satisfies the following conditions:

1. for x, $y \in M$ if $\xi(x) = \xi(y)$ then $\mu_x^\xi = \mu_y^\xi$, here $\xi(x)$ denotes a set containing x in ξ,

2. $\mu_x^\xi(\xi(x)) = 1$ for μ-almost all $x \in M$,

3. for any Borel set A a function $x \mapsto \mu_x^\xi(A)$ is measurable and $\mu(A) = \int_M \mu_x^\xi(A)d\mu(x)$.

The family $\{\mu_x^\xi | x \in M\}$ is called a *canonical system of conditional measures* for μ and ξ (see [19] for more details).

An f-invariant Borel probability measure μ is called a *Sinai-Ruelle-Bowen measure* (SRB measure for abbreviation) provided

(*A*) for μ-almost all $x \in M$, there exists a positive Lyapunov exponent of x,

(*B*) μ has a conditional measure that is absolutely continuous (with respect to the Lebesgue measure) on unstable manifolds, which is defined as follows:

From (A), the unstable manifold $W^u(x)$ is a C^2 submanifold for μ-almost all x in M. Let m_x^u denote the Lebesgue measure of $W^u(x)$. A measurable decomposition ξ of M is said to be *subordinate to unstable manifolds* if for μ-almost all x in M

(*C*) $\xi(x) \subset W^u(x)$,

(*D*) $\xi(x)$ contains an open neighborhood of x in $W^u(x)$.

We say that μ has an *absolutely continuous conditional measure* on unstable manifolds provided μ_x^ξ is absolutely continuous with respect to m_x^u for μ-almost all x in M if ξ is subordinate to unstable manifolds. It is known ([7], [8]) that μ satisfies (B) if and only if the following equation holds:

$$h_\mu(f) = \int \sum_{\lambda_i(x)>0} \lambda_i(x) \mathrm{dim} E_i(x) d\mu(x).$$

This is sometimes known as *Pesin's formula* . For the examples and the stochastic properties of diffeomorphisms with SRB measures, the readers may refer to [25]. In a similar way we can define a measurable partition subordinate to stable manifolds.

2 Preliminaries

In this section we introduce f-invariant measurable partitions each of whose elements is contained in the closure of (un)stable manifolds. Let f be a C^2-diffeomorphism of a closed C^∞-manifold M and let μ be an f-invariant ergodic Borel probability measure on M with $h_\mu(f) > 0$. Denote as $\mathcal{B}$ the family of Borel sets.

Lemma 2.1 (Proposition 3.1 [7]) *Let f and μ be as above. Then there exist measurable partitions ξ^s and ξ^u of M such that*

(a) $\xi^s \le f\xi^s$ *and* $\xi^u \le f^{-1}\xi^u$,

(b) ξ^s *and* ξ^u *are subordinate to stable manifolds and unstable manifolds respectively,*

(c) *both* $\bigvee_{n=0}^{\infty} f^n\xi^s$ *and* $\bigvee_{n=0}^{\infty} f^{-n}\xi^u$ *are the partitions into points,*

(d) *for μ-almost all $x \in M$,*

$$\bigcup_{n=0}^{\infty} f^{-n}\xi^s(f^n(x)) = W^s(x) \quad \text{and} \quad \bigcup_{n=0}^{\infty} f^n\xi^u(f^{-n}(x)) = W^u(x).$$

Lemma 2.2 (Corollary 5.3 [8]) *Let f and μ be as above and let ξ^σ ($\sigma = s, u$) be as in Lemma 2.1. Then,*

$$\begin{aligned} h_\mu(f) &= H_\mu(f\xi^s|\xi^s) = \int -\log \mu_x^{\xi^s}(f\xi^s(x))d\mu(x) \\ &= H_\mu(f^{-1}\xi^u|\xi^u) = \int -\log \mu_x^{\xi^u}(f^{-1}\xi^u(x))d\mu(x) \end{aligned}$$

where the family $\{\mu_x^{\xi^\sigma}|x \in M\}$ is a canonical system of conditional measures for μ and ξ^σ.

Let us introduce two measurable partitions defined by

$$\eta^s = \bigwedge_{i=0}^{\infty} f^{-i}\xi^s \text{ and } \eta^u = \bigwedge_{i=0}^{\infty} f^i\xi^u.$$

By Lemma 2.1(a) and (d) we can easily check that $f\eta^\sigma = \eta^\sigma$ and $\eta^\sigma(x) \subset W^\sigma(x)$ for μ-almost all x ($\sigma = s, u$). Let $\{\mu_x^\sigma|x \in M\}$ be a canonical system of conditional measures for μ and η^σ ($\sigma = s, u$). By Doob's theorem it follows that for $A \in \mathcal{B}$

$$\mu_x^s(A) = \lim_{n\to\infty} \mu_x^{f^{-n}\xi^s}(A) \quad \text{and} \quad \mu_x^u(A) = \lim_{n\to\infty} \mu_x^{f^n\xi^u}(A) \quad (\mu\text{-almost all } x).$$

Since $f\eta^\sigma = \eta^\sigma$ and f preserves μ, we have $\mu_x^\sigma(A) = \mu_{fx}^\sigma(fA)$ (μ-almost all x) for $A \in \mathcal{B}$ and $\sigma = s, u$.

Let $C(M)$ be the Banach space of continuous real-valued functions of M with the sup norm $\|\cdot\|$, and let $\mathcal{M}(M)$ be a set of all Borel probability measures on M with the weak topology. Since $C(M)$ is separable, there exists a countable set $\{\varphi_1, \varphi_2, \cdots\}$ which is dense in $C(M)$. For ν, $\nu' \in \mathcal{M}(X)$ define

$$\rho(\nu, \nu') = \sum_{n=1}^{\infty} \frac{|\int \varphi_n d\nu - \int \varphi_n d\nu'|}{2^n\|\varphi_n\|}.$$

Then ρ is a compatible metric for $\mathcal{M}(X)$ and $(\mathcal{M}(X), \rho)$ is compact (cf.[10]).

Lemma 2.3 *Let f, μ and $\{\mu_x^\sigma | x \in M\}$ be as above. Then for $\varepsilon > 0$ and $\sigma = s, u$ there exists a closed set F_ε^σ with $\mu(F_\varepsilon^\sigma) \geq 1 - \varepsilon$ satisfying the map*

$$F_\varepsilon^\sigma \ni x \mapsto \mu_x^\sigma \in \mathcal{M}(X)$$

is continuous.

Proof. Let $\{\varphi_1, \varphi_2, \cdots\}$ be as above. By the definition of conditional measures the map

$$M \ni x \mapsto \int \varphi_n d\mu_x^\sigma$$

is Borel measurable for $n \geq 1$. Thus, by Lusin's theorem, for $n \geq 1$ there exists a closed set F_n^σ with $\mu(F_n^\sigma) \geq 1 - \varepsilon/2^n$ satisfying

$$F_n^\sigma \ni x \mapsto \int \varphi_n d\mu_x^\sigma : \text{continuous.}$$

Then $F_\varepsilon^\sigma = \bigcap_{n=1}^\infty F_n^\sigma$ has the desired property.

□

Lemma 2.4 *Let f, μ and $\{\mu_x^\sigma | x \in M\}$ be as above. Then for μ-almost all $x \in M$ and $\sigma = s, u$, supp(μ_x^σ) has no isolated points.*

Proof. We will give the proof for $\sigma = u$ and so we here omit for $\sigma = s$ since the technique of the proof is similar.

If this lemma is false, supp(μ_x^u) has an isolated point for x belonging to some Borel set with positive measure. Let ξ^u be as in Lemma 2.1. Since diam$(f^{-k}\xi^u(x)) \to 0$ $(k \to \infty)$ by Lemma 2.1 (c),

$$P_{-k} = \{x \in M : \mu_x^{f^{-k}\xi^u} \text{ is a point measure }\}$$

has positive μ-measure for k large enough. Since μ is f-invariant, we have $f_*^n \mu_x^{f^{-k}\xi^u} = \mu_{f^n x}^{f^{n-k}\xi^u}$ for μ-almost all x and $n \in \mathbb{Z}$. Then

$$\begin{aligned} f^n(P_{-k}) &= \{f^n(x) \in M : \mu_x^{f^{-k}\xi^u} \text{ is a point measure }\} \\ &= \{x \in M : f_*^n \mu_{f^{-n}x}^{f^{-k}\xi^u} \text{ is a point measure }\} \\ &= \{x \in M : \mu_x^{f^{n-k}\xi^u} \text{ is a point measure }\} \\ &= P_{n-k} \qquad (n \in \mathbb{Z}). \end{aligned}$$

Put

$$P = \bigcap_{j \geq 1} \bigcup_{n \geq j} P_{n-k} = \bigcap_{j \geq 1} \bigcup_{n \geq j} f^n P_{-k}.$$

Since P is f-invariant and μ is ergodic, we have $\mu(P) = 1$.

For $x \in P$ there exists an increasing sequence $\{n_i\}_{i \geq 0}$ such that $x \in P_{n_i}$ for $i \geq 0$. Since $\mu_x^u = \lim_{n\to\infty} \mu_x^{f^n\xi^u} = \lim_{i\to\infty} \mu_x^{f^{n_i}\xi^u}$ and $\mu_x^{f^{n_i}\xi^u}$ is a point measure for i, so is μ_x^u. Therefore μ_x^u is a point measure for μ-almost all $x \in M$.

Since ξ^u is finer than η^u and μ_x^u is a point measure for μ-almost all $x \in M$, so is $\mu_x^{\xi^u}$. Thus $\mu_x^{\xi^u}(f^{-1}\xi^u(x)) = 1$ for μ-almost all x. Therefore

$$h_\mu(f) = \int -\log \mu_x^{\xi^u}(f^{-1}\xi^u(x))d\mu(x) = 0$$

by Lemma 2.2. This is a contradiction.

□

Let $B(x,r)$ and $U(x,r)$ denote the closed and open balls in M with center $x \in M$ and radius $r > 0$ respectively.

Lemma 2.5 *Let f, μ and $\{\mu_x^\sigma | x \in M\}$ be as above. Then for μ-almost all $x \in M$*

$$\overline{W^s(y)} = \overline{W^s(x)} \quad (\mu_x^s\text{-almost all } y \in M).$$

Proof. Let ξ^s be as in Lemma 2.1. Then we have that for μ-almost all $x \in M$

$$\xi^s(y) \subset \overline{W^s(x)} \quad (\mu_x^s\text{-almost all } y).$$

Indeed, let d_x^s denote the distance induced by the Riemannian metric on $W^s(x)$. Then there exist an increasing family $\{\Lambda_\ell\}_{\ell \geq 1}$ of closed sets of M with $\mu(\cup_{\ell\geq 1}\Lambda_\ell) = 1$ and a sequence $\{A_\ell\}_{\ell\geq 1}$ of positive numbers satisfying that

(e) for each $x \in \Lambda_\ell$ there exists $\varepsilon = \varepsilon(x) > 0$, such that

$$B(x,\varepsilon) \cap \Lambda_\ell \ni y \mapsto W_{A_\ell}^s(y) = \{z \in W^s(y) : d_x^s(z,y) \leq A_\ell\}$$

is continuous with respect to the Hausdorff metric d_H: i.e.

$$\lim_{\Lambda_\ell \ni y \to x} d_H(W_{A_\ell}^s(y), W_{A_\ell}^s(x)) = 0,$$

(f) for each $x \in \Lambda_\ell$, $\xi^s(x) \subset W_{A_\ell}^s(x)$

(cf.[7], [15], [16]). Take arbitrary $y \in \text{supp}(\mu_x^s|\Lambda_\ell)$ $(\ell \geq 1)$. Let $\varepsilon = \varepsilon(y) > 0$ be as in (e) and let $0 < r < \varepsilon$. Recall that for μ-almost all $x \in M$

$$\mu_x^s(\cup_{\ell\geq 1}\Lambda_\ell) = 1 \quad \text{and} \quad \mu_x^s|\Lambda_\ell = \lim_{n\to\infty} \mu_x^{f^{-n}\xi^s}|\Lambda_\ell \quad (\ell \geq 1).$$

Since $U(y,r)$ is open, we have $\mu_x^{f^{-n}\xi^s}(U(y,r)\cap\Lambda_\ell) > 0$ for n large enough. So we can take $y' \in U(y,r) \cap \Lambda_\ell \cap f^{-n}\xi^s(x)$. Since $y' \in f^{-n}\xi^s(x) \subset W^s(x)$, we have $W_{A_\ell}^s(y') \subset W^s(x)$. Since $y' \in U(y,r) \cap \Lambda_\ell$ and r is arbitrary, it follows that

$$\lim_{r\to 0} d_H(W_{A_\ell}^s(y'), W_{A_\ell}^s(y)) = 0.$$

Therefore $\xi^s(y) \subset W_{A_\ell}^s(y) \subset \overline{W^s(x)}$.

From this fact it follows that for $n \geq 0$ and μ-almost all $x \in M$

$$\xi^s(f^n(y)) \subset \overline{W^s(f^n(x))} \quad (\mu_x^s\text{-almost all } y).$$

Thus

$$W^s(y) = \cup_{n\geq 0} f^{-n}\xi^s(f^n y) \subset \cup_{n\geq 0} f^{-n}(\overline{W^s(f^n(x))}) \subset \overline{W^s(x)}$$

for μ^s_x-almost all y. On the other hand, by the definition of conditional measures, $\mu^s_y = \mu^s_x$ for $y \in \eta^s(x)$. This implies that $\overline{W^s(y)} = \overline{W^s(x)}$ for μ^s_x-almost all y. □

3 Proof of Theorem B(a)

The purpose of this section is to show Theorem B(a). Let f be a C^2-diffeomorphism of a closed C^∞-manifold M and let μ be an f-invariant ergodic Borel probability measure on M with positive entropy. As described in §1 the stable manifold $W^s(x)$ is a C^2 immersed manifold for μ-almost all $x \in M$ and so the closure of $W^s(x)$, $\overline{W^s(x)}$, is perfect.

Let η^s and $\{\mu^s_x | x \in M\}$ be as in §2. By Lemma 2.4, $\mathrm{supp}(\mu^s_x)$ has no isolated points for μ-almost all $x \in M$. Therefore, to obtain the conclusion it suffices to show the following.

Proposition 1 *If μ^s_x is not a point measure for μ-almost all $x \in M$, then $\overline{W^s(x)}$ is a $*$-chaotic set for μ-almost all $x \in M$.*

Proof. Fix $0 < \varepsilon < 1$ and let F^s_ε be as in Lemma 2.3. By assumption we can take and fix $x_0 \in \mathrm{supp}(\mu|F^s_\varepsilon)$ such that $\mu^s_{x_0}$ is not a point measure. Since $\mathrm{supp}(\mu^s_{x_0})$ is not one point, there are disjoint open sets O_1 and O_2 of M satisfying that

$$\begin{aligned} &d(O_1, O_2) = \inf\{d(x,y) : x \in O_1,\ y \in O_2\} > \delta \text{ and} \\ &\mu^s_{x_0}(O_i) > \delta \quad (i = 1, 2) \end{aligned} \tag{1}$$

for some $\delta > 0$. By Lemma 2.3 we can choose $\varepsilon' > 0$ such that

$$\mu^s_x(O_i) > \delta \quad (i = 1, 2) \tag{2}$$

for $x \in U(x_0, \varepsilon') \cap F^s_\varepsilon$. Put $K = \bigcap_{n=0}^\infty \bigcup_{k\geq n} f^{-k}(U(x_0, \varepsilon') \cap F^s_\varepsilon)$. Since $\mu(U(x_0, \varepsilon') \cap F^s_\varepsilon) > 0$, by ergodicity of μ we have $\mu(K) = 1$.

Take arbitrary δ' with $0 < \sqrt{\delta'} < \min\{\mu(U(x_0, \varepsilon') \cap F^s_\varepsilon), \delta\}$. Let ξ^s be as in Lemma 2.1 and put

$$A^s_m(n) = \left\{ x \in M \;\middle|\; \begin{array}{l} d_H(f^{-[k/2]}\xi^s(f^{[k/2]}x), \overline{W^s(x)}) \leq 1/m, \\ \mathrm{diam}(f^{k-[k/2]}\xi^s(f^{[k/2]}x)) \leq 1/m \quad (k \geq n) \end{array} \right\} \tag{3}$$

for $n, m \geq 1$. Then $A^s_m(n) \subset A^s_m(n+1)$ and $\mu(\bigcup_{n=0}^\infty A^s_m(n)) = 1$ by Lemma 2.1 (c) and (d). Thus there exists an increasing sequence $\{n_m\}$ such that $\mu(A^s_m(n_m)) \geq 1 - \delta'/2^m$ $(m \geq 1)$. Since

$$\int \mu^s_x(\cap_{m=1}^\infty A^s_m(n_m))d\mu = \mu(\cap_{m=1}^\infty A^s_m(n_m)) \geq 1 - \sum_{m=1}^\infty \delta'/2^m = 1 - \delta',$$

we can find a Borel set $C^s_{\delta'} \subset M$ with $\mu(C^s_{\delta'}) \geq 1-\sqrt{\delta'}$ satisfying

$$\mu^s_x(\cap_{m=1}^{\infty} A^s_m(n_m)) \geq 1-\sqrt{\delta'} \tag{4}$$

for $x \in C^s_{\delta}$. To obtain the conclusion it sufficies to show that $\overline{W^s(x)}$ is a $*$-chaotic set for $x \in K \cap C^s_{\delta'}$ because δ' is arbitrary.

For $x \in K \cap C^s_{\delta'}$, by the definition of K, there exists a sequence of positive integers $\{k_m\}_m$ with $k_m > n_m$ such that $f^{k_m}(x) \in U(x_0, \varepsilon') \cap F^s_\varepsilon$. Then by (4) and (2)

$$\begin{aligned} &\mu^s_x(A^s_m(k_m)) \geq \mu^s_x(\cap_{m=1}^{\infty} A^s_m(k_m)) \geq \mu^s_x(\cap_{m=1}^{\infty} A^s_m(n_m)) \geq 1-\sqrt{\delta'}, \\ &\mu^s_x(f^{-k_m}(O_i)) = \mu^s_{f^{k_m}(x)}(O_i) > \delta > \sqrt{\delta'} \quad (i=1,2,\ m \geq 1). \end{aligned}$$

Thus we have $\mu^s_x(A^s_m(k_m) \cap f^{-k_m}(O_i)) > 0$ for $i = 1,2$ and $m \geq 1$. From Lemma 2.5 we may assume that for $m \geq 1$ and $i = 1,2$ there exists a point $y_i = y_i(m) \in A^s_m(k_m) \cap f^{-k_m}(O_i)$ such that

$$\overline{W^s(y_i)} = \overline{W^s(x)}.$$

By (3) we have

$$\begin{aligned} &d_H(f^{-[k_m/2]}\xi^s(f^{[k_m/2]}y_i), \overline{W^s(x)}) = d_H(f^{-[k_m/2]}\xi^s(f^{[k_m/2]}y_i), \overline{W^s(y_i)}) \leq 1/m, \\ &\mathrm{diam}(f^{k_m-[k_m/2]}\xi^s(f^{[k_m/2]}y_i)) \leq 1/m \quad (i=1,2,\ m \geq 1). \end{aligned} \tag{5}$$

To show that $\overline{W^s(x)}$ is a $*$-chaotic set, suppose that nonempty open sets U_1 and U_2 satisfy

$$U_1 \cap U_2 \neq \emptyset, \quad U_j \cap \overline{W^s(x)} \neq \emptyset \quad (j=1,2).$$

By (5) we may assume that

$$\begin{aligned} &y_i \in f^{-[k_m/2]}\xi^s(f^{[k_m/2]}y_i) \cap U_j \neq \emptyset, \\ &f^{k_m}(y_i) \in f^{k_m-[k_m/2]}\xi^s(f^{[k_m/2]}y_i) \subset O_i \quad (1 \leq i,j \leq 2) \end{aligned} \tag{6}$$

if m is sufficiently large. Take

$$a_{i,j} = a_{i,j}(m) \in f^{-[k_m/2]}\xi^s(f^{[k_m/2]}y_i) \cap U_j$$

for $1 \leq i,j \leq 2$. Then we have that for $1 \leq i,j \leq 2$

$$a_{i,j} \in U_j, \quad d(f^{k_m}(a_{1,1}), f^{k_m}(a_{2,2})) > \tau \ \text{ and } \ d(f^{k_m}(a_{1,1}), f^{k_m}(a_{1,2})) < 1/m$$

by (1), (5) and (6). Since m is arbitrary, $\overline{W^s(x)}$ is a $*$-chaotic set for $x \in K \cap C^s_{\delta'}$.

□

4 Proof of Theorem B(b)

In this section we will prove Theorem B (b). Let f, μ, η^u and $\{\mu_x^u | x \in M\}$ be as in §2. By Lemma 2.4, $\mathrm{supp}(\mu_x^u)$ has no isolated points for μ-almost all $x \in M$. Therefore, to obtain the conclusion it sufficies to show the following.

Proposition 2 *If μ_x^u is not a point measure for μ-almost all $x \in M$, then $\mathrm{supp}(\mu_x^u)(\subset \overline{W^u(x)})$ is a $*$-chaotic set for μ-almost all $x \in M$.*

Proof. Fix $0 < \varepsilon < 1$ and let F_ε^u be as in Lemma 2.3. By assumption we can take and fix $x_0 \in \mathrm{supp}(\mu | F_\varepsilon^u)$ such that $\mu_{x_0}^u$ is not a point measure. Choose two distinct points $y_1, y_2 \in \mathrm{supp}(\mu_{x_0}^u)$ and put $\tau = d(y_1, y_2)/2 (> 0)$. Take arbitrarily $0 < r < \tau/2$ and choose $\delta = \delta(r) > 0$ such that

$$\mu_{x_0}^u(U(y_i, r)) > \delta \quad (i = 1, 2). \tag{7}$$

Remark that

$$d(U(y_1, r), U(y_2, r)) = \inf\{d(x, y) : d(x, y_1) < r,\ d(y, y_2) < r\} > \tau. \tag{8}$$

Since $U(y_i, r)$ $(i = 1, 2)$ are open, by (7) there exists a large integer $M = M(r) > 0$ such that

$$\nu(U(y_i, r)) > \delta = \delta(r) \quad (i = 1, 2) \tag{9}$$

for $\nu \in \mathcal{M}(M)$ with $\rho(\nu, \mu_{x_0}^u) < 1/M$. We can find $\epsilon' = \epsilon'(r) > 0$ such that

$$\rho(\mu_x^u, \mu_{x_0}^u) < 1/2M = 1/2M(r) \quad (x \in U(x_0, \varepsilon') \cap F_\varepsilon^u). \tag{10}$$

Note that ϵ' depends on r.

Let ξ^u be as in Lemma 2.1 and put

$$B_m^u(n) = \left\{ x \in M \;\middle|\; \begin{array}{l} \rho(\mu_x^{f^k \xi^u}, \mu_x^u) \le 1/m, \\ \mathrm{diam}(f^{-k+[k/2]} \xi^u (f^{-[k/2]} x)) \le 1/m \quad (k \ge n) \end{array} \right\} \tag{11}$$

for $n, m \ge 1$.Then $B_m^u(n) \subset B_m^u(n+1)$ and $\mu(\bigcup_{n=0}^\infty B_m^u(n)) = 1$, by Lemma 2.1 (c) and Doob's theorem. Thus there exists an increasing sequence $\{n_m\}$ such that $\mu(B_m^u(n_m)) \ge 1 - 1/2^{m+1}$ $(m \ge 1)$. Since $\mu(\cap_{k=m}^\infty B_k^u(n_k)) \ge 1 - 1/2^m$ for $m \ge 1$, we can find a Borel set D_m^u with $\mu(D_m^u) \ge 1 - 2^{-m/2}$ satisfying

$$\mu_x^u(\cap_{k=m}^\infty B_k^u(n_k)) \ge 1 - 2^{-m/2} \quad (x \in D_m^u). \tag{12}$$

Put

$$K_r = \bigcap_{k=1}^\infty \bigcup_{m=k}^\infty \left(\bigcap_{n=0}^\infty \bigcup_{\ell \ge n} f^{-\ell}(U(x_0, \varepsilon'(r)) \cap F_\varepsilon^u \cap D_m^u) \right) \quad (0 < r < \tau/2).$$

Then $\mu(K_r) = 1$ $(0 < r < \tau/2)$ by ergodicity of μ. To obtain the conclusion it sufficies to show that $\mathrm{supp}(\mu_x^u)$ is a $*$-chaotic set for $x \in K = \cap_{n \ge 1} K_{1/n}$.

To do this fix $x \in K_r$ $(r = 1/n, n \geq 1)$ and suppose that nonempty open sets U_1 and U_2 satisfy

$$U_1 \cap U_2 \neq \emptyset, \quad U_j \cap \mathrm{supp}(\mu_x^u) \neq \emptyset \quad (j = 1, 2).$$

Choose $m_0 > 0$ with

$$0 < 2^{-m_0/2} < \min\{\mu_x^u(U_j) : j = 1, 2\} \quad \text{and} \quad m_0 \geq 2M.$$

Since $x \in K_r$, by the definition of K_r, there exist $m_1 > m_0$ and a sequence of positive integers $\{\ell_k\}_k$ with $\ell_k > n_k$ such that

$$f^{\ell_k}(x) \in U(x_0, \varepsilon'(r)) \cap F_\varepsilon^u \cap D_{m_1}^u \quad (k \geq 1). \tag{13}$$

Since

$$\begin{aligned} \mu_x^u(f^{-\ell_k}(B_k^u(n_k))) &\geq \mu_x^u(f^{-\ell_k}(\cap_{k=m_1}^\infty B_k^u(n_k))) \\ &= \mu_{f^{\ell_k}(x)}^u(\cap_{k=m_1}^\infty B_k^u(n_k)) \\ &\geq 1 - 2^{-m_1/2} \geq 1 - 2^{-m_0/2} \quad (k \geq m_1) \end{aligned}$$

by (12), we have

$$\mu_x^u(U_j \cap f^{-\ell_k}(B_k^u(n_k))) \geq \mu_x^u(U_j) - 2^{-m_0/2} > 0 \quad (k \geq m_1).$$

Then, by the definition of $\{\mu_x^u | x \in M\}$, we can choose

$$z_j = z_j(k) \in U_j \cap f^{-\ell_k} B_k^u(n_k)$$

with $\eta^u(x) = \eta^u(z_j)$ for $j = 1, 2$ and $k \geq m_1$. Thus we have $\eta^u(f^{\ell_k}(x)) = \eta^u(f^{\ell_k}(z_j))$ and so $\mu_{f^{\ell_k}(x)}^u = \mu_{f^{\ell_k}(z_j)}^u$. Since $f^{\ell_k}(z_j) \in B_k^u(n_k) \subset B_k^u(\ell_k)$, by (11) we have

$$\rho(\mu_{f^{\ell_k}(z_j)}^{f^{\ell_k}\xi^u}, \mu_{f^{\ell_k}(x)}^u) = \rho(\mu_{f^{\ell_k}(z_j)}^{f^{\ell_k}\xi^u}, \mu_{f^{\ell_k}(z_j)}^u) \leq 1/k \leq 1/m_0 \leq 1/2M, \tag{14}$$

$$\mathrm{diam}(f^{-\ell_k+[\ell_k/2]}\xi^u(f^{\ell_k-[\ell_k/2]}(z_j))) \leq 1/k \tag{15}$$

for $j = 1, 2$ and $k \geq m_1$. Thus, by (9), (10), (13) and (14),

$$\mu_{z_j}^{\xi^u}(f^{-\ell_k}U(y_i, r)) = \mu_{f^{\ell_k}(z_j)}^{f^{\ell_k}\xi^u}(U(y_i, r)) > \delta.$$

Since $\mathrm{supp}(\mu_{z_j}^{\xi^u}) \subset \xi^u(z_j)$, we have

$$\xi^u(z_j) \cap f^{-\ell_k}U(y_i, r) \neq \emptyset$$

for $1 \leq i, j \leq 2$ and $k \geq m_1$. For k large enough, by (15) we may assume

$$z_j \in f^{-\ell_k+[\ell_k/2]}\xi^u(f^{\ell_k-[\ell_k/2]}(z_j)) \subset U_j.$$

Therefore

$$\begin{aligned} U_j \cap f^{-\ell_k}U(y_i, r) &\supset f^{-\ell_k+[\ell_k/2]}\xi^u(f^{\ell_k-[\ell_k/2]}(z_j)) \cap f^{-\ell_k}U(y_i, r) \\ &\supset \xi^u(z_j) \cap f^{-\ell_k}U(y_i, r) \neq \emptyset \end{aligned}$$

for $1 \leq i,j \leq 2$ and k large enough. Take

$$b_{i,j} = b_{i,j}(k) \in f^{-\ell_k}(U(y_i, r)) \cap U_j$$

for $1 \leq i,j \leq 2$. Then we have that for $1 \leq i,j \leq 2$

$$b_{i,j} \in U_j, \quad d(f^{\ell_k}(b_{1,1}), f^{\ell_k}(b_{2,2})) > \tau \text{ and } d(f^{\ell_k}(b_{1,1}), f^{\ell_k}(b_{1,2})) < r = 1/n$$

by (8). This implies that $\mathrm{supp}(\mu^u_x)$ is a $*$-chaotic set for $x \in K = \cap_{n\geq 1} K_{1/n}$.

□

5 Proof of Theorem C

The purpose of this section is to show Theorem C. Let f be a C^2-diffeomorphism of a closed C^∞-manifold M and let μ be an ergodic SRB measure. As described in §1 the Pesin's formula holds: i.e. $h_\mu(f) = \sum_{\lambda_i>0} \lambda_i m_i$. Thus we have $h_\mu(f) \geq \max\{\lambda_i\} > 0$ because μ satisfies the condition (A) mentioned in §1. Therefore, by Theorem B, $\overline{W^s(x)}$ is a $*$-chaotic set for μ-almost all $x \in M$.

To show that $\overline{W^u(x)}$ is a $*$-chaotic set we need the following lemma.

Lemma 5.1 ([8], Corollary 6.1.4) *Let μ be an ergodic measure satisfying Pesin's formula, let ξ^u be as in Lemma 2.1 and let ψ be the density of $\mu^{\xi^u}_x$ with respect to m^u_x. Then at μ-almost all x, ψ is a strictly positive function on $\xi^u(x)$ and $\log\psi$ is Lipschitz along W^u-leaves.*

Let η^u and $\{\mu^u_x | x \in M\}$ be as in §2. Then, by Proposition 2, $\mathrm{supp}(\mu^u_x)(\subset \overline{W^u(x)})$ is a $*$-chaotic set for μ-almost all $x \in M$. Therefore, to obtain the conclusion it sufficies to show the following.

Proposition 3 *If μ is an SRB measure, then $\mathrm{supp}(\mu^u_x) = \overline{W^u(x)}$ for μ-almost all $x \in M$.*

Proof. We first show that $\xi^u(x) \subset \mathrm{supp}(\mu^u_x)$ for μ-almost all $x \in M$. Since ξ^u is finer than η^u, for μ-almost all $z \in M$

$$\int \mu^{\xi^u}_x(\mathrm{supp}(\mu^u_z))d\mu^u_z(x) = \mu^u_z(\mathrm{supp}(\mu^u_z)) = 1.$$

Then $\mu^{\xi^u}_x(\mathrm{supp}(\mu^u_z)) = 1$ for μ^u_z-almost all x. Since $\mathrm{supp}(\mu^u_z)$ is closed, by Lemma 5.1 we have that

$$\xi^u(x) \subset \mathrm{supp}(\mu^u_z) = \mathrm{supp}(\mu^u_x)$$

for μ^u_z-almost all x. Therefore $\xi^u(x) \subset \mathrm{supp}(\mu^u_x)$ for μ-almost all x.

Since $f_*\mu^u_x = \mu^u_{fx}$ for μ-almost all x, we have $f(\mathrm{supp}(\mu^u_x)) = \mathrm{supp}(\mu^u_{fx})$ for μ-almost all x. By Lemma 2.1 (d)

$$\begin{aligned} W^u(x) &= \cup_{n=0}^{\infty} f^n \xi^u(f^{-n}(x)) \\ &\subset \cup_{n=0}^{\infty} f^n(\mathrm{supp}(\mu^u_{f^{-n}(x)})) \\ &\subset \mathrm{supp}(\mu^u_x) \end{aligned}$$

for μ-almost all $x \in M$. Therefore $\mathrm{supp}(\mu^u_x) = \overline{W^u(x)}$ for μ-almost all $x \in M$.

□

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