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Notes on the examples for certain starlike functions and convex functions of order α

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Abstract

T. Sekine and T. Yamanaka showed some examples of stalike functions and convex functions of order α with negative coefficients. However those examples have the coeffients which the second terms are zero. In this notes we show some examples that the coefficients of the second terms are not zero.

1 Introduction and Examples

Let A denote the class of functions $f(z)$ of the form

(1.1)
$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \quad (a_n\in C)$$

that are analytic in the unit disk $U=\{z\in C:|z|<1\}$.

Let S be the subclass of A consisting of functions which are univalent in the unit disk U. We denote by S^* the subclass of S consisting of starlike functions. Further we denote by K the subclass of S consisting of convex functions.

A function $f(z)$ in A is said to be starlike of order α if

(1.2)
$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha$$

for some $\alpha(0\leq\alpha<1)$, and for all $z\in U$. The subclass of A consisting of all starlike functions of order α is denoted by $S^*(\alpha)$.

A function $f(z)$ in A is said to be convex of order α if

(1.3)
$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha$$

for some $\alpha(0\leq\alpha<1)$, and for all $z\in U$. The subclass of A consisting of such functions is denoted by $K(\alpha)$.

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Let $A(1)$ denote the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0). \tag{1.4}$$

A function $f(z)$ of the above form is called an analytic function with negative coefficients.

We denote by T, $T^*(\alpha)$ and $C(\alpha)$ the subclasses of $A(1)$ that are, respectively, univalent, starlike of order α, and convex of order α. That is,

$$\begin{aligned} T &= S \cap A(1), \\ T^*(\alpha) &= S^*(\alpha) \cap A(1), \\ C(\alpha) &= K(\alpha) \cap A(1). \end{aligned}$$

We note that

$$f(z) \in C(\alpha) \text{ if and only if } zf'(z) \in T^*(\alpha). \tag{1.5}$$

In [2], H.Silverman gave the following coefficient inequalities for functions belonging to $T^*(\alpha)$ and $C(\alpha)$, respectively.

Theorem A([2], Theorem 2). *A function $f(z)$ in $A(1)$ is in $T^*(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} (n - \alpha) a_n \leq 1 - \alpha. \tag{1.6}$$

Theorem B([2], Corollary 2). *A function $f(z)$ in $A(1)$ is in $C(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} n(n - \alpha) a_n \leq 1 - \alpha. \tag{1.7}$$

Using the above theorems and (1.5), Sekine and Yamanaka[1] showed the following examples of functions in $T^*(\alpha)$ and $C(\alpha)$, respectively.

Example 1.1. ([1], Theorem 2.1) *The function*

$$\begin{aligned} f(z) &= \frac{1}{2}(1-z)^2 \log(1-z) + \frac{3}{2}z - \frac{3}{4}z^2 \\ &= z - \sum_{n=2}^{\infty} a_n z^n, \end{aligned} \tag{1.8}$$

where $a_2 = 0$, $a_n = \dfrac{1}{(n-2)(n-1)n} \quad (n \geq 3)$

belongs to $T^*(0)$.

Example 1.2. ([1], Corollary 2.1) *The function*

$$(1.9) \qquad \begin{aligned} g(z) &= \frac{1}{2}\int_0^z \frac{\log(1-\xi)}{\xi}d\xi + \left(\frac{1}{4}z^2 - z + \frac{3}{4}\right)\log(1-z) + \frac{9}{4}z - \frac{1}{2}z^2 \\ &= z - \sum_{n=2}^{\infty} a_n z^n, \end{aligned}$$

where $a_2 = 0,\ a_n = \dfrac{1}{(n-2)(n-1)n^2} \quad (n \geq 3)$
belongs to $C(0)$.

Example 1.3. ([1], Theorem 2.2) *The function*

$$(1.10) \qquad \begin{aligned} f(z) &= z - (1-\alpha)^2 z^\alpha \int_0^z \frac{(z-\xi)^2}{2}\frac{\xi^{-\alpha}}{1-\xi}d\xi \\ &= z - \sum_{n=2}^{\infty} a_n z^n, \end{aligned}$$

where $a_2 = 0,\ a_n = \dfrac{(1-\alpha)^2}{(n-2-\alpha)(n-1-\alpha)(n-\alpha)} \quad (n \geq 3)$
belongs to $T^*(\alpha) \quad (0 < \alpha < 1)$.

The following theorem is a new result.

Theorem 1.1. *The function*

$$(1.11) \qquad \begin{aligned} f(z) &= z - \frac{(1-\alpha)^2}{\alpha} z^\alpha \int_0^z \frac{(z-\xi)^2}{2}\frac{\xi^{-\alpha}}{1-\xi}d\xi \\ &= z - \sum_{n=2}^{\infty} a_n z^n, \end{aligned}$$

where $a_2 = 0,\ a_n = \dfrac{(1-\alpha)^2}{(n-2-\alpha)(n-1-\alpha)(n-\alpha)n} \quad (n \geq 3)$
belongs to $C(\alpha) \quad (0 < \alpha < 1)$.

Proof. By virtue of Example 1.3 and (1.5), we may complete the proof. □

2 Examples of the case $a_2 \neq 0$

Theorem 2.1. *Let* $0 < \delta \leq 1$. *Then the function*

$$(2.1) \qquad \begin{aligned} f(z) &= \frac{\delta}{2}(1-z)^2\log(1-z) + \left(1 + \frac{\delta}{2}\right)z - \left(\frac{1}{2} + \frac{\delta}{4}\right)z^2 \\ &= z - \sum_{n=2}^{\infty} a_n z^n, \end{aligned}$$

where $a_2 = \dfrac{1-\delta}{2},\ a_n = \dfrac{\delta}{(n-2)(n-1)n} \quad (n \geq 3)$
belongs to $T^*(0)$.

Proof. We first prove that the function

$$f(z) = z - \frac{1-\delta}{2}z^2 - \sum_{n=3}^{\infty} \frac{\delta}{(n-2)(n-1)n} z^n \quad (0 < \delta \leq 1)$$

belongs to $T^*(0)$.
For the functionb $f(z)$, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} n a_n &= \frac{2(1-\delta)}{2} + \sum_{n=3}^{\infty} \frac{n\delta}{(n-2)(n-1)n} \\
&= 1 - \delta + \delta \sum_{n=3}^{\infty} \frac{1}{(n-2)(n-1)} \\
&= 1 - \delta + \delta \sum_{n=3}^{\infty} \left(\frac{1}{n-2} - \frac{1}{n-1} \right) \\
&= 1 - \delta + \delta \\
&= 1.
\end{aligned}$$

Hence by virtue of Theorem A, the function $f(z)$ belongs to $T^*(0)$.
Next we use the equality (1.8) in Example 1.1.
Multiplying the both sides of (1.8) by δ, we have

$$\frac{\delta}{2}(1-z)^2 \log(1-z) + \frac{3}{2}\delta z - \frac{3}{4}\delta z^2 = \delta z - \sum_{n=3}^{\infty} \frac{\delta}{(n-2)(n-1)n} z^n. \tag{2.2}$$

Further adding

$$z - \delta z - \frac{1-\delta}{2} z^2$$

to the both sides of (2.2), we have that

$$\begin{aligned}
f(z) &= z - \frac{1-\delta}{2}z^2 - \sum_{n=3}^{\infty} \frac{\delta}{(n-2)(n-1)n} z^n \\
&= \frac{\delta}{2}(1-z)^2 \log(1-z) + \left(1 + \frac{\delta}{2}\right) z - \left(\frac{1}{2} + \frac{\delta}{4}\right) z^2.
\end{aligned}$$

□

The following Theorem 2.2, Theorem 2.3 and Therom 2.4 are also derived from Example 1.2, Example 1.3 and Theorem 1.1, respectively. We omit the proofs of the following theorems.

Theorem 2.2. *Let* $0 < \delta \leq 1$. *Then the function*

$$\begin{aligned}
g(z) &= \frac{\delta}{2} \int_0^z \frac{\log(1-\xi)}{\xi} d\xi + \delta \left(\frac{1}{4}z^2 - z + \frac{3}{4} \right) \log(1-z) + \frac{5\delta}{4} z - \frac{1+\delta}{4} z^2 \\
&= z - \sum_{n=2}^{\infty} a_n z^n,
\end{aligned} \tag{2.3}$$

where $a_2 = \dfrac{1-\delta}{4}$, $a_n = \dfrac{\delta}{(n-2)(n-1)n^2}$ $(n \geq 3)$
belongs to $C(0)$.

Theorem 2.3. *Let* $0 < \delta \leq 1$ *and* $0 < \alpha < 1$. *Then the function*

$$
\begin{aligned}
(2.4) \qquad f(z) &= z - \frac{(1-\alpha)(1-\delta)}{2-\alpha}z^2 - \delta(1-\alpha)^2 z^\alpha \int_0^z \frac{(z-\xi)^2}{2}\frac{\xi^{-\alpha}}{1-\xi}d\xi \\
&= z - \sum_{n=2}^{\infty} a_n z^n,
\end{aligned}
$$

where $a_2 = \dfrac{(1-\alpha)(1-\delta)}{2-\alpha}$, $a_n = \dfrac{\delta(1-\alpha)^2}{(n-2-\alpha)(n-1-\alpha)(n-\alpha)}$ $(n \geq 3)$
belongs to $T^*(\alpha)$.

Theorem 2.4. *Let* $0 < \delta \leq 1$ *and* $0 < \alpha < 1$. *Then the function*

$$
\begin{aligned}
(2.5) \qquad f(z) &= z - \frac{(1-\alpha)(1-\delta)}{2(2-\alpha)}z^2 - \delta\frac{(1-\alpha)^2}{\alpha} z^\alpha \int_0^z \frac{(z-\xi)^2}{2}\frac{\xi^{-\alpha}}{1-\xi}d\xi \\
&= z - \sum_{n=2}^{\infty} a_n z^n,
\end{aligned}
$$

where $a_2 = \dfrac{(1-\alpha)(1-\delta)}{2(2-\alpha)}$, $a_n = \dfrac{\delta(1-\alpha)^2}{(n-2-\alpha)(n-1-\alpha)(n-\alpha)n}$ $(n \geq 3)$
belongs to $C(\alpha)$.

References

[1] T. Sekine and T. Yamanaka, Starlike functions and convex functions of order α with negative coefficients, Math. Sci. Res. Hot-Line 1(1997), 7-12.

[2] H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51(1975), 109 - 116.