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Title	On some conditions of starlikeness
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Citation	数理解析研究所講究録 (1999), 1112: 138-141
Issue Date	1999-09
URL	http://hdl.handle.net/2433/63350
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

Kyoto University

数理解析研究所講究録
1112 巻 1999 年 138-141

On some conditions of starlikeness

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Abstract

Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be analytic in the unit disk U. Yuan Chun Fang proved that

$$|\frac{f''(z)}{f'(z)}| < m \Longrightarrow Re\frac{zf'(z)}{f(z)} > 0 \quad (z \in U),$$

where $m(= 2.83\cdots)$ is the best possible. In this paper, we generalize this theorem.

1. Introduction

Let A_p denote the class of functions of the form

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2}\cdots\cdots \quad (p \in N), \tag{1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$, and M_p denote the class of functions

$$f(z) = z^{-p} + a_{-p+1}z^{-p+1} + a_{-p+2}z^{-p+2}\cdots\cdots \quad (p \in N), \tag{2}$$

which are analytic in punctured disk $U - \{0\}$. Then we denote two classes of starlike functions as follows:

$$A_p^* = \{f \in A_p : \Re\frac{zf'(z)}{f(z)} > 0, \ z \in U\}, \tag{3}$$

$$M_p^* = \{f \in M_p : \Re\frac{zf'(z)}{f(z)} < 0, \ z \in U\}. \tag{4}$$

For the class A_1, Singh and Singh [4] showed the following theorem.

Theorem A *Let $f(z) \in A_1$, then*

$$|\frac{f''(z)}{f'(z)}| < \frac{3}{2} \quad (z \in U) \Longrightarrow f(z) \in A_1^*. \tag{5}$$

He used Jack's Lemma. Mocanu [3,p.338] showed

Theorem B *Let $g(z) = \frac{e^{\lambda z}-1}{\lambda}$.*

$$g(z) \in A_1^* \Longleftrightarrow |\lambda| \le m = 2.8329\cdots . \tag{6}$$

Where m is the least positive solutin of the following equation

$$\cos\sqrt{x^2-1} + \sqrt{x^2-1}\sin\sqrt{x^2-1} - \frac{1}{e} = 0. \tag{7}$$

Miller and Mocanu [2] proved, using their theory of first order differential subordination, that

Theorem C *Let $f(z) \in A_1$.Then*

$$|\frac{f''(z)}{f'(z)}| < 2 \Longrightarrow f(z) \in A_1^*. \tag{8}$$

In the same article [2], they posed the interesting question of finding the maximum value of k for which

$$|\frac{f''(z)}{f'(z)}| < k \Longrightarrow f(z) \in A_1^*. \tag{9}$$

From the above two theorems, $2 \le k \le m$. Some Mathematicians improved the lower bound of k. And recently, Yuan Chun Fang [4] showed

Theorem D *Let $f(z) \in A_1$. Then*

$$|\frac{f''(z)}{f'(z)}| < m \quad (z \in U) \Longrightarrow f(z) \in A_1^*. \tag{10}$$

The result is sharp, with the extremal function

$$G(z) = \frac{e^{mz}-1}{m}. \tag{11}$$

The purpose of this paper is to obtain simillar theorems for A_p and M_p.

Theorem 1 *If $f(z) \in A_p$ satisfies*

$$|\frac{f''(z)}{f'(z)} - \frac{p-1}{p}\frac{f'(z)}{f(z)}| \leq m \quad (z \in U), \tag{12}$$

then f belongs to A_p^*. The result is sharp, with the extremal function

$$G_1(z) = (\frac{e^{mz}-1}{m})^p. \tag{13}$$

Theorem 2 *If $f(z) \in M_p$ satisfies*

$$|\frac{f''(z)}{f'(z)} - \frac{p+1}{p}\frac{f'(z)}{f(z)}| \leq m \quad (z \in U), \tag{14}$$

then f belongs to M_p^*. The result is sharp, with the extremal function

$$G_2(z) = (\frac{m}{e^{mz}-1})^p. \tag{15}$$

2. Proof of Theorem 1

We use the following lemma due to Miller and Mocanu [2, Theorem 2].

Lemma *let h be convex in U and θ and ϕ be analytic in a domain D. Let p be analytic in U, with $p(0) = h(0) = \theta(p(0))$ and $p(U) \subset D$. If the differential equation*

$$\theta(q(z)) + zq'(z)\phi(q(z)) = h(z) \tag{16}$$

has a univalent solution in U that satisfies $q(0) = h(0)$ and

$$\theta(q(z)) \prec h(z), \tag{17}$$

then the relation

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec h(z) \tag{18}$$

implies $p(z) \prec q(z)$. The function q is the best dominant of (18).

Suppose $f(z) \in A_p^*$ satisfies (12), then we have

$$\frac{zf''(z)}{f'(z)} - \frac{p-1}{p}\frac{zf'(z)}{f(z)} \prec mz. \tag{19}$$

Let put

$$p(z) = \frac{1}{p}\frac{zf'(z)}{f(z)}, \qquad q(z) = \frac{1}{p}\frac{zG_1'(z)}{G_1(z)}$$

$$h(z) = 1 + mz, \qquad \theta(z) = z, \quad and \quad \phi(z) = \frac{1}{z}.$$

Then we have

$$q(z) = \frac{zG'(z)}{G(z)} = m\frac{ze^{mz}}{e^{mz}-1}. \tag{20}$$

$$q(z) + \frac{zq'(z)}{q(z)} = 1 + mz, \tag{21}$$

and

$$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} - \frac{p-1}{p}\frac{zf'(z)}{f(z)}. \tag{22}$$

From (19) and (22) we obtain (18), and from (21) we obtain (16). It yields $p(z) \prec q(z)$, and so $\frac{zf'(z)}{f(z)} \prec \frac{zG'(z)}{G(z)}$. Therefore, from Theorem B we obtain that

$$\frac{zf'(z)}{f(z)} > 0 \quad (z \in U).$$

Concernig the exremal function, we have

$$\frac{zG_1''(z)}{G_1'(z)} - \frac{p-1}{p}\frac{G_1 f'(z)}{G_1(z)} = m.$$

This implies that $G_1(z)$ is exremal.

Proof of Theorem 2 is similar, so we omit.

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