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Author(s)	Akiyama, Mai; Yoshida, Hiroaki
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THE DISTRIBUTIONS FOR LINEAR COMBINATIONS OF A FREE FAMILY OF PROJECTIONS AND THEIR APPLICATIONS

お茶の水女子大 理　秋山 麻衣 (MAI AKIYAMA)
吉田 裕亮 (HIROAKI YOSHIDA)

1. Introduction

In [22], Voiculescu began studying the operator algebra free products from the probabilistic point of view. His idea is to look at free products as an analogue of tensor products and to develop a corresponding highly noncommutative probabilistic framework, where freeness is given as the notion of independence (see [21]). It has been introduced in [23] the operation of the additive free convolution as analogue of the usual convolution. In order to compute it, it was also introduced the R-transform which linearlizes the additive free convolution. Thus it is the free analogue of the logarithm of the Fourier transform of a probability distribution or the free cumulants. An alternative, combinatorial approach to the R-transform was found by Speicher in [19]. The most important advantages of this combinatorial approach is that it can be generalized in a straightforward way to multi-dimensional situations as in [16]. The machinery of the R-transform was found independently and simultaneously by Woess in [25], by Soardi in [18], and by Cartwright and Soardi in [6] and [7], from the studies of the random walks on free product groups to obtain the walk generating function or the Plancherel measures.

The spectral theory of the infinite graphs such as the homogeneous tree or the infinite distance regular graphs, has been studied in [4], [11], [13], and [15], for example. The survey on the spectra of infinite graphs is now available in [16]. Especially, many authors have contributed to spectral theory and to harmonic analysis for the homogeneous tree T_m. If m is even, then T_m is the Cayley graph of a free group, and many papers have dealt with this structure. The ancestor is Kesten in [14], who calculated the closed walk generating function of the transition operator. In [24], Voiculescu has also treated it by using the R-transform, which is called generally free harmonic analysis.

In this note, we calculate the probability measures associated to the linear combinations of the above free families of projections, explicitly, by using the R-transform and the Stieltjes inversion formula. Some applications to the spectral theory for the infinite distance regular graphs, and to the Plancherel measures of the free products of two cyclic groups as in [6], are also discussed. Then we find the recursion formula for the orthogonal polynomials of the measures obtained in above.

2. The linear combinations of a free family of projections

Let $\{p_i\}_{i=1}^n$ be a free family of projections with $\phi(p_i) = \alpha_i$ for each i. We consider the linear combination, $\ell = \sum_{i=1}^n \lambda_i p_i$ of these projections, where λ_i is assumed to be positive. Using the properties of the R-transform (see [21]), it can be given for the element ℓ that

$$R_\ell(z) = \sum_{i=1}^n \frac{1}{2z}\left\{(\lambda_i z - 1) + \sqrt{(\lambda_i z - 1)^2 + 4\alpha_i\lambda_i z}\right\}, \tag{2.1}$$

which implies

$$K_\ell(z) = -\left(\frac{n-2}{2z}\right) + \frac{1}{2z}\sum_{i=1}^n\left\{\lambda_i z + \sqrt{(\lambda_i z - 1)^2 + 4\alpha_i\lambda_i z}\right\}. \tag{2.2}$$

If we solve the equation $\zeta = K_\ell(z)$ in z then we can have the G-series, $G_\ell(\zeta)$, the Cauchy transform of the compactly supported probability measure on $\mathbb{R}$ associated with the self-adjoint element ℓ. It is immediately seen that $G_\ell(\zeta)$ is an algebraic, but in general case, it can not be solved in radicals. However, this can be done, for instance, in the cases where at most two different square roots will appear in the right hand side of the equation (2.2). That is, in the cases where the family $\{(\alpha_i, \lambda_i)\}_{i=1}^n$ is constituted from at most two different pairs. From now on, we will concentrate our attention upon the following typical two cases and find the probability measure of the random variable ℓ in each case:

Case 1) $(\alpha_i, \lambda_i) = (\alpha, \lambda)$ for $i = 1, 2, ..., n$, with $n \geq 2$, $0 < \alpha < 1$, $\lambda > 0$,

Case 2) $n = 2$ and $\{(\alpha_i, \lambda_i)\}_{i=1,2}$ with $0 < \alpha_i < 1$, $\lambda_i > 0$.

First we shall investigate Case 1). In this case, the equation $\zeta = K_\ell(z)$ yields the quadratic equation in z that

$$\zeta(\zeta - n\lambda)z^2 + ((n-2)\zeta + n\lambda(1 - n\alpha))\,z + (1-n) = 0, \tag{2.3}$$

and the G-series of the element ℓ can be obtained as

$$G_\ell(\zeta) = \frac{-\{(n-2)\zeta + n\lambda(1-n\alpha)\} + n\sqrt{(\zeta-\gamma_+)(\zeta-\gamma_-)}}{2\zeta(\zeta - n\lambda)}, \tag{2.4}$$

where $\gamma_\pm = \lambda\{(n-2)\alpha + 1\} \pm 2\lambda\sqrt{(n-1)\alpha(1-\alpha)}$ and it holds the inequalities that $0 \leq \gamma_- < \gamma_+ \leq n\lambda$. Here the branch of the analytic square root in (2.4) should be determined by the condition

$$\mathrm{Im}\zeta > 0 \implies \mathrm{Im}G(\zeta) \leq 0. \tag{2.5}$$

We shall determine the probability measure ν of ℓ by using the Stieltjes inversion formula on $G_\ell(\zeta)$. It says that ν has point masses where $G_\ell(\zeta)$ has poles on $\mathbb{R}$ and the mass equals the residue there, and ν is absolutely continuous with respect to Lebesgue measure where $G_\ell(\zeta)$ has non-zero imaginary part on the real axis with the density

$$-\frac{1}{\pi}\lim_{\varepsilon\to+0}\operatorname{Im} G_\ell(t+i\varepsilon). \tag{2.6}$$

In our case, it is easily seen that ν is absolutely continuous on the interval $[\gamma_-,\gamma_+]$ with the density

$$f(t)=\frac{-n\sqrt{-(t-\gamma_+)(t-\gamma_-)}}{2\pi t(t-n\lambda)}. \tag{2.7}$$

Concerning with the poles, we have that $\zeta=0$ is removable singularity if $1-n\alpha\le 0$ and it is a simple pole with residue $1-n\alpha$ if $1-n\alpha>0$. Similarly, $\zeta=n\lambda$ is removable singularity if $1-n(1-\alpha)\le 0$ and it is a simple pole with residue $1-n(1-\alpha)$ if $1-n(1-\alpha)>0$. From the above observations, we have the measure as follows:

Theorem 2.1. *Let $\{p_i\}_{i=1}^n$ be a free family of projections with $\phi(p_i)=\alpha$ for all i. Then the distribution ν for the element $\ell=\lambda\sum_{i=1}^n p_i$ where $\lambda>0$, is given by*

$$\begin{aligned} d\nu=&\frac{-n\sqrt{-(t-\gamma_+)(t-\gamma_-)}}{2\pi t(t-n\lambda)}\chi_{[\gamma_-,\gamma_+]}dt\\ &+\max(0,1-n\alpha)\delta_0+\max(0,1-n(1-\alpha))\delta_{n\lambda}, \end{aligned} \tag{2.8}$$

where dt denotes the Lebesgue measure, δ_t is the Dirac unit mass at t, and χ_I means the characteristic function for the interval I.

Next we shall consider Case 2). That is $\ell=\lambda p+\mu q$ where p and q are free projections with $\phi(p)=\alpha$ and $\phi(q)=\beta$, and λ and μ are positive scalars. In this case, the equation $\zeta=K_\ell(z)$ becomes

$$\zeta=\frac{1}{2z}\left\{(\lambda+\mu)z+\sqrt{(\lambda z-1)^2+4\alpha\lambda z}+\sqrt{(\mu z-1)^2+4\beta\mu z}\right\}. \tag{2.9}$$

After some more tedious calculation, we can see that the equation (2.9) will be reduced to the quadratic equation $Az^2+Bz+C=0$, where

$$\begin{aligned} A&=\zeta(\zeta-\lambda)(\zeta-\mu)(\zeta-\lambda-\mu),\\ B&=\{\lambda(1-2\alpha)+\mu(1-2\beta)\}\zeta(\zeta-\lambda-\mu)+\lambda\mu(\lambda+\mu)(1-\alpha-\beta),\\ C&=-\{(\zeta-\mu)-(\lambda\alpha-\mu\beta)\}\{(\zeta-\lambda)+(\lambda\alpha-\mu\beta)\}. \end{aligned} \tag{2.10}$$

If we put $D=B^2-4AC$ then it follows by direct calculation that

$$D=(2\zeta-\lambda-\mu)^2(\zeta-\gamma_1)(\zeta-\gamma_2)(\zeta-\gamma_3)(\zeta-\gamma_4), \tag{2.11}$$

where γ_i's are given by

$$\begin{aligned}
\gamma_1 &= \frac{1}{2}\left((\lambda+\mu) - \sqrt{(\lambda+\mu)^2 - 4\lambda\mu(\sqrt{(1-\alpha)(1-\beta)} - \sqrt{\alpha\beta}\,)^2}\right),\\
\gamma_2 &= \frac{1}{2}\left((\lambda+\mu) - \sqrt{(\lambda+\mu)^2 - 4\lambda\mu(\sqrt{(1-\alpha)(1-\beta)} + \sqrt{\alpha\beta}\,)^2}\right),\\
\gamma_3 &= \frac{1}{2}\left((\lambda+\mu) + \sqrt{(\lambda+\mu)^2 - 4\lambda\mu(\sqrt{(1-\alpha)(1-\beta)} + \sqrt{\alpha\beta}\,)^2}\right),\\
\gamma_4 &= \frac{1}{2}\left((\lambda+\mu) + \sqrt{(\lambda+\mu)^2 - 4\lambda\mu(\sqrt{(1-\alpha)(1-\beta)} - \sqrt{\alpha\beta}\,)^2}\right).
\end{aligned} \tag{2.12}$$

Swap p and q, and replace p by $1-p$ or q by $1-q$ if necessary, we may assume that $\lambda \geq \mu$ and $\alpha \leq \beta \leq \frac{1}{2}$ without any loss of generality. First we shall pay our attention upon the case where strictly $\lambda > \mu$. It can be seen the inequalities

$$0 \leq \gamma_1 \leq \gamma_2 \leq \mu < \lambda \leq \gamma_3 \leq \gamma_4 \leq \lambda + \mu, \tag{2.13}$$

and that $G_\ell(\zeta)$ can be given as

$$\begin{aligned}
G_\ell(\zeta) =& \frac{1}{2\zeta(\zeta-\lambda)(\zeta-\mu)(\zeta-\lambda-\mu)} \times \\
&\Big(-\{\lambda(1-2\alpha)+\mu(1-2\beta)\}\zeta(\zeta-\lambda-\mu) - \lambda\mu(\lambda+\mu)(1-\alpha-\beta) \\
&+ \sqrt{(2\zeta-\lambda-\mu)^2(\zeta-\gamma_1)(\zeta-\gamma_2)(\zeta-\gamma_3)(\zeta-\gamma_4)}\Big),
\end{aligned} \tag{2.14}$$

where the branch of the analytic square root should be determined by the same condition as in (2.5). If $\alpha < \beta$, it is the most generic case where $G_\ell(\zeta)$ has two removable singularities and two simple poles. Taking care of the choices of the branch of the analytic square root in $G_\ell(\zeta)$, it follows that $G_\ell(\zeta)$ has simple poles at 0 and λ with the residues $\mathrm{Res}(0) = 1-\alpha-\beta$ and $\mathrm{Res}(\lambda) = \beta - \alpha$. Note that $z = \mu$ and $z = \lambda + \mu$ are removable singularities. By the Stieltjes inversion formula, it follows that ν is absolutely continuous with respect to the Lebesgue measure on the intervals $[\gamma_1, \gamma_2]$ and $[\gamma_3, \gamma_4]$ with the densities for $t \in [\gamma_1, \gamma_2]$,

$$f_1(t) = \frac{\left(t - \frac{\lambda+\mu}{2}\right)\sqrt{-(t-\gamma_1)(t-\gamma_2)(t-\gamma_3)(t-\gamma_4)}}{\pi t(t-\lambda)(t-\mu)(t-\lambda-\mu)}, \tag{2.15}$$

and for $t \in [\gamma_3, \gamma_4]$,

$$f_2(t) = \frac{-\left(t - \frac{\lambda+\mu}{2}\right)\sqrt{-(t-\gamma_1)(t-\gamma_2)(t-\gamma_3)(t-\gamma_4)}}{\pi t(t-\lambda)(t-\mu)(t-\lambda-\mu)}. \tag{2.16}$$

Hence, we have the probability measure as

$$d\nu = f_1(t)\chi_{[\gamma_1,\gamma_2]}dt + f_2(t)\chi_{[\gamma_3,\gamma_4]}dt + (1-\alpha-\beta)\delta_0 + (\beta-\alpha)\delta_\lambda. \tag{2.17}$$

For the other cases, we can also find the probability measure without much difficulties via the similar arguments and finally we have the following results:

Theorem 2.2. *Let $\{p, q\}$ be a free pair of projections with $\phi(p) = \alpha$ and $\phi(q) = \beta$, and let λ and μ are positive scalars. Then the distribution ν for the element $\ell = \lambda p + \mu q$ is given in the following:*

(I) $\lambda > \mu$;

(i) $\alpha < \beta$,

$$d\nu = \frac{-\left|t - \frac{\lambda+\mu}{2}\right| \sqrt{-(t-\gamma_1)(t-\gamma_2)(t-\gamma_3)(t-\gamma_4)}}{\pi t(t-\lambda)(t-\mu)(t-\lambda-\mu)} \chi_{[\gamma_1,\gamma_2]\cup[\gamma_3,\gamma_4]} dt + (1-\alpha-\beta)\delta_0 + (\beta-\alpha)\delta_\lambda, \tag{2.18}$$

(ii) $\alpha = \beta \neq \frac{1}{2}$,

$$d\nu = \frac{-\left|t - \frac{\lambda+\mu}{2}\right|}{\pi t(t-\lambda-\mu)} \sqrt{-\frac{(t-\gamma_1)(t-\gamma_4)}{(t-\lambda)(t-\mu)}} \chi_{[\gamma_1,\mu]\cup[\lambda,\gamma_4]} dt + (1-2\alpha)\delta_0, \tag{2.19}$$

(iii) $\alpha = \beta = \frac{1}{2}$,

$$d\nu = \frac{\left|t - \frac{\lambda+\mu}{2}\right|}{\pi\sqrt{-t(t-\lambda)(t-\mu)(t-\lambda-\mu)}} \chi_{[0,\mu]\cup[\lambda,\lambda+\mu]} dt, \tag{2.20}$$

(II) $\lambda = \mu$;

(i) $\alpha < \beta$,

$$d\nu = \frac{\sqrt{-(t-\gamma_1)(t-\gamma_2)(t-\gamma_3)(t-\gamma_4)}}{\pi|t(t-\lambda)(t-2\lambda)|} \chi_{[\gamma_1,\gamma_2]\cup[\gamma_3,\gamma_4]} dt + (1-\alpha-\beta)\delta_0 + (\beta-\alpha)\delta_\lambda, \tag{2.21}$$

(ii) $\alpha = \beta \neq \frac{1}{2}$,

$$d\nu = \frac{\sqrt{-(t-\gamma_1)(t-\gamma_4)}}{-\pi t(t-2\lambda)} \chi_{[\gamma_1,\gamma_4]} dt + (1-2\alpha)\delta_0, \tag{2.22}$$

(iii) $\alpha = \beta = \frac{1}{2}$,

$$d\nu = \frac{1}{\pi\sqrt{-t(t-2\lambda)}} \chi_{[0,2\lambda]} dt, \tag{2.23}$$

where γ_i's are given by (2.12).

Of course, the last two cases in are included in the case of $n = 2$ of Theorem 2.1 and the last one is nothing but the arcsin law on the interval $[0, 2\lambda]$.

3. Some applications

The special cases of the measures which we have given in the previous section, have been obtained as the spectral measures of the adjacency operators of some infinite graphs and the Plancherel measures for some infinite discrete groups. In this section, we shall show how they connect to our measures.

Definition 3.1. Let $\mathcal{G} = (V, E)$ be an unoriented infinite graphs with the set of vertices V and one of edges E. One consider the Hilbert space $\ell^2(V)$ of all the square summable functions on V. Suppose $\mathcal{G}$ is *uniformly locally finite*, that is, $\deg(\mathcal{G}) = \sup\{\deg(u) : u \in V\} < \infty$, where $\deg(u)$ is the number of edges emanating from u. Then the bounded self-adjoint operator A on $\ell^2(V)$, called the *adjacency operator* of $\mathcal{G}$, is defined by

$$(Af)(u) = \sum_{(u,v)} f(v) \qquad f \in \ell^2(V), \tag{3.1}$$

where (u, v) forms an edge.

Concerning with the measures in Theorem 2.1, the spectral measures of the adjacency operators of the infinite distance-regular graphs can be obtained as its special case.

Definition 3.2. A connected graph $\mathcal{G}$ is called *distance-regular* if there exists a function $f : (\mathbb{N}_0)^3 \longrightarrow \mathbb{N}_0$ such that for all $u, v \in V(\mathcal{G})$ and $j, k \in \mathbb{N}_0$,

$$^{\#}\{w \in V(\mathcal{G}) : d(u, w) = j, d(v, w) = k\} = f(j, k, d(u, v)), \tag{3.2}$$

where $V(\mathcal{G})$ is the set of all vertices of the graph $\mathcal{G}$ and, as usual, $d(u, v)$ is the distance between u and v, the length of the shortest walk from u to v.

The infinite distance-regular graphs have been completely characterized in [10]. They are tree-like graphs and parameterized by two integers $m, s \geq 2$. The infinite distance-regular graph $D_{m,s}$ can be obtained from the biregular tree $T_{m,s}$. Here, the biregular tree $T_{m,s}$ is an infinite tree where the vertex degree is constant on each of the two bipartite classes, with values m and s, respectively. The set of vertices of the infinite distance-regular graph $D_{m,s}$ is the bipartite block of degree m, and two vertices constitute an edge if and only if their distance in $T_{m,s}$ is two. Hence, each vertex of $D_{m,s}$ lies in the intersection of exactly m copies of the finite complete graph K_s, in particular, $D_{m,2}$ is nothing but the m-homogeneous tree T_m.

We consider the free product group $G = \underbrace{\mathbb{Z}_s * \mathbb{Z}_s * \cdots * \mathbb{Z}_s}_{m}$ and the reduced group C^*-algebra $C_r^*(G)$. Let u_i $(i = 1, 2, \ldots, m)$ be the unitary generator of each cyclic group in $C_r^*(G)$. Then it is easy to see that, for all i, $p_i = \frac{1}{s}\sum_{j=1}^{s}(u_i)^j$ is a projection with $\tau_G(p_i) = 1/s$. Furthermore, $\{p_i\}_{i=1}^m$ is a free family of projections in a C^*-probability space $(C_r^*(G), \tau_G)$, where $\tau_G(\cdot) = \langle \cdot\ \delta_e | \delta_e \rangle$ is the canonical faithful tracial state by the characteristic function δ_e at the identity e.

From the definitions of the free product and of the infinite distance-regular graph, it is clear that there exists a bijection between the set of vertices of the graph $D_{m,s}$ and the group G, Furthermore the adjacency operator A can be represented as

$$A = \sum_{i=1}^{m} \left(\sum_{j=1}^{s-1} (u_i)^j \right) = \sum_{i=1}^{m} (sp_i - 1) = s \sum_{i=1}^{m} p_i - m \cdot 1 \tag{3.3}$$

in $C_r^*(G)$. Now Theorem 2.1 is applicable with $n = m$, $\lambda = s$, and $\alpha = 1/s$. Making m-shift, we have the spectral measure $\nu_{m,s}$ for the adjacency operator of $D_{m,s}$ in the following: We put the interval as

$$I_{m,s} = [s - 2 - 2\sqrt{(m-1)(s-1)}, s - 2 + 2\sqrt{(m-1)(s-1)}] \tag{3.4}$$

and the function

$$f_{m,s}(t) = \frac{-m\sqrt{-(t-s+2)^2 + 4(m-1)(s-1)}}{2\pi(t+m)(t-m(s-1))}, \tag{3.5}$$

then we obatain the measure

$$d\nu_{m,s} = \begin{cases} f_{m,s}(t)\chi_{I_{m,s}} dt & \text{if } m \geq s, \\ f_{m,s}(t)\chi_{I_{m,s}} dt + \left(1 - \frac{m}{s}\right) \delta_{-m} & \text{if } m < s. \end{cases} \tag{3.6}$$

Remark 3.3. The measures that we obtained in Theorem 2.1 can be also found in [5] and [8]. Especially in [8], they calculated the measure for which a sequence of polynomials generated from a constant recursion formula, is orthogonal.

Let us state an application of the measures in Theorem 2.2. In [6], Cartwright and Soardi considered the free product group $G = \mathbb{Z}_r * \mathbb{Z}_s$, where $r > s \geq 2$ and the length for the elements of G was defined. They studied the convolution C^*-algebra generated by the characteristic function χ_1 on the elements of the length 1 and obtain the associated Plancherel measure. This measure can be regarded as the special case of ours as follows :

Let u_1 and u_2 be the unitary generators of the cyclic groups for $\mathbb{Z}_r$ and $\mathbb{Z}_s$ in the reduced C^*-algebra $C_r^*(G)$, respectively. Then the convolution operator T_{χ_1} associated to the characteristic function χ_1 is in the form

$$T_{\chi_1} = \sum_{i=1}^{r-1} (u_1)^i + \sum_{j=1}^{s-1} (u_2)^j. \tag{3.7}$$

As we mentioned before, $\sum_{i=1}^{r-1}(u_1)^i$ can be written as $rp_1 - 1$ with a projection p_1 of trace $1/r$. Similarly, we have $\sum_{j=1}^{s-1}(u_2)^j = sp_2 - 1$ where p_2 is a projection of trace $1/s$. Hence it follows that

$$T_{\chi_1} = rp_1 + sp_2 - 2 \tag{3.8}$$

and $\{p_1, p_2\}$ is a free pair of projections. Now it is clear that the Plancherel measure can be obtained as the special case of Theorem 2.2, see also [7].

As we mentioned at the beginning of Section 2, if the family $\{(\alpha_i, \lambda_i)\}_{i=1}^n$ is constituted from at most two different pairs then we can find the G-series explicitly. Thus, for instance, we can also obtain the Plancherel measure for the group of the free product of k copies of $\mathbb{Z}_r$ and m copies of $\mathbb{Z}_s$.

4. The orthogonal polynomials for a simple sum of n-projections

In this section, we will give the orthogonal polynomials with respect to the probability measures obtained in Theorem 2.1. As we mentioned in Remark 3.3, Cohen and Trenholme considered in [8] the the sequence of polynomials determined by the following constant recursion formula:

$$\begin{aligned} &P_0(X) = c, \quad P_1(X) = X - \alpha_0, \\ &P_{m+1}(X) = (X - a)P_m(X) - bP_{m-1}(X) \quad (m \geq 1), \end{aligned} \tag{4.1}$$

where α_0 and a are arbitrary real numbers, and b and c are positive numbers. Furthermore, they calculated the measure ν, explicitly, for which the sequence of polynomials $\{P_m(X)\}$ is orthogonal. Their normalization for the measure, however, is not one for the probability measure in general. Here we should note that there is a typological error in [8] that we have to multiplicate by c on the continuous part or divide by c on the discrete part in their original result (Thorem 3 in [8]).

We consider the element

$$\begin{aligned} x &= \lambda(p_1 + p_2 + \cdots + p_n) - \lambda n\alpha \cdot 1 \\ &= \lambda(p_1 - \alpha \cdot 1) + \lambda(p_2 - \alpha \cdot 1) + \cdots + \lambda(p_n - \alpha \cdot 1), \end{aligned} \tag{4.2}$$

translated so as to be zero-expectation. The probability measure for this element x is the same as one in Theorem 2.1 but $\lambda n\alpha$ left shifted. We shall derive that the orthogonal polynomials for the probability measure of x can be given as the constant recursion formula (4.1) with parameters

$$a = \lambda(1 - 2\alpha), \quad b = (n-1)\lambda^2\alpha(1-\alpha), \quad c = \frac{n}{n-1}, \quad \alpha_0 = 0. \tag{4.3}$$

from the combinatorial nature of the element x. If we set $y_i = \lambda(p_i - \alpha 1)$ then $\{y_i\}_{i=1}^n$ is a free family with $\phi(y_i) = 0$ and we have

$$y_i^2 = \lambda(1 - 2\alpha)y_i + \lambda^2\alpha(1-\alpha). \tag{4.4}$$

We denote by s_m, the sum of all reduced words (adjacently distinct product) of y_i's, of length m, that is,

$$s_m = \sum_{i_\ell \neq i_{\ell+1}} y_{i_1} y_{i_2} \cdots y_{i_m}. \tag{4.5}$$

Proposition 4.1. *The set* $S = \{1, s_m \mid m \geq 1\}$ *is an orthogonal system with respect to the inner product* $\langle x|y\rangle = \phi(y^* x)$.

Proof. By the freeness of y_i's, it is clear that $\phi(s_m) = 0$ for all $m \geq 1$. We also note the following fact which follows from the freeness and the relation (4.4) by induction: Let w_1 and w_2 be reduced words of y_i's such that

$$w_1 = y_{i_1} y_{i_2} \cdots y_{i_m} \quad (i_\ell \neq i_{\ell+1}), \qquad w_2 = y_{j_1} y_{j_2} \cdots y_{j_k} \quad (j_\ell \neq j_{\ell+1}). \tag{4.6}$$

Then we have

$$\begin{aligned} \langle w_1 | w_2 \rangle &= \phi\left((y_{j_1} y_{j_2} \cdots y_{j_k})^* (y_{i_1} y_{i_2} \cdots y_{i_m})\right) \\ &= \delta_{m,k} \delta_{i_1,j_1} \delta_{i_2,j_2} \cdots \delta_{i_m,j_m} \left(\lambda^2 \alpha(1-\alpha)\right)^m, \end{aligned} \tag{4.7}$$

where δ means Kronecker's delta. Now it is clear that

$$\langle s_m | s_k \rangle = \phi\left(s_k^* s_m\right) = \begin{cases} 0 & \text{if } m \neq k \\ n(n-1)^{m-1} \left(\lambda^2 \alpha(1-\alpha)\right)^m & \text{if } m = k \end{cases} \tag{4.8}$$

since s_m has $n(n-1)^{m-1}$ terms. □

Let $P_m(X) \in \mathbb{R}[X]$ $(m \geq 0)$ be the orthogonal polynomials with respect to the probability measure ν of the element x. For this sequence of the polynomials, we make the self-adjoint elements $P_m(x)$, where $P_0(x)$ should be regarded as scalar. The relations (4.4) ensures by induction that the monomial $x^m = (y_1 + y_2 + \cdots + y_n)^m$ can be expanded as the linear combination of $\{1, s_1, s_2, ..., s_m\}$. Hence, we can write $P_m(x)$ in the form that

$$P_m(x) = \gamma_{m,0} \cdot 1 + \sum_{j=1}^{\infty} \gamma_{m,j} s_j \quad (m \geq 1), \tag{4.9}$$

where $\gamma_{m,j} = 0$ for $j > m$ and $\gamma_{m,m} = 1$.

Proposition 4.2. *For all* $m \geq 1$, *we have*

$$P_m(x) = s_m. \tag{4.10}$$

Proof. Since the elements $P_m(x)$ $(m \geq 0)$ are self-adjoint and $\{P_m(X)\}$ is a system of the orthogonal polynomials with respect to the measure ν, if $0 \leq k < m$ then we obtain

$$\int_{\mathbb{R}} P_m(t) P_k(t) d\nu(t) = \phi(P_m(x) P_k(x)) = \langle P_k(x) | P_m(x) \rangle = 0. \tag{4.11}$$

Hence for $0 \leq k < m$ we have

$$\gamma_{m,0} \gamma_{k,0} + \sum_{j=1}^{\infty} \gamma_{m,j} \gamma_{k,j} ||s_j||_2^2 = 0 \tag{4.12}$$

by the orthogonality of the set $S = \{1, s_j \mid j \geq 1\}$, where $||s_j||_2^2 = \langle s_j | s_j \rangle$. Setting $k = 0$, we can see that $\gamma_{m,0} = 0$ for all $m \geq 1$. Take $k = 1$ in (4.12) then we have $\gamma_{1,1} \gamma_{m,1} ||s_1||_2^2 = 0$, thus $\gamma_{m,1} = 0$ for $m \geq 2$. Increasing k, we can conclude that $P_m(x) = s_m$ by induction. □

Proposition 4.3. *For all $m \geq 2$, we have the relation*

$$s_{m+1} = (x - \lambda(1-2\alpha))\, s_m - (n-1)\lambda^2\alpha(1-\alpha)s_{m-1}. \tag{4.13}$$

Proof. We denote by $s_m^{(j)}$ the sum of all reducced words of y_i's of length m, starting not with y_j, that is,

$$s_m^{(j)} = \sum_{\substack{i_\ell \neq i_{\ell+1} \\ i_1 \neq j}} y_{i_1} y_{i_2} \cdots y_{i_m}, \qquad j = 1, 2, ..., n. \tag{4.14}$$

Then it is easy to see that

$$s_m = s_m^{(j)} + y_j s_{m-1}^{(j)}, \quad s_m = \sum_{j=1}^{n} y_j s_{m-1}^{(j)}, \quad \sum_{j=1}^{n} s_m^{(j)} = (n-1)s_m. \tag{4.15}$$

Hence we obtain

$$\begin{aligned} y_j s_m &= y_j s_m^{(j)} + y_j^2 s_{m-1}^{(j)} \\ &= y_j s_m^{(j)} + \lambda(1-2\alpha) y_j s_{m-1}^{(j)} + \lambda^2\alpha(1-\alpha)s_{m-1}^{(j)}. \end{aligned} \tag{4.16}$$

Taking the summation for j, we have

$$x s_m = s_{m+1} + \lambda(1-2\alpha)s_m + (n-1)\lambda^2\alpha(1-\alpha)s_{m-1}. \tag{4.17}$$

□

Moreover it follows by the relation (4.4) that

$$\begin{aligned} x s_1 &= (y_1 + y_2 + \cdots + y_n)^2 = \sum_{i \neq j} y_i y_j + \sum_i y_i^2 \\ &= s_2 + \lambda(1-2\alpha)\sum_i y_i + n\lambda^2\alpha(1-\alpha) \\ &= s_2 + \lambda(1-2\alpha)s_1 + (n-1)\lambda^2\alpha(1-\alpha)\left(\frac{n}{n-1}\right). \end{aligned} \tag{4.18}$$

Hence we obtain the orthogonal polynomials with the constant recursion parameters (4.3) for the probability measure of the element x.

Comparing the measure in Theorem 2.1 with the renormalized result in [8], we can also obtain the above parameters but the above method is constructive and applicable for the case of semiradial (for the measures in Theorem 2.2) in later.

Example 4.4. (The free de Moivre-Laplace theorem) It is obvious that if we take $\lambda = (n\alpha(1-\alpha))^{-1/2}$ then the element x is standardized to be of variance 1. In this case, the recursion formula can be given by

$$\begin{aligned} P_0(X) &= \frac{n}{n-1}, \quad P_1(X) = X, \\ P_{m+1}(X) &= \left(X - \frac{(1-2\alpha)}{\sqrt{n\alpha(1-\alpha)}}\right) P_m(X) - \frac{n-1}{n} P_{m-1}(X). \end{aligned} \tag{4.19}$$

Taking the limit as $n \to \infty$, the relation will become one for the well-known Chebychev polynomials, which are orthogonal with respect to the standard semicircle law, $\frac{1}{2\pi}\sqrt{4-t^2}$. It is nothing but the free analogue of de Moivre-Laplace theorem.

Example 4.5. (The free Poisson distribution) If we put the polynomial $Q_m(X) = P_m(X - \lambda n\alpha)$ then $\{Q_m(X)\}$ satisfies the recursion formula

$$\begin{aligned} &Q_0(X) = \frac{n}{n-1}, \quad Q_1(X) = X - \lambda n\alpha, \\ &Q_{m+1}(X) = (X - \lambda n\alpha - \lambda(1-2\alpha))\, Q_m(X) - (n-1)\lambda^2\alpha(1-\alpha)Q_{m-1}(X), \end{aligned} \tag{4.20}$$

and it must be the orthogonal polynomials for the probability measure of the element $\lambda(p_1 + p_2 + \cdots + p_n)$. The free Poisson distribution can be introduced as the weak limit distribution that

$$\lim_{n\to\infty} \left(\left(1 - \frac{\alpha}{n}\right) \delta_0 + \frac{\alpha}{n}\delta_\lambda \right)^{\boxplus n}, \tag{4.21}$$

where $\boxplus n$ means n-fold free convolution with itself (See, for instance, [21]). It is obvious that the distribustion (4.21) is the same one for the scalar multiple of the simple sum of free n-projections, $\lambda \sum_{i=1}^{n} p_i$ with $\phi(p_i) = \frac{\alpha}{n}$. Thus, substitute α in (4.20) by $\frac{\alpha}{n}$ and take the limit as $n \to \infty$, we have the recursive relation for the free Poisson distribution (4.21) that

$$\begin{aligned} &Q_0(X) = 1, \quad Q_1(X) = X - \lambda\alpha, \\ &Q_{m+1}(X) = (X - \lambda(\alpha+1))\, Q_m(X) - \lambda^2\alpha Q_{m-1}(X). \end{aligned} \tag{4.22}$$

We can also give the orthogonal polynomials for the measures in Theorem 2.2 by determining the Jacobi parameters for the (non-constant) recursive relation (See [3] for detail) .

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