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Algebraic Curves and Balanced n-ary Designs

筑波大学社会工学系 藤原 良 (Ryoh Fuji-Hara)
筑波大学社会工学研究科 篠原 聡 (Satoshi Shinohara)

Abstract

Let $\mathcal{V}$ be a set of points and $\mathcal{B}$ a collection of multi-subsets (called blocks) of $\mathcal{V}$ each of size k. A balanced n-ary design is a pair $(\mathcal{V}, \mathcal{B})$ such that each pointt occurs at most $n-1$ times in any block, and each unordered pair of distinct points occurs λ times in the blocks of $\mathcal{B}$. Note that in the block $\{x, x, y\}$, the pair $\{x, y\}$ is counted twice in the block. We show here some constructions of balanced n-ary designs by using algebraic curves over finite fields.

1 Introduction

Let $\mathcal{V}$ be a set of v points and $\mathcal{B}$ a collection of multi-subsets, called blocks, of $\mathcal{V}$. A balanced n-ary design is a pair $(\mathcal{V}, \mathcal{B})$ satisfying

(1) each block is of a constant size k,

(2) each point occurs at most $n-1$ times in any block $B \in \mathcal{B}$, and

(3) each unordered pair of distinct points occurs exactly λ times in the blocks of $\mathcal{B}$.

Note that, for example, the block size of $B = \{x, x, x, y, y, z\}$ is 6 since the points x, y and z occur 3 times, twice and once, respectively. And in the block B the pairs $\{x, y\}$, $\{y, z\}$ and $\{x, z\}$ are counted 6, 2 and 3 times, respectively.

Let $N = (n_{ij})$ be a $|\mathcal{V}| \times |\mathcal{B}|$ matrix, where n_{ij} is the number of occurrences of the i-th point in the j-th block. We consider N as the incidence matrix of a balanced n-ary design. Using the incidence matrix, the conditions in the definition of a balanced n-ary design can be rewritten as follows:

(1′) $\sum_i n_{ij} = k$ for any j,

(2′) $0 \leq n_{ij} \leq n - 1$ for any i, j, and

(3′) $\sum_j n_{ij} n_{i'j} = \lambda$ for any unordered pair $\{i, i'\}$, $i \neq i'$.

Example 1.1. Let $\mathcal{V} = \{a, b, c, d, e\}$ and $\mathcal{B}$ be the collection of the following blocks:

$$\{a, b, b, d, d, e, e, e\}, \{a, a, c, c, c, d, e, e\},$$
$$\{b, b, b, c, c, d, d, e\}, \{a, a, b, c, c, d, d, d\},$$
$$\{a, a, a, b, b, c, e, e\}, \{a, c, c, d, d, d, e, e\},$$
$$\{a, a, b, b, b, c, d, d\}, \{b, b, c, c, d, e, e, e\},$$
$$\{a, a, b, b, c, c, c, e\}, \{a, a, a, b, d, d, e, e\}.$$

Then $(\mathcal{V}, \mathcal{B})$ is a balanced 4-ary (quaternary) design with 5 points, 10 blocks and the block size is 8. The incidence matrix of the above balanced 4-ary design is

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 3 & 1 & 2 & 0 & 2 & 3 \\ 2 & 0 & 3 & 1 & 2 & 0 & 3 & 2 & 2 & 1 \\ 0 & 3 & 2 & 2 & 1 & 2 & 1 & 2 & 3 & 0 \\ 2 & 1 & 2 & 3 & 0 & 3 & 2 & 1 & 0 & 2 \\ 3 & 2 & 1 & 0 & 2 & 2 & 0 & 3 & 1 & 2 \end{bmatrix},$$

and for any pair $\{i, i'\}$, $i \neq i'$,

$$\sum_j n_{ij} n_{i'j} = 23 = \lambda.$$

Let $\rho_s^{(i)}$ be the number of blocks containing the i-th point exactly s times, i.e. the number of the entry s in the i-th row of the incidence matrix. If $\rho_s^{(i)} = \rho_s$ for all i then the design is said to be *regular*. If a balanced n-ary design is regular then the replication number R_i of the i-th point is a constant number R, since

$$R_i = \sum_{r=0}^{n-1} r\rho_r^{(i)} = \sum_r r\rho_r = R.$$

Balanced n-ary designs were first introduced by Tocher [3] in a statistical paper. The interested reader is referred to [1, 2] for their excellent surveys.

2 Algebraic curves

Let q be a prime power, and $GF(q)$ a finite field of order q. A *divisor* D on a curve C is a formal sum $\sum_{P \in C} m_P P$, $m_P \in \boldsymbol{Z}$. The set of divisors on C is denoted by Div(D). The *support* of a divisor D, denoted by Supp(D), is the set of points satisfying $m_P \neq 0$. A divisor D is said to be *efficient* if $m_P \geq 0$ for all P, and denoted by $D \geq 0$. The *degree* of D is $\deg D = \sum_P m_P$. Let $D = \sum_P m_P P$ and $E = \sum_P m'_P P$. Then $D + E = \sum_P (m_P + m'_P)P$.

Let Rat(C) be the set of rational functions over a curve C. The divisor of a rational function $f \in \text{Rat}(C)$ is $\text{div}(f) = \sum_{P \in C} m_P P$, where m_P is the order of f at P. For a divisor D, a divisor E is said to be *equivalent to* D, denoted by $E \sim D$, if there exists a rational function $f \in \text{Rat}(C)$ satisfying $E = D + \text{div}(f)$.

Let C be a curve defined by $F = 0$, where F is a polynomial over a finite field $GF(q)$. The divisor of the curve C is the divisor of F. Note that $\text{div}(C) = \text{div}(F) \geq 0$. When the divisor of C is written as $\text{div}(C) = \sum_p m_p p$, the *intersection multiplicity* of a point p on a curve C' with the curve C is the order of F at p, say m_p.

Let $L(D) = \{f \in \text{Rat}(C) : \text{div}(f) + D \geq 0\} \cup \{0\}$. It is well-known that $L(D)$ is a linear space with a finite dimension over an extension field $GF(q^m)$ of $GF(q)$.

3 Algebraic curves and balanced n-ary designs

Let C be an irreducible curve defined over $GF(q)$ and $\mathcal{C} = \{C_1, \cdots, C_b\}$ a set of b curves defined over an extension $GF(q^m)$ of $GF(q)$. Let V_j be a set of intersection points of $C_j \in \mathcal{C}$ with C, and $\mathcal{V} = \bigcup_j V_j = \{p_1, \cdots, p_v\}$.

Let n_{ij} denotes the intersection multiplicity of a point p_i with a curve C_j. We consider the $v \times b$ matrix (n_{ij}) as the incidence matrix. To satisfy the conditions of balanced n-ary designs, we have to choose suitable C, $\mathcal{C}$ and $\mathcal{V}$.

The first condition is required for the block size to be constant, i.e., $\sum_i n_{ij} = k$ for any curve $C_j \in \mathcal{C}$. Let D be a divisor on C and $\mathcal{D} = \{\mathrm{div}(f) + D : f \in L(D) \setminus \{0\}\} = \{E_1, \cdots, E_b\}$. Note that each element of $\mathcal{D}$ is the divisor of a curve.

Lemma 3.1. *Let $\mathcal{C}$ be the set of curves such that their divisors are the elements of $\mathcal{D}$. For any curve of $\mathcal{C}$, the total number of multiplicities of its intersection points with C is a constant number k.*

Proof. The set $\mathcal{D}$ is the set of efficient divisors being equivalent to D, i.e.,

$$\begin{aligned}\mathcal{D} &= \{\mathrm{div}(f) + D : f \in L(D) \setminus \{0\}\} \\ &= \{E \in \mathrm{Div}(C) : E \geq 0, E \sim D\}.\end{aligned}$$

It is well-known that if $E \sim D$ then $\deg D = \deg E$. Hence for any j

$$\sum_i n_{ij} = \deg E_j = \deg D = k.$$

□

We assume here that we choose $\mathcal{C}$ as above, and that the point set $\mathcal{V}$ of a design is the set $\mathcal{V} = \bigcup_{E \in \mathcal{D}} \text{Supp}(E)$. The second condition says that each point of the design occurs at most $n-1$ times. Since the intersection multiplicities of points on curves are always positive, this condition is automatically satisfied from Lemma 3.1.

Theorem 3.2. *Let C be a curve defined over a finite field, D a divisor on C, $\mathcal{D} = \{div(f) + D : f \in L(D) \setminus \{0\}\} = \{E_1, \cdots, E_b\}$, $\mathcal{V} = \bigcup_{E \in \mathcal{D}} Supp(E)$ $E_j = \sum n_{ij} P_i$, $P_i \in V$. If $\deg C \leq 2$ and $D \geq 0$ then (n_{ij}) is the incidence matrix of a balanced n-ary design $(\mathcal{V}, \mathcal{D})$.*

Proof. We only have to check whether the third condition of balanced n-ary designs is satisfied. Let E_j be the j-th element of $\mathcal{D}$. Assume that D and E_j's have the forms $D = \sum_i m_i P_i$ and $E_j = \text{div}(f_j) + D$, respectively. Let $\text{div}(f_j) = \sum_i e_{ij} P_i$. Then each entry n_{ij} of the incidence matrix (n_{ij}) is $m_i + e_{ij}$, since $\text{div}(f_j) + D = \sum_i e_{ij} P_i + D = \sum_i (m_i + e_{ij}) P_i$. For any pair

$\{i, i'\}$, we have

$$\begin{aligned}
&\sum_j n_{ij} n_{i'j} \\
=&\sum_j (m_i + e_{ij})(m_{i'} + e_{i'j}) \\
=&\sum_j (m_i m_{i'} + m_i e_{i'j} + m_{i'} e_{ij} + e_{ij} e_{i'j}) \\
=&b m_i m_{i'} + m_i \sum_j e_{i'j} + m_{i'} \sum_j e_{ij} + \sum_j e_{ij} e_{i'j}.
\end{aligned}$$

Let $\lambda(a, b)$ be the number of j satisfying $(e_{ij}, e_{i'j}) = (a, b)$. Then we have

$$\sum_j e_{ij} e_{i'j} = \sum_{(a,b)} ab\lambda(a, b).$$

When the degree of base curve C is less than or equal to 2, it can be easily seen that if $a + b = c + d$ then $\lambda(a, b) = \lambda(c, d)$. Moreover we can see that

$$\sum_j e_{i'j} = \sum_j e_{ij} = \sum_a a\lambda(a),$$

where $\lambda(a)$ is the number of j satisfying $n_{ij} = a$. Since both of $\lambda(a, b)$ and $\lambda(a)$ are independent of $\{i, i'\}$ chosen, we have

$$\sum_j n_{ij} n_{i'j} = \lambda,$$

and we can conclude that the third condition in the definition of a balanced n-ary design is also satisfied. □

References

[1] E. J. Billington, *Balanced n-ary designs: combinatorial survey and some new results*, Ars Combin. 17A (1984), 37 – 72.

[2] E. J. Billington, *Designs with repeated elements in blocks: a survey and some recent results*, Congr. Numer. 68 (1989), 123 – 146.

[3] K. D. Tocher, *Design and analysis of block experiments*, J. R. Statist. Soc. B14 (1952), 45 – 100.